Chromatic cohomology of finite groups 4

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Duality

- Recall: for a finite groupoid G we put $M^*(G) = \mathrm{Map}(\pi_0(G), \mathbb{Q})$. We define $\theta\colon M^*(G) \to \mathbb{Q}$ by $\theta(h) = \sum_{i<r} |G(a_i, a_i)|^{-1} h(a_i)$, where $\{a_i \mid i < r\}$ contains one representative of each isomorphism class. We then define $\langle f, g\rangle_G = \theta(fg)$.This is a perfect pairing on $M^*(G)$.
- The HKR theorem says $L \otimes_{E^0} E^0(BG) = L \otimes_{\mathbb{Q}} M^*(\Lambda G)$, where $\Lambda G = [\Theta^*, G] = [\mathbb{Z}_p^n, G]$ is again a finite groupoid.
- We therefore have $\theta\colon L \otimes_{E^0} E^0(BG) \to L$ giving a perfect pairing.
- **Theorem:** this comes from a map $\theta\colon E^0(BG) \to E^0$ which also gives a perfect pairing (at least when $E^0(BG)$ is a free module over E^0).
- This is like Poincaré duality for oriented manifolds: the map θ is like the map $u \mapsto \langle u, [M]\rangle$ from $H^d(M)$ to $\mathbb{Z}$.
- It is also like the map $\theta\colon R(G) \to \mathbb{Z}$ given by $\theta([V]) = \dim(V^G)$, where $R(G)$ is the complex representation ring. This also gives a perfect pairing.
- The above theorem (due to Greenlees and Sadofsky) was the first known example of chromatic ambidexterity; there is now a more general theory.
- The proof of the theorem uses transfers and Tate spectra.

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Collapse and transfer

- Consider a map $f: X \to Y$ of finite sets.
  This induces a map $f: \mathbb{Z}[X] \to \mathbb{Z}[Y]$,
  and also a map $f^t: \mathbb{Z}[Y] \to \mathbb{Z}[X]$ given by $f^t([y]) = \sum_{f(x)=y} [x]$.
- The suspension spectrum $\Sigma^\infty X_+$ is a kind of refinement of $\mathbb{Z}[X]$,
  and we again have an easy map $f: \Sigma^\infty X_+ \to \Sigma^\infty Y_+$.
  We also want a map $f^T: \Sigma^\infty Y_+ \to \Sigma^\infty X_+$.
- Put $V = \mathbb{R}[X]$, giving $i: X \to V \subset S^V = V \cup \{\infty\}$.
- Put $s(v) = v/\sqrt{2(1 + \|v\|^2)}$,
  giving a homeomorphism from V to an open ball of radius $1/\sqrt{2}$.
- Define $\widetilde{f}: V \times X \to V \times Y$ by $\widetilde{f}(v, x) = (s(v) + i(x), f(x))$,
  so $\widetilde{f}$ is an open embedding covering f.
- Define $c: S^V \wedge Y_+ = (V \times Y) \cup \{\infty\} \to (V \times X) \cup \{\infty\} = S^V \wedge X_+$
  by $c(\widetilde{f}(v, x)) = (v, x)$ and $c(v, y) = \infty$ for $(v, y) \notin \mathrm{image}(\widetilde{f})$.
- This is completely natural and so is compatible with any group actions.
- In the world of spectra we have a negative sphere S^{-V} and we can take
  the smash product with this to get $f^t: \Sigma^\infty Y_+ \to \Sigma^\infty X_+$.
- Take $f = (G/H \to G/G = 1)$ and apply $EG_+ \wedge_G (-)$ to f^t and use
  $EG/H \simeq BH$ to get a map $\Sigma^\infty BG_+ \to \Sigma^\infty BH_+$ (the *transfer*).

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- The unit sphere $S(\infty L)$ is contractible and has free G-action so we can take $EG = S(\infty L)$. Thus, the reduced suspension $\widetilde{E}G$ is the same as $S^{\infty L} = \varinjlim_k S^{kL}$, where $S^{kL} \simeq S^{2k}$ is the one-point compactification of nL.
- It is not hard to check that $G_+ \wedge \widetilde{E}G = F(G_+, \widetilde{E}G) = EG_+ \wedge \widetilde{E}G = F(EG_+, \widetilde{E}G) = 0$. Using this one can identify $t_G(K)$ with $(\widetilde{E}G \wedge F(EG_+, K))^G$.
- Using $\widetilde{E}G = \varinjlim S^{kL}$ and $S^{kL} \wedge F(EG_+, K) = F(S^{-kL} \wedge EG_+, K)$ we get $t_G(K) = \varinjlim_k F(BG^{-kL}, K)$, where BG^{-kL} is the Thom spectrum.
- Using the fact that the Euler class of kL is x^k, we find that $\pi_*(t_G(K)) = K^{-*}(BG)[x^{-1}] = (K^*[x]/x^{p^n})[x^{-1}] = 0$ as required.
- For general G: combine fairly similar arguments with an induction on $|G|$.
- Conclusion: $K_*(BG)$ maps isomorphically to $K^*(BG)$, which is dual to $K_*(BG)$, so $K^*(BG)$ is self-dual.
- By rearranging the argument slightly: the K-local spectrum $L_K\Sigma^\infty BG_+$ is self-dual, and $E^*(BG)$ is self-dual provided that it is a free E^*-module.

More about duality

- We have shown that $K^*(BG)$ is naturally self-dual when G is a finite group. There is an easy generalisation to groupoids.
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- **Theorem:** the Frobenius form $\theta\colon E^0(BC_{p^m}) = E^0[\![x]\!]/[p^m]_F(x) \to E^0$ is induced by ρ_m.
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Semirings with λ-operations

- The representation semiring $R_+(G)$ is the set of isomorphism classes of complex representations.
- This has addition $[V]+[W]=[V\oplus W]$ and multiplication $[V][W]=[V\otimes W]$ but no subtraction.
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- If h is the number of conjugacy classes then h is also the number of isomorphism classes of irreducible representations. These form a basis giving $R_+(G)\simeq\mathbb{N}^h$ and $R(G)\simeq\mathbb{Z}^h$ additively.
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Chern approximations

- We define $\mathrm{Ch}(G)$ to be the scheme of morphism $R_+(G) \to \mathrm{Div}^+(\mathbb{G})$ of λ-semirings, and $C(E,G) = \mathcal{O}_{\mathrm{Ch}(G)}$.
- If we understand everything about $R_+(G)$ then we can write down a presentation of $C(E,G)$ by generators and relations, partly determined by the formal group law. But it is easier to work with schemes where possible.
- Like $E^0(BG)$, the ring $C(E,G)$ is finitely generated as an E^0-module.
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- Like $E^0(BG)$, the ring $C(E,G)$ is finitely generated as an E^0-module.
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- There may be a kernel in general, consisting of relations between Chern classes that do not follow automatically from representation theory.
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The group $G = \Sigma_4$ with $n = p = 2$

The character table of Σ_4 is as follows:

class	size	1	ϵ	σ	ρ	$\epsilon\rho$
1^4	1	1	1	2	3	3
$1^2 2$	6	1	-1	0	1	-1
2^2	3	1	1	2	-1	-1
13	8	1	1	-1	0	0
4	6	1	-1	0	-1	1

The ring structure, Adams operations and λ-operations are described in the following table.

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\begin{array}{lll}
\epsilon^2 = 1 & \psi^k(\epsilon) = \epsilon^k & \lambda^2(\sigma) = \epsilon \\
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$\mathrm{Ch}(\Sigma_4)$ is the scheme of pairs $(d, D) \in \mathbb{G} \times \mathrm{Div}_3^+(\mathbb{G})$ such that

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$K^0(B\Sigma_4) = C(K, \Sigma_4) = \mathbb{F}_2[w, c_2, c_3]/J$ where

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