Chromatic cohomology of finite groups 4

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Duality

- Recall: for a finite groupoid G we put $M^*(G) = \mathrm{Map}(\pi_0(G), \mathbb{Q})$. We define $\theta\colon M^*(G) \to \mathbb{Q}$ by $\theta(h) = \sum_{i<r} |G(a_i, a_i)|^{-1} h(a_i)$, where $\{a_i \mid i < r\}$ contains one representative of each isomorphism class. We then define $\langle f, g\rangle_G = \theta(fg)$.This is a perfect pairing on $M^*(G)$.
- The HKR theorem says $L \otimes_{E^0} E^0(BG) = L \otimes_{\mathbb{Q}} M^*(\Lambda G)$, where $\Lambda G = [\Theta^*, G] = [\mathbb{Z}_p^n, G]$ is again a finite groupoid.
- We therefore have $\theta\colon L \otimes_{E^0} E^0(BG) \to L$ giving a perfect pairing.
- **Theorem:** this comes from a map $\theta\colon E^0(BG) \to E^0$ which also gives a perfect pairing (at least when $E^0(BG)$ is a free module over E^0).
- This is like Poincaré duality for oriented manifolds: the map θ is like the map $u \mapsto \langle u, [M]\rangle$ from $H^d(M)$ to $\mathbb{Z}$.
- It is also like the map $\theta\colon R(G) \to \mathbb{Z}$ given by $\theta([V]) = \dim(V^G)$, where $R(G)$ is the complex representation ring. This also gives a perfect pairing.
- The above theorem (due to Greenlees and Sadofsky) was the first known example of chromatic ambidexterity; there is now a more general theory.
- The proof of the theorem uses transfers and Tate spectra.

Collapse and transfer

- Consider a map $f\colon X \to Y$ of finite sets.
  This induces a map $f\colon \mathbb{Z}[X] \to \mathbb{Z}[Y]$,
  and also a map $f^t\colon \mathbb{Z}[Y] \to \mathbb{Z}[X]$ given by $f^t([y]) = \sum_{f(x)=y}[x]$.
- The suspension spectrum $\Sigma^\infty X_+$ is a kind of refinement of $\mathbb{Z}[X]$,
  and we again have an easy map $f\colon \Sigma^\infty X_+ \to \Sigma^\infty Y_+$.
  We also want a map $f^T\colon \Sigma^\infty Y_+ \to \Sigma^\infty X_+$.
- Put $V = \mathbb{R}[X]$, giving $i\colon X \to V \subset S^V = V \cup \{\infty\}$.
- Put $s(v) = v/\sqrt{2(1+\|v\|^2)}$,
  giving a homeomorphism from V to an open ball of radius $1/\sqrt{2}$.
- Define $\widetilde{f}\colon V \times X \to V \times Y$ by $\widetilde{f}(v,x) = (s(v) + i(x), f(x))$,
  so $\widetilde{f}$ is an open embedding covering f.
- Define $c\colon S^V \wedge Y_+ = (V \times Y) \cup \{\infty\} \to (V \times X) \cup \{\infty\} = S^V \wedge X_+$
  by $c(\widetilde{f}(v,x)) = (v,x)$ and $c(v,y) = \infty$ for $(v,y) \notin \text{image}(\widetilde{f})$.
- This is completely natural and so is compatible with any group actions.
- In the world of spectra we have a negative sphere S^{-V} and we can take the smash product with this to get $f^t\colon \Sigma^\infty Y_+ \to \Sigma^\infty X_+$.
- Take $f = (G/H \to G/G = 1)$ and apply $EG_+ \wedge_G (-)$ to f^t and use $EG/H \simeq BH$ to get a map $\Sigma^\infty BG_+ \to \Sigma^\infty BH_+$ (the *transfer*).

Transfers

- Let M be an abelian group with G-action, and let H be a subgroup of G.
- There is an evident inclusion $\mathrm{res}_H^G\colon M^G \to M^H$.
- There is a natural map $\mathrm{tr}_H^G\colon M^H \to M^G$ given by $\mathrm{tr}_H^G(m) = \sum_{t\in T} tm$, where T is any subset of G containing one element of every H-coset.
- Now let E be an even periodic ring spectrum.
- The inclusion $H \to G$ gives a map $BH \to BG$ and thus a ring map $E^0(BG) \to E^0(BH)$, called res_H^G.
- The transfer $\Sigma^\infty BG_+ \to \Sigma^\infty BH_+$ gives a map $E^0(BH) \to E^0(BG)$, called tr_H^G; this is $E^0(BG)$-linear.
- Formal properties of these maps are similar to those of the maps $M^G \to M^H \to M^G$ mentioned above.
- Put $\Delta = \{(g,g) \mid g \in G\} \leq G^2$ and $u = \mathrm{tr}_\Delta^{G^2}(1) \in E^0(BG^2)$.
- Take $E = K =$ Morava K-theory, so $K^*(BG)$ and $K_*(BG)$ are finitely generated free modules and dual to each other.
  Then u is adjoint to a map $u^\#\colon K_*(BG) \to K^{-*}(BG)$.
- This arises from a map $K \wedge BG_+ \to F(BG_+, K)$ of spectra, whose cofibre is called the Tate spectrum $t_G(K)$.
- **Theorem** (Greenlees-Sadofsky): $t_G(K) = 0$.

Sketch proof of Tate vanishing

- Claim: the cofibre $t_G(K)$ of the map $K \wedge BG_+ \to F(BG_+, K)$ is zero.
- First suppose $|G| = p$, so G has a faithful one-dimensional complex representation L with Euler class $x \in K^0(BG)$, and $K^*(BG) = K^*[x]/x^{p^n}$.
- The unit sphere $S(\infty L)$ is contractible and has free G-action so we can take $EG = S(\infty L)$. Thus, the reduced suspension $\widetilde{E}G$ is the same as $S^{\infty L} = \varinjlim_k S^{kL}$, where $S^{kL} \simeq S^{2k}$ is the one-point compactification of nL.
- It is not hard to check that $G_+ \wedge \widetilde{E}G = F(G_+, \widetilde{E}G) = EG_+ \wedge \widetilde{E}G = F(EG_+, \widetilde{E}G) = 0$. Using this one can identify $t_G(K)$ with $(\widetilde{E}G \wedge F(EG_+, K))^G$.
- Using $\widetilde{E}G = \varinjlim S^{kL}$ and $S^{kL} \wedge F(EG_+, K) = F(S^{-kL} \wedge EG_+, K)$ we get $t_G(K) = \varinjlim_k F(BG^{-kL}, K)$, where BG^{-kL} is the Thom spectrum.
- Using the fact that the Euler class of kL is x^k, we find that $\pi_*(t_G(K)) = K^{-*}(BG)[x^{-1}] = (K^*[x]/x^{p^n})[x^{-1}] = 0$ as required.
- For general G: combine fairly similar arguments with an induction on $|G|$.
- Conclusion: $K_*(BG)$ maps isomorphically to $K^*(BG)$, which is dual to $K_*(BG)$, so $K^*(BG)$ is self-dual.
- By rearranging the argument slightly: the K-local spectrum $L_K\Sigma^\infty BG_+$ is self-dual, and $E^*(BG)$ is self-dual provided that it is a free E^*-module.

More about duality

- We have shown that $K^*(BG)$ is naturally self-dual when G is a finite group. There is an easy generalisation to groupoids.
- Any functor $\alpha\colon G \to H$ induces $\alpha^*\colon K^*(BH) \to K^*(BG)$. As everything is self-dual, there is a unique $\alpha_!\colon K^*(BG) \to K^*(BH)$ adjoint to α^*, i.e. $\langle \alpha_!(a), b\rangle_H = \langle a, \alpha^*(b)\rangle_G$ for $a \in K^*(BG)$ and $b \in K^*(BH)$.
- If α is an inclusion of groups, then $\alpha_!$ is just the transfer.
- Given a homotopy pullback square of groupoids as shown on the left, we have a commutative diagram as shown on the right.

$$
\begin{array}{ccc}
G & \xrightarrow{\alpha} & H \\
{\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\
K & \xrightarrow[\delta]{} & L
\end{array}
\qquad\qquad
\begin{array}{ccc}
K^*(BG) & \xleftarrow{\alpha^*} & K^*(BH) \\
{\scriptstyle\beta_!}\downarrow & & \downarrow{\scriptstyle\gamma_!} \\
K^*(BK) & \xleftarrow[\delta^*]{} & K^*(BL)
\end{array}
$$

  This generalises the classical Mackey property of transfers.
- Recall $u = \mathrm{tr}_\Delta^{G^2}(1) \in K^0(BG^2) = K^0(BG) \otimes_{K^0} K^0(BG)$.
- From the duality theorem it follows that there is a unique *Frobenius form* $\theta\colon K^0(BG) \to K^0$ such that $(\theta \otimes 1)(u) = 1$ in $K^0(BG)$.
- Using the Mackey property: $\langle u, v\rangle_G = \theta(uv)$.
- There are similar statements for E^* when $E^0(BG)$ is free.

Duality in the abelian case

- Let E be Morava E-theory.
- There is a power series $\log_F(x) = \sum_{k>0} m_k x^k$ with $m_1 = 1$ and $m_k \in \mathbb{Q} \otimes E^0$ and $\log_F(x +_F y) = \log_F(x) + \log_F(y)$.
- The series $d\log_F(x) = \sum_k k\, m_k x^{k-1}\, dx$ actually lies in $E^0[\![x]\!].dx$.
- Given any $f(x) \in E^0[\![x]\!]$ we can expand $f(x)\omega/[p^m]_F(x)$ in positive and negative powers of x, and define $\rho_m(f(x)) = \operatorname{res}(f(x)\omega/[p^m]_F(x))$ to be the coefficient of $x^{-1}dx$.
- **Theorem:** the Frobenius form $\theta\colon E^0(BC_{p^m}) = E^0[\![x]\!]/[p^m]_F(x) \to E^0$ is induced by ρ_m.
- For a general finite abelian group A we can decompose A as a product of cyclic groups and thus determine the Frobenius form.
- **Open problem**: do this more naturally in terms of higher-dimensional residues and local cohomology.
- **Theorem** (Hopkins-Lurie): for all k the space $B^kA = K(k, A)$ has $E^*(B^kA)$ naturally self-dual (but it is trivial for $k > n$).
- **Open problem**: give a residue formula for $\theta\colon E^0(B^kA) \to E^0$.

Semirings with λ-operations

- The representation semiring $R_+(G)$ is the set of isomorphism classes of complex representations.
- This has addition $[V]+[W]=[V\oplus W]$ and multiplication $[V][W]=[V\otimes W]$ but no subtraction.
- There are also operations λ^k sending $[V]$ to $[\Lambda^k V]$, where $\Lambda^k V$ is the k'th exterior power of V.
- We also have a ring $R(G)$ of virtual representations, which is the group completion of $R(G)$.
- If h is the number of conjugacy classes then h is also the number of isomorphism classes of irreducible representations. These form a basis giving $R_+(G)\simeq\mathbb{N}^h$ and $R(G)\simeq\mathbb{Z}^h$ additively.
- The scheme $\mathrm{Div}^+(\mathbb{G})=\coprod_{k\geq 0}\mathrm{Div}_k^+(\mathbb{G})=\coprod_{k\geq 0}\mathbb{G}^k/\Sigma_k=\mathrm{spf}(E^0(\coprod_k BU(k)))$ is a semiring object in the category of schemes, with λ-operations.
- For divisors $D=\sum_{i<r}[a_i]$ and $E=\sum_{j<s}[b_j]$ we have $DE=\sum_{i,j}[a_i+b_j]$ and $\lambda^k D=\sum_{i_1<\cdots<i_k<r}[a_{i_1}+\cdots+a_{i_k}]$.
- The λ operations on $R_+(G)$ induce Adams operations on $R(G)$, e.g. $\psi^2(x)=x^2-2\,\lambda^2(x)$ and $\psi^3(x)=x^3-3x\lambda^2(x)+3\lambda^3(x)$. These are ring maps with $\psi^k\psi^m=\psi^{km}$.

Chern approximations

- We define $\mathrm{Ch}(G)$ to be the scheme of morphism $R_+(G) \to \mathrm{Div}^+(\mathbb{G})$ of λ-semirings, and $C(E, G) = \mathcal{O}_{\mathrm{Ch}(G)}$.
- If we understand everything about $R_+(G)$ then we can write down a presentation of $C(E, G)$ by generators and relations, partly determined by the formal group law. But it is easier to work with schemes where possible.
- Like $E^0(BG)$, the ring $C(E, G)$ is finitely generated as an E^0-module.
- There is a natural map $\alpha_G \colon C(E, G) \to E^0(BG)$, whose image is the subring generated by all Chern classes of all representations.
- There may be a kernel in general, consisting of relations between Chern classes that do not follow automatically from representation theory.
- If the chromatic height n is one, then α_G is an isomorphism for all G. This is because Morava E-theory at height one is the p-completion of KU and so is very close to representation theory.
- If G is abelian, or is a general linear group over a finite field of characteristic not equal to p, then α_G is iso.
  However, this fails for the symmetric group Σ_6 when $p = n = 2$.
  It also fails for certain central extensions $C_p \to G \to C_p^{2d}$ with $d > 1$.

The group $G = \Sigma_4$ with $n = p = 2$

The character table of Σ_4 is as follows:

class	size	1	ϵ	σ	ρ	$\epsilon\rho$
1^4	1	1	1	2	3	3
$1^2 2$	6	1	-1	0	1	-1
2^2	3	1	1	2	-1	-1
13	8	1	1	-1	0	0
4	6	1	-1	0	-1	1

The ring structure, Adams operations and λ-operations are described in the following table.

$$
\begin{array}{lll}
\epsilon^2 = 1 & \psi^k(\epsilon) = \epsilon^k & \lambda^2(\sigma) = \epsilon \\
\epsilon\sigma = \sigma & \psi^2(\sigma) = 1 - \epsilon + \sigma & \lambda^2(\rho) = \epsilon\rho \\
\sigma^2 = 1 + \epsilon + \sigma & \psi^3(\sigma) = 1 + \epsilon & \lambda^3(\rho) = \epsilon \\
\sigma\rho = \rho + \epsilon\rho & \psi^2(\rho) = 1 + \sigma + \rho - \epsilon\rho & \\
\rho^2 = 1 + \sigma + \rho + \epsilon\rho & \psi^3(\rho) = 1 + \epsilon - \sigma + \rho. &
\end{array}
$$

$\mathrm{Ch}(\Sigma_4)$ is the scheme of pairs $(d, D) \in \mathbb{G} \times \mathrm{Div}_3^+(\mathbb{G})$ such that

$$2d = 0 \qquad \lambda^3(D) = [0] \qquad \psi^{-1}(D) = D \qquad \psi^2(D) + D = 2[0] + [d] + [d]D$$

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The formal group law for $n = p = 2$ satisfies

$$\begin{aligned} [2](x) &= x^4 \\ [-1](x) &= x + x^4 + x^{10} + x^{16} + x^{22} \pmod{x^{32}} \\ x +_F y &= x + y + x^2 y^2 \pmod{x^4 y^4}. \end{aligned}$$

$K^0(B\Sigma_4) = C(K, \Sigma_4) = \mathbb{F}_2[w, c_2, c_3]/J$ where

$$J = (w^4\ ,\ c_3^2\ ,\ c_2 c_3\ , c_2^4 + w^2 c_2^3 + w c_2^2 + w^2 c_3, w c_2^3 + w^2 c_2 + w c_3).$$

The following 17 monomials form a basis for this ring over $\mathbb{F}_2$:

1	c_2	c_2^2	c_2^3	c_3
w	wc_2	wc_2^2		wc_3
w^2	$w^2 c_2$	$w^2 c_2^2$		$w^2 c_3$
w^3	$w^3 c_2$	$w^3 c_2^2$		$w^3 c_3$