Chromatic cohomology of finite groups 1

Neil Strickland

December 5, 2023

Introduction

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- For any space X there are graded rings $K^*(X)$ and $E^*(X)$, called the (2-periodic) *Morava K-theory* and *Morava E-theory* of X.
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- The ring $K^0(BG)$ is a finite algebra over the finite field $\mathbb{F}_p$, so it is very amenable to explicit calculation, sometimes by computer.
- Good answers are known for abelian groups, symmetric groups, finite general linear groups of characteristic different from p, and various groups that are not far from being abelian.
- Kriz and Lee produced examples of groups G with $|G| = p^6$ and $K^1(BG) \neq 0$ and $E^0(BG)$ not free. Probably generic groups are like that. But many of the most interesting examples have $K^1(BG) = 0$.

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  This gives an even periodic theory with $MP^0(1) = \mathbb{Z}[a_1, a_2, a_3, \dots]$. This is called *periodic complex cobordism*.
- The Nilpotence (pre)Theorem of Hopkins-Devinatz-Smith: if $MP^*(u) = 0$ then $u^k = 0$ for large k. This is the most powerful known theorem of the type algebra $\Rightarrow$ topology.
- Fix a prime p and an integer $n > 0$. There is then an even periodic theory K with $K^*(1) = \mathbb{F}_p[u, u^{-1}]$. This is called *Morava K-theory*.
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  This gives an even periodic theory with $MP^0(1) = \mathbb{Z}[a_1, a_2, a_3, \dots]$. This is called *periodic complex cobordism*.
- The Nilpotence (pre)Theorem of Hopkins-Devinatz-Smith: if $MP^*(u) = 0$ then $u^k = 0$ for large k. This is the most powerful known theorem of the type algebra $\Rightarrow$ topology.
- Fix a prime p and an integer $n > 0$. There is then an even periodic theory K with $K^*(1) = \mathbb{F}_p[u, u^{-1}]$. This is called *Morava K-theory*.
- The K's together (for all p and n) carry $\sim$ the same information as MP.

Generalised cohomology

- A *generalised cohomology theory* is a contravariant, homotopy invariant functor $E^* \colon \text{Spaces} \to \text{Rings}^*$ with properties similar to H^*, but $E^*(1)$ need not be $\mathbb{Z}$. It takes work to provide interesting examples.
- We often work with *even periodic theories* where $E^1(1) = 0$ and $E^{-2}(1)$ contains a unit. Here it is natural to focus on $E^0(X)$.
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Formal groups — what are they good for?

- Every even periodic theory E gives a formal group $\mathbb{G} = P_E = \mathrm{spf}(E^0(\mathbb{C}P^\infty))$.
- The functor $E \mapsto P_E$ is not too far from being an equivalence.
- The most elementary examples of formal groups are the additive and multiplicative formal groups; these correspond to HP and KU. (Here $HP^i(X) = \prod_j H^{i+2j}(X)$.)
- Steenrod operations in $HP^0(X; \mathbb{F}_p)$ and Adams operations in $KU^0(X)$ are closely related to endomorphisms of the associated formal groups.
- The ring $MP^0(1)$ is naturally isomorphic to the Lazard ring, which plays a central role in formal group theory.
- The Morava K-theories $K(p, n)$ all have different formal groups.
- Together with $HP^0(X; \mathbb{F}_p)$ and $HP^0(X; \mathbb{Q})$ this gives all formal groups over fields up to Galois twisting.
- For many spaces X the scheme X_E can be described naturally in terms of $\mathbb{G}$. For example, if $X = BU(n) = \{n-\text{dimensional subspaces of } \mathbb{C}^\infty\}$ then $X_E = \mathbb{G}^n/\Sigma_n$.

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Examples of formal groups

- For any ring R we define commutative groups as follows:
  - $G_a(R) = \{a \in R \mid a \text{ is nilpotent }\}$ (under addition)
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- Axioms: $F(s, 0) = s$, $F(s, t) = F(t, s)$ and $F(F(s, t), u) = F(s, F(t, u))$.
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