Chromatic cohomology of finite groups 5

Neil Strickland

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Symmetric groups

- Let $\mathcal{F}$ be the groupoid of finite sets and bijections, and $\mathcal{F}_d$ the subgroupoid of sets of order d.
- Then $B\mathcal{F}_d \simeq B\Sigma_d$ and $B\mathcal{F} \simeq \coprod_d B\Sigma_d$.
- There is a diagonal functor $\delta\colon \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ given by $\delta(X) = (X, X)$, and functors $\sigma, \mu\colon \mathcal{F}^2 \to \mathcal{F}$ given by $\sigma(X, Y) = X \amalg Y$ and $\mu(X, Y) = X \times Y$.
- These give maps $\Sigma^\infty_+ B\mathcal{F} \xleftarrow{\sigma,\mu} \Sigma^\infty_+ B\mathcal{F}^2 \xleftarrow{\delta} \Sigma^\infty_+ B\mathcal{F}$ and also transfers $\Sigma^\infty_+ B\mathcal{F} \xrightarrow{\sigma_!,\mu_!} \Sigma^\infty_+ B\mathcal{F}^2 \xrightarrow{\delta_!} \Sigma^\infty_+ B\mathcal{F}$.
- These satisfy many relations and give rich algebraic structure on $E^0(B\mathcal{F})$.
- Everything is easy to understand in generalised character theory.
- Recall that $\Theta^* = \mathbb{Z}_p^n$, and let $\mathbb{A} = \pi_0[\Theta^*, \mathcal{F}]$ be the set of isomorphism
- classes of finite sets with Θ^*-action. Then $L\widehat{\otimes}_{E^0} E^0(B\mathcal{F}) = \mathrm{Map}(\mathbb{A}, L)$.

$$\sigma^*(f)(X, Y) = f(X \amalg Y) \qquad \sigma^!(f \otimes g)(X) = \sum_{X = Y \amalg Z} f(Y)g(Z)$$

$$\mu^*(f)(X, Y) = f(X \times Y) \qquad \mu^!(f \otimes g)(X) \sim \sum_{X = Y \times Z} f(Y)g(Z)$$

$$\delta^*(f \otimes g)(X) = f(X)g(X) \qquad \delta^!(f)(X, Y) = |\,\mathrm{Iso}(X, Y)| f(X).$$

Symmetric groups

- $\mathbb{A}$ is the set of isomorphism classes of finite sets with action of $\Theta^* = \text{Hom}(\Theta, \mathbb{Q}/\mathbb{Z}_{(p)}) \simeq \mathbb{Z}_p^n$; then $L\widehat{\otimes}_{E^0} E^0(B\mathcal{F}) = \text{Map}(\mathbb{A}, L)$.
- Analogy: the group $\Theta = (\mathbb{Q}/\mathbb{Z}_{(p)})^n$ is like the formal group scheme $\mathbb{G}$.
- We would like to say: $\text{spf}(E^0(B\mathcal{F}))$ is the scheme of iso classes of sets with action of $\text{Hom}(\mathbb{G}, \mathbb{Q}/\mathbb{Z}_{(p)})$. But we do not know how to interpret that.
- For a finite subgroup $A < \Theta$ we have a surjective map $\Theta^* \to A^*$ and thus an action of Θ^* on A^* so $[A^*] \in \mathbb{A}$.
- Any $[X] \in \mathbb{A}$ can be expressed uniquely as a disjoint union of $[A^*]$'s.
- Put $I = \ker(E^0(B\mathcal{F}) \to E^0)$ and $I^{*2} = \sigma_!(I \otimes I)$ and $Q = I/I^{*2}$. This is still a ring using δ^*, with $L\widehat{\otimes}_{E^0} Q = \text{Map}(\text{Sub}(\Theta), L)$.
- We do know how to interpret $\text{Sub}(\mathbb{G})$ as a moduli scheme of finite subgroups of $\mathbb{G}$, and the main theorem is that $\text{spf}(Q) = \text{Sub}(\mathbb{G})$.
- Also $E^0(B\mathcal{F})$ is polynomial over E^0 under the $\sigma^!$-product, with Q as the module of indecomposables.
- This is enough to give a basis for $E^0(B\Sigma_d)$ as a module over E^0, together with most of the algebraic structure.

Finite subgroups of formal groups

- Let $\mathbb{G}$ be a formal group over a base scheme S, so $\mathcal{O}_{\mathbb{G}} \simeq \mathcal{O}_S[\![x]\!]$.
- The addition $\sigma\colon \mathbb{G} \times_S \mathbb{G} \to \mathbb{G}$ gives $\sigma^*\colon \mathcal{O}_{\mathbb{G}} \to \mathcal{O}_{\mathbb{G}} \widehat{\otimes}_{\mathcal{O}_S} \mathcal{O}_{\mathbb{G}} \simeq \mathcal{O}_S[\![y,z]\!]$ with $\sigma^*(x) = F(y,z)$ for some formal group law F.
- A (globally defined) *finite subgroup* of $\mathbb{G}$ is a subscheme $A = \text{spf}(\mathcal{O}_{\mathbb{G}}/I) < \mathbb{G}$ with $\sigma(A \times_S A) \leq A$ such that $\mathcal{O}_A$ is a finitely generated free module over $\mathcal{O}_S$.
- The condition $\sigma(A \times_S A) \leq A$ is equivalent to $\sigma^*(I) \leq I \widehat{\otimes}_S \mathcal{O}_{\mathbb{G}} + \mathcal{O}_{\mathbb{G}} \widehat{\otimes}_S I$.
- The finite rank condition says that A is a divisor.
- **Example:** $\mathcal{O}_S = \mathbb{Z}_p$, $\mathbb{G} = \{u \mid u-1 \text{ is nilpotent}\}$, $A_n = \{u \in \mathbb{G} \mid u^{p^n} = 1\}$.
- **Example:** $\mathcal{O}_S = \mathbb{F}_p$, $\mathbb{G} = \{x \mid x \text{ is nilpotent }\}$, $A_n = \{x \in \mathbb{G} \mid x^{p^n} = 0\}$.
- More generally: consider $\mathcal{O}_S$-algebras R together with schemes $A = \text{spf}(R[\![x]\!]/I) < \text{spec}(R) \times_S \mathbb{G}$ where $\mathcal{O}_A$ is a finitely generated free module over R and A is closed under addition.
- Free module condition: $\mathcal{O}_A = R[\![x]\!]/f_A(x)$ for some polynomial $f_A(x) = \sum_{i=0}^n c_i x^{n-i}$ with $c_0 = 1$ and c_i nilpotent for $i > 0$.
- Closure under addition: certain relations among the coefficients c_i.
- Thus, there is a universal example $\mathcal{O}_{\text{Sub}_n(\mathbb{G})} = \mathcal{O}_S[\![c_1, \ldots, c_n]\!]/\text{relations}$.
- In fact $\mathcal{O}_{\text{Sub}_n(\mathbb{G})} = 0$ unless $n = p^d$ for some d.

Finite subgroups of formal groups

- Now consider the formal group $\mathbb{G}$ for Morava E-theory.
- There is a universal ring $R_d = \mathcal{O}_{\mathrm{Sub}_{p^d}(\mathbb{G})}$ for E^0-algebras equipped with a finite subgroup $A < \mathrm{spf}(R_d) \times_S \mathbb{G}$ of order p^d.
- Put $R'_d = E^0(B\Sigma_{p^d})/J$, where J is the sum of images of transfer maps from $E^0(B(\Sigma_i \times \Sigma_j))$ with $i, j > 0$ and $i + j = p^d$.
- The standard representation V of Σ_{p^d} on $\mathbb{C}^{p^d}$ gives a divisor $A = \mathrm{spf}(E^0((PV)_{h\Sigma_{p^d}}))$ on $\mathbb{G}$ over $\mathrm{spf}(E^0(B\Sigma_{p^d}))$.
- This does not satisfy the coefficient relations for A to be closed under addition, so A is not a subgroup scheme.
- If we work mod J then the relations are satisfied, so we get a subgroup defined over R'_d, and thus a map $R_d \to R'_d$.
- **Theorem:** $R_d \simeq R'_d$.
- If we tensor with L then the relationship between $\mathbb{G}$ and Θ becomes close, so $L \otimes_{E^0} R_d \simeq L \otimes_{E^0} R'_d \simeq \mathrm{Map}(\mathrm{Sub}_{p^d}(\Theta), L)$.
- Topological methods show that R'_d is a free module over E^0.
- By the above, the rank is $|\mathrm{Sub}_{p^d}(\Theta)|$, which can be computed by algebra.

Finite subgroups of formal groups

- Recall $R'_d = E^0(B\Sigma_{p^d})/J$; now put $\overline{R}_d = R_d/(u_0, \dots, u_{n-1})$ and $\overline{R}'_d = K^0(B\Sigma_{p^d})/J$.
- Suppose we have a ring P which is a free module of finite rank over $\mathbb{F}_p$, and we have a relation $ab^k = 0$ in P.
- Then $b^k P$ is a cyclic module over P/a and $b^i P/b^{i+1} P$ is a cyclic module over P/b so $\dim(P) \leq \dim(P/a) + k \dim(P/b)$.
- Using formal group theory we can define lots of quotients of $\overline{R}_d$ to which this applies, and so get an upper bound for $\dim(\overline{R}_d)$.
- By some combinatorics, this upper bound is the same as $|\operatorname{Sub}_{p^d}(\Theta)|$, which is the rank of R'_d over E^0, or of $\overline{R}'_d$ over $\mathbb{F}_p$.
- Now we know that all the ranks are the same, we can show that $R'_d = R_d$.

Power operations

- Under mild conditions we have $E^0(X \times Y) = E^0(X) \otimes_{E^0} E^0(Y)$ and so $E^0(X^p) = E^0(X)^{\otimes p}$.
- Thus for $u \in E^0(X)$ we have $u^{\otimes p} \in E^0(X^p)$, invariant under permutation.
- Commutativity of ring spectra is subtle; but the conclusion is that there is a power operation $Pu \in E^0(X^p_{h\Sigma_p})$ mapping to $u^{\otimes p}$.
- This has $P0 = 0$ and $P1 = 1$ and $P(uv) = P(u)\, P(v)$ and $P(u + v) = P(u) + P(v) + \text{transfers}$.
- It follows that P induces a ring map $E^0(X) \to E^0(X^p_{h\Sigma_p})/J \simeq R_1 \otimes E^0(X)$.
- There is a similar story involving Σ_{p^d} for $d > 1$.
- Taking $X = \mathbb{C}P^\infty$ we get a ring map $\mathcal{O}_{\mathbb{G}} \to \mathcal{O}_{\mathrm{Sub}_{p^d}(\mathbb{G})} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathbb{G}}$, corresponding to a map $\mathrm{Sub}_{p^d}(\mathbb{G}) \times_S \mathbb{G} \to \mathbb{G}$.
- You should think of $\mathbb{G}$ as a bundle of groups over S. Given a point $a \in S$ and a subgroup $A < \mathbb{G}_a$, it turns out that there is a canonical point $b \in S$ and a surjective homomorphism $q_{a,A} \colon \mathbb{G}_a \to \mathbb{G}_b$ with kernel A. The above map sends (A, x) to $q_A(x)$.

Finite general linear groups

- Let F be a finite field of characteristic not equal to p.
- To simplify bookkeeping, we will assume that $|F| = q$ with $v_p(q-1) = r > 0$ so $q = 1 \pmod{p^r}$ but $q \neq 1 \pmod{p^{r+1}}$. This implies that $v_p(q^m - 1) = v_p(m) + r$ for all $m > 0$.
- Let $\overline{F}$ be an algebraic closure of F. This has a Frobenius automorphism $\phi\colon x \mapsto x^q$, and the Galois group Γ is isomorphic to $\widehat{\mathbb{Z}}$, topologically generated by ϕ.
- We put $\mathbb{H} = BGL_1(\overline{F})_E$, which has a natural group structure. One can choose an isomorphism

$$GL_1(\overline{F}) \simeq \{u \in S^1 \mid u^r = 1 \text{ for some } r \in \mathbb{Z},\ (r,q) = 1\},$$

  and using this we find that $\mathbb{H}$ is noncanonically isomorphic to $\mathbb{G} = (BS^1)_E$, and canonically isomorphic to $\mathrm{Tor}(\overline{F}^\times, \mathbb{G})$.
- Generalised character theory compares $\mathbb{G}$ with $\Theta = (\mathbb{Z}/p^\infty)^n$. We will also compare $\mathbb{H}$ with $\Phi = \mathrm{Tor}(\overline{F}^\times, \Theta) \simeq \mathrm{Hom}(\Theta^*, \overline{F}^\times)$ (so Φ is noncanonically isomorphic to Θ).

Theorem

The inclusion $GL_1(\overline{F})^d \to GL_d(\overline{F})$ induces $GL_d(\overline{F})_E \simeq \mathbb{H}^d/\Sigma_d \simeq \mathrm{Div}_d^+(\mathbb{H})$. Equivalently,

$$E^0(BGL_1(\overline{F})^d) = E^0[\![x_1, \ldots, x_d]\!],$$

and $E^0 BGL_d(\overline{F})$ is the subring of symmetric functions, generated by elementary symmetric functions $c_1, \ldots, c_d$.

Proof.

This is built into the foundations of étale homotopy theory.
The main point is that one can build a torsion-free local ring $\overline{W}$ (the Witt ring of $\overline{F}$) with residue field $\overline{F}$.
One can then choose an embedding $\overline{W} \to \mathbb{C}$.
Using the fact that $|F|$ is coprime to p, one can check that the maps

$$BGL_d(\overline{F}) \leftarrow BGL_d(\overline{W}) \to BGL_d(\mathbb{C})$$

induce isomorphisms in mod p cohomology.
The claim follows easily from this. □

The theorem of Tanabe

Recall that the group $\Gamma = \mathrm{Gal}(\overline{F}/F)$ is generated by the Frobenius map ϕ.

Theorem (Tanabe)

The elements

$$\phi^*(c_k) - c_k \in E^0 BGL_d(\overline{F}) = E^0[\![c_1, \dots, c_d]\!]$$

form a regular sequence, and

$$E^0 BGL_d(F) = \frac{E^0[\![c_1, \dots, c_d]\!]}{(\phi^*(c_1) - c_1, \dots, \phi^*(c_d) - c_d)} = (E^0 BGL_d(\overline{F}))_\Gamma.$$

Equivalently, we have $BGL_d(F)_E = \mathrm{Div}_d^+(\mathbb{H})^\Gamma$.

In many respects this is very satisfactory, but there are many natural questions that cannot be answered without more detailed algebraic analysis.

Groupoids

- Let $\mathcal{V}$ be the groupoid of finite dimensional vector spaces over F, and their isomorphisms. Then $B\mathcal{V} \simeq \coprod_d BGL_d(F)$.
- We write $\overline{\mathcal{V}}$ for the corresponding groupoid for $\overline{F}$, so $B\overline{\mathcal{V}} \simeq \coprod_d BGL_d(\overline{F})$.
- Now $B\overline{\mathcal{V}}_E = \coprod_d \mathrm{Div}_d^+(\mathbb{H}) = \mathrm{Div}^+(\mathbb{H})$,
  and the functor $V \mapsto \overline{F} \otimes_F V$ gives $B\mathcal{V}_E = \mathrm{Div}^+(\mathbb{H})^\Gamma$.
- The functors $\oplus, \otimes \colon \mathcal{V}^2 \to \mathcal{V}$ make $B\mathcal{V}$ a commutative semiring in the homotopy category of spaces.This in turn makes $B\mathcal{V}_E$ a commutative semiring in the category of formal schemes.This matches an obvious commutative semiring structure on $\mathrm{Div}^+(\mathbb{H})^\Gamma$.
- Alternatively, $E_*^\vee(B\mathcal{V})$ and $K_*(B\mathcal{V})$ are Hopf rings.
- Some other groupoids are also relevant, for example

  $$\mathcal{L} = \{(X, L) \mid X \text{ is a finite set, and } L \text{ is an } F\text{-linear line bundle over } X\}.$$

  This has $B\mathcal{L} \simeq \coprod_d E\Sigma_d \times_{\Sigma_d} BGL_1(F)^d$.
  There is a functor $\pi \colon \mathcal{L} \to \mathcal{V}$ given by $\pi(X, L) = \bigoplus_x L_x$.
- The index of $\Sigma_d \wr GL_1(F)^d$ in $GL_d(F)$ has index coprime to p, so $B\mathcal{L} \to B\mathcal{V}$ gives an epimorphism in E-cohomology. Earlier work on symmetric groups gives a good understanding of $E^0 B\mathcal{L}$.

Generalised character theory

- There is a diagonal functor $\delta\colon \mathcal{V} \to \mathcal{V} \times \mathcal{V}$ given by $\delta(X) = (X, X)$, and functors $\sigma, \mu\colon \mathcal{V}^2 \to \mathcal{V}$ given by $\sigma(X, Y) = X \oplus Y$ and $\mu(X, Y) = X \otimes Y$.
- These give maps $\Sigma^\infty_+ B\mathcal{V} \xleftarrow{\sigma,\mu} \Sigma^\infty_+ B\mathcal{V}^2 \xleftarrow{\delta} \Sigma^\infty_+ B\mathcal{V}$ and also transfers $\Sigma^\infty_+ B\mathcal{V} \xrightarrow{\sigma_!,\mu_!} \Sigma^\infty_+ B\mathcal{V}^2 \xrightarrow{\delta_!} \Sigma^\infty_+ B\mathcal{V}$.
- These satisfy many relations and give rich algebraic structure on $E^0(B\mathcal{V})$.
- Everything is easy to understand in generalised character theory.
- Recall that $\Theta^* = \mathbb{Z}_p^n$, and let $\mathbb{B} = \pi_0[\Theta^*, \mathcal{V}]$ be the set of isomorphism classes of finite-dimensional F-linear representations of Θ^*.
  Then $L \widehat{\otimes}_{E^0} E^0(B\mathcal{V}) = \mathrm{Map}(\mathbb{B}, L)$.
- .
$$\sigma^*(f)(X, Y) = f(X \oplus Y) \qquad \sigma^!(f \otimes g)(X) = \sum_{X = Y \oplus Z} f(Y) g(Z)$$
$$\delta^*(f \otimes g)(X) = f(X) g(X) \qquad \delta^!(f)(X, Y) = |\operatorname{Iso}(X, Y)| f(X).$$

- For finite sets, any subset of $Y \amalg Z$ is $Y_0 \amalg Z_0$ with $Y_0 \subseteq Y$ and $Z_0 \subseteq Z$.
- But a subspace of $Y \oplus Z$ need not be $Y_0 \oplus Z_0$ with $Y_0 \leq Y$ and $Z_0 \leq Z$.
- This causes a lot of trouble with adapting the symmetric group proof.

The Atiyah-Hirzebruch Spectral Sequence

- Theorem: $E_0^\vee B\mathcal{V}$ is also polynomial.
- It is enough to prove that $K_0 B\mathcal{V}$ is polynomial.
- We use the Atiyah-Hirzebruch spectral sequence $H_*(B\mathcal{V}; K_*) \Longrightarrow K_*(B\mathcal{V})$ and its dual.
- Quillen: $H_*(B\mathcal{V}; K_*)$ is generated by $B\mathcal{V}_1$ and has countably many polynomial generators b_i and exterior generators e_i.
- Let $F(k)$ be the extension of F of degree p^k, so $GL_d(F(k))$ maps to $GL_{p^k d}(F)$. The group $GL_1(F(k))$ is cyclic so the AHSS is well understood, with only one differential. This gives some information about the AHSS for $GL_{p^k}(F)$.
- Tanabe and HKR also tell us that $K_*(B\mathcal{V})$ is concentrated in even degrees, with known rank.
- The ordinary ring structure on $K^*(BGL_d(F))$ also gives some information.
- At the E_∞ page, all exterior generators have been killed, and $b_i^{p^{m_i}}$ survives. This leaves a polynomial algebra, and it follows that $K_*(B\mathcal{V})$ is also polynomial.
- This is the most complex pattern of AHSS differentials that we have seen.

Irreducibles

- Put $I = \ker(E^0(B\mathcal{V}) \to E^0)$ and $I^{*2} = \sigma^!(I \otimes I)$ and $Q = I/I^{*2}$.
- This is still a ring with $L \otimes_{E^0} Q = \mathrm{Map}(\mathrm{Irr}(\Theta^*), L)$, where $\mathrm{Irr}(\Theta^*) = \mathrm{Hom}(\Theta^*, GL_1(\overline{F}))/\mathrm{Gal}$ is the set of isomorphism classes of irreducible F-linear representations of Θ^*.
- We find that $Q \simeq \prod_m D_m^\Gamma$, where $D_m^\Gamma = E^0[\![y]\!]/g_m(y)$ for a certain monic polynomial $g_m(y)$.
- All this and many more details have nice interpretations in formal group theory.