Chromatic cohomology of finite groups 5

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December 6, 2023

Symmetric groups

- Let $\mathcal{F}$ be the groupoid of finite sets and bijections, and $\mathcal{F}_d$ the subgroupoid of sets of order d.
- Then $B\mathcal{F}_d \simeq B\Sigma_d$ and $B\mathcal{F} \simeq \coprod_d B\Sigma_d$.
- There is a diagonal functor $\delta\colon \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ given by $\delta(X) = (X, X)$, and functors $\sigma, \mu\colon \mathcal{F}^2 \to \mathcal{F}$ given by $\sigma(X, Y) = X \amalg Y$ and $\mu(X, Y) = X \times Y$.
- These give maps $\Sigma^\infty_+ B\mathcal{F} \xleftarrow{\sigma,\mu} \Sigma^\infty_+ B\mathcal{F}^2 \xleftarrow{\delta} \Sigma^\infty_+ B\mathcal{F}$ and also transfers $\Sigma^\infty_+ B\mathcal{F} \xrightarrow{\sigma_!,\mu_!} \Sigma^\infty_+ B\mathcal{F}^2 \xrightarrow{\delta_!} \Sigma^\infty_+ B\mathcal{F}$.
- These satisfy many relations and give rich algebraic structure on $E^0(B\mathcal{F})$.
- Everything is easy to understand in generalised character theory.
- Recall that $\Theta^* = \mathbb{Z}_p^n$, and let $\mathbb{A} = \pi_0[\Theta^*, \mathcal{F}]$ be the set of isomorphism classes of finite sets with Θ^*-action. Then $L\widehat{\otimes}_{E^0} E^0(B\mathcal{F}) = \mathrm{Map}(\mathbb{A}, L)$.

$$\sigma^*(f)(X, Y) = f(X \amalg Y) \qquad \sigma^!(f \otimes g)(X) = \sum_{X = Y \amalg Z} f(Y)g(Z)$$

$$\mu^*(f)(X, Y) = f(X \times Y) \qquad \mu^!(f \otimes g)(X) \sim \sum_{X = Y \times Z} f(Y)g(Z)$$

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- Analogy: the group $\Theta = (\mathbb{Q}/\mathbb{Z}_{(p)})^n$ is like the formal group scheme $\mathbb{G}$.
- We would like to say: $\mathrm{spf}(E^0(B\mathcal{F}))$ is the scheme of iso classes of sets with action of $\mathrm{Hom}(\mathbb{G}, \mathbb{Q}/\mathbb{Z}_{(p)})$. But we do not know how to interpret that.
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- Put $I = \ker(E^0(B\mathcal{F}) \to E^0)$ and $I^{*2} = \sigma_!(I \otimes I)$ and $Q = I/I^{*2}$. This is still a ring using δ^*, with $L\widehat{\otimes}_{E^0} Q = \mathrm{Map}(\mathrm{Sub}(\Theta), L)$.
- We do know how to interpret $\mathrm{Sub}(\mathbb{G})$ as a moduli scheme of finite subgroups of $\mathbb{G}$, and the main theorem is that $\mathrm{spf}(Q) = \mathrm{Sub}(\mathbb{G})$.
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- If we tensor with L then the relationship between $\mathbb{G}$ and Θ becomes close, so $L \otimes_{E^0} R_d \simeq L \otimes_{E^0} R'_d \simeq \mathrm{Map}(\mathrm{Sub}_{p^d}(\Theta), L)$.
- Topological methods show that R'_d is a free module over E^0.
- By the above, the rank is $|\mathrm{Sub}_{p^d}(\Theta)|$, which can be computed by algebra.

Finite subgroups of formal groups

- Now consider the formal group $\mathbb{G}$ for Morava E-theory.
- There is a universal ring $R_d = \mathcal{O}_{\mathrm{Sub}_{p^d}(\mathbb{G})}$ for E^0-algebras equipped with a finite subgroup $A < \mathrm{spf}(R_d) \times_S \mathbb{G}$ of order p^d.
- Put $R'_d = E^0(B\Sigma_{p^d})/J$, where J is the sum of images of transfer maps from $E^0(B(\Sigma_i \times \Sigma_j))$ with $i, j > 0$ and $i + j = p^d$.
- The standard representation V of Σ_{p^d} on $\mathbb{C}^{p^d}$ gives a divisor $A = \mathrm{spf}(E^0((PV)_{h\Sigma_{p^d}}))$ on $\mathbb{G}$ over $\mathrm{spf}(E^0(B\Sigma_{p^d}))$.
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- Then $b^k P$ is a cyclic module over P/a and $b^i P/b^{i+1}P$ is a cyclic module over P/b so $\dim(P) \leq \dim(P/a) + k \dim(P/b)$.
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- Thus for $u \in E^0(X)$ we have $u^{\otimes p} \in E^0(X^p)$, invariant under permutation.
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- Commutativity of ring spectra is subtle; but the conclusion is that there is a power operation $Pu \in E^0(X^p_{h\Sigma_p})$ mapping to $u^{\otimes p}$.
- This has $P0 = 0$ and $P1 = 1$ and $P(uv) = P(u)\,P(v)$ and $P(u+v) = P(u) + P(v) + \text{transfers}$.
- It follows that P induces a ring map $E^0(X) \to E^0(X^p_{h\Sigma_p})/J \simeq R_1 \otimes E^0(X)$.
- There is a similar story involving Σ_{p^d} for $d > 1$.
- Taking $X = \mathbb{C}P^\infty$ we get a ring map $\mathcal{O}_{\mathbb{G}} \to \mathcal{O}_{\text{Sub}_{p^d}(\mathbb{G})} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathbb{G}}$, corresponding to a map $\text{Sub}_{p^d}(\mathbb{G}) \times_S \mathbb{G} \to \mathbb{G}$.
- You should think of $\mathbb{G}$ as a bundle of groups over S.
  Given a point $a \in S$ and a subgroup $A < \mathbb{G}_a$, it turns out that there is a canonical point $b \in S$ and a surjective homomorphism $q_{a,A} \colon \mathbb{G}_a \to \mathbb{G}_b$ with kernel A. The above map sends (A, x) to $q_A(x)$.

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General linear groups over $\overline{F}$

Theorem

The inclusion $GL_1(\overline{F})^d \to GL_d(\overline{F})$ induces $GL_d(\overline{F})_E \simeq \mathbb{H}^d/\Sigma_d \simeq \mathrm{Div}_d^+(\mathbb{H})$.

Equivalently,

$$E^0(BGL_1(\overline{F})^d) = E^0[\![x_1, \dots, x_d]\!],$$

and $E^0 BGL_d(\overline{F})$ is the subring of symmetric functions, generated by elementary symmetric functions $c_1, \dots, c_d$.

Proof.

This is built into the foundations of étale homotopy theory.

The main point is that one can build a torsion-free local ring $\overline{W}$ (the Witt ring of $\overline{F}$) with residue field $\overline{F}$.

One can then choose an embedding $\overline{W} \to \mathbb{C}$.

Using the fact that $|F|$ is coprime to p, one can check that the maps

$$BGL_d(\overline{F}) \leftarrow BGL_d(\overline{W}) \to BGL_d(\mathbb{C})$$

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The claim follows easily from this. $\square$

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Recall that the group $\Gamma = \mathrm{Gal}(\overline{F}/F)$ is generated by the Frobenius map ϕ.

Theorem (Tanabe)

The elements

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form a regular sequence, and

$$E^0 BGL_d(F) = \frac{E^0[\![c_1, \dots, c_d]\!]}{(\phi^*(c_1) - c_1, \dots, \phi^*(c_d) - c_d)} = (E^0 BGL_d(\overline{F}))_\Gamma.$$

Equivalently, we have $BGL_d(F)_E = \mathrm{Div}_d^+(\mathbb{H})^\Gamma$.

In many respects this is very satisfactory, but there are many natural questions that cannot be answered without more detailed algebraic analysis.

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Groupoids

- Let $\mathcal{V}$ be the groupoid of finite dimensional vector spaces over F, and their isomorphisms. Then $B\mathcal{V} \simeq \coprod_d BGL_d(F)$.
- We write $\overline{\mathcal{V}}$ for the corresponding groupoid for $\overline{F}$, so $B\overline{\mathcal{V}} \simeq \coprod_d BGL_d(\overline{F})$.
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and the functor $V \mapsto \overline{F} \otimes_F V$ gives $B\mathcal{V}_E = \mathrm{Div}^+(\mathbb{H})^\Gamma$.
- The functors $\oplus, \otimes \colon \mathcal{V}^2 \to \mathcal{V}$ make $B\mathcal{V}$ a commutative semiring in the homotopy category of spaces.This in turn makes $B\mathcal{V}_E$ a commutative semiring in the category of formal schemes.This matches an obvious commutative semiring structure on $\mathrm{Div}^+(\mathbb{H})^\Gamma$.
- Alternatively, $E_*^\vee(B\mathcal{V})$ and $K_*(B\mathcal{V})$ are Hopf rings.
- Some other groupoids are also relevant, for example

$$\mathcal{L} = \{(X, L) \mid X \text{ is a finite set, and } L \text{ is an } F\text{-linear line bundle over } X\}.$$

This has $B\mathcal{L} \simeq \coprod_d E\Sigma_d \times_{\Sigma_d} BGL_1(F)^d$.
There is a functor $\pi \colon \mathcal{L} \to \mathcal{V}$ given by $\pi(X, L) = \bigoplus_x L_x$.
- The index of $\Sigma_d \wr GL_1(F)^d$ in $GL_d(F)$ has index coprime to p, so $B\mathcal{L} \to B\mathcal{V}$ gives an epimorphism in E-cohomology. Earlier work on symmetric groups gives a good understanding of $E^0 B\mathcal{L}$.

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- Everything is easy to understand in generalised character theory.
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The Atiyah-Hirzebruch Spectral Sequence

- Theorem: $E_0^\vee B\mathcal{V}$ is also polynomial.
- It is enough to prove that $K_0 B\mathcal{V}$ is polynomial.
- We use the Atiyah-Hirzebruch spectral sequence $H_*(B\mathcal{V}; K_*) \Longrightarrow K_*(B\mathcal{V})$ and its dual.
- Quillen: $H_*(B\mathcal{V}; K_*)$ is generated by $B\mathcal{V}_1$ and has countably many polynomial generators b_i and exterior generators e_i.
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- Tanabe and HKR also tell us that $K_*(B\mathcal{V})$ is concentrated in even degrees, with known rank.
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Irreducibles

- Put $I = \ker(E^0(B\mathcal{V}) \to E^0)$ and $I^{*2} = \sigma^!(I \otimes I)$ and $Q = I/I^{*2}$.
- This is still a ring with $L \otimes_{E^0} Q = \mathrm{Map}(\mathrm{Irr}(\Theta^*), L)$, where $\mathrm{Irr}(\Theta^*) = \mathrm{Hom}(\Theta^*, GL_1(\overline{F}))/\mathrm{Gal}$ is the set of isomorphism classes of irreducible F-linear representations of Θ^*.
- We find that $Q \simeq \prod_m D_m^\Gamma$, where $D_m^\Gamma = E^0[\![y]\!]/g_m(y)$ for a certain monic polynomial $g_m(y)$.
- All this and many more details have nice interpretations in formal group theory.

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