Chromatic cohomology of finite groups 3

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Flag manifolds

- Again let $V \to X$ be a complex vector bundle of dimension d.
- $\mathrm{Flag}_k(V) = \{(a, W_0, \dots, W_k) \mid x \in X, W_i < W_{i+1} \leq V_a, \dim(W_i) = i\}$.
- For $0 \leq i < k$ we have a line bundle $(Q_i)_{(a,\underline{W})} = W_{i+1}/W_i$ and an Euler class $x_i = e(Q_i) \in E^0(\mathrm{Flag}_k(V))$.
- We also have a bundle R_k over $\mathrm{Flag}_k(V)$ with $(R_k)_{(a,\underline{W})} = V_a/W_k$ (so $\dim(R_k) = d - k$), and $\mathrm{Flag}_{k+1}(V)$ is the projective bundle $P(R_k)$.
- By induction based on this: the set of monomials $x^\alpha = \prod_{i<k} x_i^{\alpha_i}$ with $0 \leq \alpha_i < d - i$ is a basis for $E^0(\mathrm{Flag}_k(V))$ over $E^0(X)$.
- For the ring structure: put $g_k(t) = \prod_{i<k}(t - x_i) \in E^0(X)[x_0, \dots, x_{k-1}]$, then divide $f_V(t)$ by $g(t)$ with remainder to get $f_V(t) = g(t)q(t) + r(t)$ with $\deg(r(t)) < k$, then let I be the ideal generated by the coefficients of $r(t)$.We then have $E^0(\mathrm{Flag}_k(V)) = E^0(X)[x_0, \dots, x_{k-1}]/I$ as rings.
- Let G be a group with $|G| = n$. The representation $\mathbb{C}[G]$ gives a bundle $V = EG \times_G \mathbb{C}[G]$ over BG and a space $\mathrm{Flag}_n(V)$ with $E^0(\mathrm{Flag}_n(V)) \simeq E^0(BG)^{n!}$.
- $\mathrm{Flag}_n(V) = EG \times_G F$, where $F = \{\underline{W} \mid W_0 < \cdots < W_n = \mathbb{C}[G]\}$.
- **Key fact:** all stabiliser groups in F are abelian. Indeed, $\mathrm{stab}_G(\underline{W})$ injects in the abelian group $\prod_{i=0}^{n-1} \mathrm{Aut}(W_{i+1} \ominus W_i) = (\mathbb{C}^\times)^n$.

Finiteness

- **Theorem:** if X is a finite simplicial complex with simplicial G-action, then the ring $E^*(EG \times_G X) = E^*(X_{hG})$ is finitely generated as an E^*-module.
- **Proof:** First treat the case $\mathrm{stab}_G(x)$ is abelian for all $x \in X$.
- If $X = G/H$ then H must be abelian and $X_{hG} = BH$ and $E^*(BH)$ is finitely generated by previous calculation.
- If X is just a finite discrete G-set then it is a disjoint union of G/H's and the same applies.
- In general, if X^k is the k-skeleton of X then $X^k/X^{k-1} = \Sigma^k W_+$ for some finite G-set W, giving an exact sequence of $E^*(1)$-modules

  $$E^{*-k}(W_{hG}) \to E^*(X^k_{hG}) \to E^*(X^{k-1}_{hG}).$$

  As $E^*(1)$ is noetherian, it follows inductively that $E^*(X^k_{hG})$ is finitely generated for all k, so $E^*(X_{hG})$ is finitely generated.
- Now remove the abelian stabiliser condition.
- Put $n = |G|$ and $F = \{(W_0, \dots, W_n) \mid W_0 < \cdots < W_n = \mathbb{C}[G]\}$.
- Then $X \times F$ has abelian stabilisers, so $A^* = E^*((X \times F)_{hG})$ is finitely generated; enough to show that $B^* = E^*(X_{hG})$ is a retract of this.
- But $(X \times F)_{hG}$ is $\mathrm{Flag}_n(V)$ for a bundle V over BG, so $A^* \simeq (B^*)^{n!}$, so B^* is a retract of A^* $\square$.

Finite groupoids

- Many things about Morava E-theory are more convenient using groupoids.
- A *groupoid* is a category G in which all morphisms are invertible.
- Say G is *finite* if all Hom sets $G(a,b)$ are finite, and the set $\pi_0(G)$ of isomorphism classes is finite.
- If so, we can choose $a_1,\dots,a_m$ containing one element of each isomorphism class, and put $G_i = G(a_i,a_i)$, and we get $BG \simeq \coprod_i BG_i$.
- Thus $E^*(BG) = \prod_i E^*(BG_i)$, which is a finitely generated E^*-module.
- Any group can be regarded as a groupoid with one object.
- A *representation* of G is a functor V from G to the category $\mathcal{V}$ of finite-dimensional complex vector spaces.
- This again gives spaces $\mathrm{Flag}_k(V)$ and $P(V) = \mathrm{Flag}_1(V)$ over BG.
- Given groupoids G and H, the functor category $[G,H]$ is also a groupoid.
- If G,H are groups then $\mathrm{obj}([G,H]) = \mathrm{Hom}(G,H)$ and morphisms $\alpha \to \beta$ in $[G,H]$ are elements $h \in H$ with $\beta(g) = h\alpha(g)h^{-1}$ for all $g \in G$.
- So $\alpha \simeq \beta$ iff α and β are conjugate, and $\pi_0([G,H])$ is the set of conjugacy classes of homomorphisms.

Naive groupoid duality

- For a finite groupoid G put $M(G) = \mathbb{Q}\{\pi_0(G)\}$ and $M^*(G) = \mathrm{Hom}(M(G), \mathbb{Q}) = \mathrm{Map}(\pi_0(G), \mathbb{Q})$.
- Define an inner product on $M(G)$ by $([a], [b])_G = |G(a, b)|$ (so $([a], [b]) = 0$ unless $a \simeq b$).
- The induced inner product on $M^*(G)$ is $\langle f, g\rangle_G = \sum_{i=1}^{r} |G(a_i, a_i)|^{-1} f(a_i) g(a_i)$, where $a_1, \dots, a_r$ contains one member of each isomorphism class.
- This is also $\langle f, g\rangle_G = \theta(fg)$, where $\theta(h) = \sum_i |G(a_i, a_i)|^{-1} h(a_i)$.
- Given $q \colon G \to H$ we define $q_! \colon M(G) \to M(H)$ by $q_!([a]) = [q(a)]$, and $q^* \colon M^*(H) \to M^*(G)$ by $q^*(g)(a) = g(q(a))$.
- Define $q^* \colon M(H) \to M(G)$ and $q_! \colon M^*(G) \to M^*(H)$ to be adjoint, so $(q_!(u), v)_H = (u, q^*(v))_G$ and $\langle q_!(f), g\rangle_H = \langle f, q^*(g)\rangle_G$.
- This is compatible with the isomorphisms $M(G) \simeq M^*(G) \simeq \mathrm{Hom}(M(G), \mathbb{Q})$.

Generalised characters

- Fix a prime p and $n > 0$ and let E be Morava E-theory.
- Then $[p^k]_E(x) = g_k(x)h_k(x)$, where $h_k(x) \in E^0[\![x]\!]^\times$ and $g_k(x) \in E^0[x]$ is a monic polynomial of degree p^{nk} and $E^0(BC_{p^k}) = E^0[x]/g_k(x)$.
- Construct L from $\mathbb{Q} \otimes E^0$ by adjoining a full set of roots of $g_k(x)$ for all k.
- Put $\mathbb{Z}/p^\infty = \varinjlim_k \mathbb{Z}/p^k = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} = \mathbb{Q}/\mathbb{Z}_{(p)} = \mathbb{Q}_p/\mathbb{Z}_p = \bigcup_k \sqrt[p^k]{1} \subset S^1$. (Exercise: $\text{Hom}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty) \simeq \mathbb{Z}_p \simeq \text{Hom}(\mathbb{Z}/p^\infty, S^1)$.)
- Put $\Theta = \{\text{all roots of all } g_k(x)\} \subset L$. This is a group under $+_E$, iso to $(\mathbb{Z}/p^\infty)^n$, analogous to the formal group scheme $\text{spf}(E^0(\mathbb{C}P^\infty))$.
- Put $\Theta^* = \text{Hom}(\Theta, S^1) \simeq \mathbb{Z}_p^n$, regarded as a groupoid with one object.
- Put $\Lambda G = [\Theta^*, G] = \varinjlim_k [\Theta^*/p^k, G]$, $\quad C(G) = L \otimes M^*\Lambda G = \text{Map}(\pi_0\Lambda G, L)$.
- Recall $E^0(B(\Theta^*/p^k)) = E^0[\![x_1, \dots, x_n]\!]/(g_k(x_1), \dots, g_k(x_n))$; there is a canonical map ϕ_k from this to L.
- Thus any $u\colon \Theta^*/p^k \to G$ gives $\phi_k \circ E^0(Bu)\colon E^0BG \to L$. Assembling these gives $\chi\colon L \otimes_{E^0} E^0(BG) \to C(G)$.
- Theorem (Hopkins, Kuhn, Ravenel): χ is an isomorphism.

Proof of the generalised character theorem

- Reduce to the case of a finite group G.
- Generalise: for a finite G-CW complex Z, we have

$$\chi_{G,Z}\colon L \otimes_{E^0} E^*(Z_{hG}) \to L \otimes_{\mathbb{Q}} \left(\prod_{\theta\colon \Theta^* \to G} H^*(Z^{\mathrm{image}(\theta)}; \mathbb{Q}) \right)^G$$

- Prove by calculation that $\theta_{G,Z}$ is iso when $Z = G/A$ with $A \leq G$ abelian. (Here $Z_{hG} = BA$, and $Z^{\mathrm{image}(\theta)}$ is Z (if $\mathrm{image}(\theta) \leq A$) or $\emptyset$ (otherwise).)
- Deduce by Mayer-Vietoris that $\chi_{G,Z}$ is iso if Z has abelian isotropy.
- Let $F = \{\underline{W} \mid W_0 < \cdots < W_n = \mathbb{C}[G]\}$ be the space of complete flags in $\mathbb{C}[G]$, so $Z \times F$ and $Z \times F^2$ have abelian isotropy, and we have projections $p\colon (Z \times F)_{hG} \to Z_{hG}$ and $q_0, q_1\colon (Z \times F^2)_{hG} \to (Z \times F)_{hG}$
- We saw before that $E^*((Z \times F)_{hG})$ has a canonical basis $e_1 = 1, e_2, \ldots, e_{n!}$ over $E^*(Z_{hG})$. Similarly, the elements $q_0^*(e_i)q_1^*(e_j)$ form a basis for $E^*((Z \times F^2)_{hG})$ over $E^*(Z_{hG})$.
- It follows that the diagram $E^*(Z_{hG}) \to E^*((Z \times F)_{hG}) \rightrightarrows E^*((Z \times F^2)_{hG})$ is an equaliser.
- Deduce the general case from this.

Divisors and vector bundles

- A *divisor of degree d* on $\mathbb{G} = \mathrm{spf}(E^0[\![x]\!])$ is a closed subscheme $D < \mathbb{G}$ such that $\mathcal{O}_D$ is free of rank d as a module over $\mathcal{O}_S = E^0$.
- Equivalently, $\mathcal{O}_D = E^0[\![x]\!]/f(x)$ for some monic polynomial $f(x) = \sum_{i=0}^d c_i x^{d-i}$ with c_i in the maximal ideal for $i > 0$.
- $[p^k]_E(x)$ is a unit multiple of a polynomial of degree p^{nk}, so the scheme $\mathbb{G}[p^k] = \ker(p^k.1\colon \mathbb{G} \to \mathbb{G}) = \mathrm{spf}(E^0[\![x]\!]/[p^k]_E(x)) = \mathrm{spf}(E^0(BC_{p^k}))$ is a divisor of degree p^{nk}.
- More generally, for $T \to S$, a *divisor of degree d on $\mathbb{G}$ over T* is a closed subscheme $D < T \times_S \mathbb{G}$ such that $\mathcal{O}_D$ is free of rank d over $\mathcal{O}_T$.
- Equivalently, $\mathcal{O}_D = \mathcal{O}_T[\![x]\!]/f(x)$ for some monic polynomial $f(x) = \sum_{i=0}^d c_i x^{d-i}$ with $c_i \in \mathcal{O}_T$ topologically nilpotent for $i > 0$.
- Example: if $V \to Z$ is a complex bundle of dimension d, then the scheme $D_V = \mathrm{spf}(E^0(P(V)))$ is a divisor of degree d on $\mathbb{G}$ over $\mathrm{spf}(E^0(Z))$ (by earlier calculation of $E^0(P(V))$).
- There is a sum operation for divisors: if $\mathcal{O}_{D_i} = \mathcal{O}_T[\![x]\!]/f_i(x)$ for $i = 0, 1$ then $\mathcal{O}_{D_0+D_1} = \mathcal{O}_T[\![x]\!]/(f_0(x)f_1(x))$. For this: $D_{V_0 \oplus V_1} = D_{V_0} + D_{V_1}$.
- An element $a \in \mathbb{G}$ gives a divisor $[a]$ of degree one. A list $a_1, \dots, a_d$ gives a divisor $\sum_i [a_i]$ of degree d, symmetric in $a_1, \dots, a_d$. Using this: $\mathrm{Div}_d^+(\mathbb{G}) = \mathbb{G}^d/\Sigma_d$.

Level structures

- We have identified $\mathrm{spf}(E^0(BA))$ with $\mathrm{Hom}(A^*, \mathbb{G})$.
- Is there a subscheme of monomorphisms from A^* to $\mathbb{G}$?
- Monomorphisms of schemes are closely related to epimorphisms of commutative rings, which have poor behaviour. Note that $\mathbb{Z} \to \mathbb{Q}$ and $\mathbb{Z} \to \mathbb{Z}/n$ are epimorphisms, but behave quite differently.
- We will define $\mathrm{Level}(A^*, \mathbb{G}) \subseteq \mathrm{Hom}(A^*, \mathbb{G})$ which is motivated by the above; but do not take the analogy too seriously.
- Write $A = \prod_{i=0}^{s-1} C_{p^{m_i+1}}$, put $D'_A = E^0[\![x_i \mid i < s]\!]$ and $y_i = [p^{m_i}](x_i) \in R'$.
- Let U be the set of all terms $\sum_{i<s}^F [k_i]_E(y_i)$ with $0 \leq k_i < p$ for all i.
- Put $g(t) = \prod_{u \in U}(t - u)$ and $[p]_E(t) = q(t)g(t) + r(t)$ with $\deg(r(t))) < s$. Let I be the ideal generated by the coefficients of $r(t)$, and $D_A = D'_A/I$, and $\mathrm{Level}(A^*, \mathbb{G}) = \mathrm{spf}(D_A)$.
- Schematically:
  $\mathrm{Level}(A^*, \mathbb{G}) = \{\phi \in \mathrm{Hom}(A^*, \mathbb{G}) \mid \sum_{\alpha \in A^*[p]}[\phi(\alpha)] \leq \mathbb{G}[p]\}$.
- From HKR: $\mathbb{Q} \otimes E^0(BG) = u_0^{-1}E^0(BG) \simeq \left(\prod_{A \leq G} \mathbb{Q} \otimes D_A\right)^G$.
- There is a similar map $u_k^{-1}E^0(BG)/I_k \to \left(\prod_{A \leq G} u_k^{-1}D_{k,A}\right)^G$ for $k > 0$, which is an F-isomorphism (Greenlees-Strickland; see also Stapleton).