Chromatic cohomology of finite groups 2

Neil Strickland

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The Lazard ring

- Consider a formal power series $F(s,t) = \sum_{i,j} b_{ij} s^i t^j \in k[\![s,t]\!]$. When is this an FGL?
- For $F(s,0) = s$ we need $b_{i0} = \delta_{i,1}$. For $F(s,t) = F(t,s)$ we need $b_{ij} = b_{ji}$.
- Now
  $F(s,t) = s + t + b_{11}st + b_{12}(st^2 + s^2t) + b_{22}s^2t^2 + b_{13}(st^3 + s^3t) + O(5)$
- Using this we get
  $F(F(s,t),u) - F(s,F(t,u)) = (2b_{11}b_{12} + 3b_{13} - 2b_{22})(s-u)stu + O(5)$
- For an FGL we must have $2b_{11}b_{12} + 3b_{13} - 2b_{22}$. In terms of the parameters $a_1 = b_{11}$ and $a_2 = b_{12}$ and $a_3 = b_{22} - b_{13}$ we get
  $F(s,t) = s+t+a_1st+a_2st(s+t)+2(a_3-a_1a_2)st(s^2+st+t^2)+a_3s^2t^2+O(5)$.
- There are no more relations: any power series of the above form satisfies the FGL conditions up to errors of order 5.
- Lazard's theorem: we can continue to define $a_4, a_5, \dots$ so that $F(s,t)$ can be expressed in terms of the a_i, and no further relations are required to make the associativity axiom hold.
- Reformulation: over the Lazard ring $L = \mathbb{Z}[a_1, a_2, \dots]$ there is a universal formal group law F_u such that the resulting map $\text{Rings}(L,k) \to \text{FGL}(k)$ is bijective for all k.

Quillen's theorem

- Recall $MP^0(X) = \varinjlim_n [\Sigma^{2n}X, MP(n)]$ (for X a finite complex). Both P and $MP(n)$ are defined using complex linear algebra so it is not hard to give an explicit x with $MP^0(P) = MP^0(1)[\![x]\!]$.
(We do not need to know $MP^0(1)$ for this.)
- Using this we get a formal group law F over $MP^0(1)$.
- Recall that $\mathrm{FGL}(k) = \mathrm{Rings}(L, k)$ so we get a ring map $L \to MP^0(1)$.
- Quillen's theorem: this is an isomorphism (and $MP^1(1) = 0$).
- This is the heart of a close connection between formal groups and algebraic topology.

Morava K and E

- Fix a prime p and $n > 0$.
- Put $l(x) = \sum_{k \geq 0} x^{p^{nk}} / p^k \in \mathbb{Q}[\![x]\!]$, $\quad F(x, y) = l^{-1}(l(x) + l(y)) \in \mathrm{FGL}(\mathbb{Q})$.
- In fact $F \in \mathrm{FGL}(\mathbb{Z})$ so we can reduce mod p to get $F_K \in \mathrm{FGL}(\mathbb{F}_p)$.
- There is a unique $\phi_K \colon MP_0 \to \mathbb{F}_p$ carrying F_{MP} to F_K.
- Write $x +_F y = F(x, y)$ and $[n]_F(x) = x +_F \cdots +_F x$ (n terms).
- We find that $[p]_K(x) = [p]_{F_K}(x) = x^{p^n}$ i.e. F_K has *height* n.
- Define $E_0 = \mathbb{Z}_p[\![u_1, \ldots, u_{n-1}]\!]$ with $u_0 = p$, $u_n = 1$.
- For $I = (i_1, \ldots, i_r)$ in $\{1, \ldots, n\}^r$ we put $|I| = r$ and $\|I\| = i_1 + \cdots + i_r$ and $\pi_t(I) = \prod_{s<t} p^{i_s}$ and $u_I = \prod_{t=1}^{r} u_{i_t}^{\pi_t(I)}$.Then put $l_E(x) = \sum_I u_I x^{p^{\|I\|}} / p^{|I|} \in (\mathbb{Q} \otimes E_0)[\![x]\!]$ and $F_E(x, y) = l_E^{-1}(l_E(x) + l_E(y))$.
- Using the *Functional Equation Lemma:* $F_E \in \mathrm{FGL}(E_0)$.
- Key fact: $[p]_E(x) = u_k x^{p^k} \pmod{u_i \mid i < k}$.
- There is a unique $\phi_E \colon MP_0 \to E_0$ carrying F_{MP} to F_E.
- Using Landweber exactness and Brown representability: there is a commutative ring spectrum E with $E_0 X = \pi_0(E \wedge X) = E_0 \otimes_{MP_0} (MP_0 X)$.
- There is also a ring spectrum K with $K^0 X = (E^0 X)/(u_0, \ldots, u_{n-1})$ whenever the sequence is regular (and same for $K_0 X$).

Morava K and E of BC_m

- Put $U = \mathbb{C}[t] \setminus \{0\}$ so $\mathbb{C}P^\infty = U/\mathbb{C}^\times$
- Define $\phi_m \colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ by $\phi_m([f]) = [f^m]$, so $\phi_m^*(x) = [m]_{F_{MP}}(x) = [m]_{MP}(x) \in MP^0(\mathbb{C}P^\infty)$ (and same for E, K).
- The map $h(s,f)(t) = s + (1-s)(1+st)f(t)$ gives a contraction of U.
- Put $C_m = \langle e^{2\pi i/m} \rangle < \mathbb{C}^\times$ and $BC_m = U/C_m$.
- $\mathbb{C}P^\infty$ has a tautological bundle T with $T_{[f]} = \mathbb{C}f$ and $\phi_m^*(T) \simeq T^{\otimes m}$.
- Then $BC_m = E(T^{\otimes m}) \setminus$ (zero section) so cofibre$(BC_m \to \mathbb{C}P^\infty) = \text{Thom}(T^{\otimes m})$
- Using the Thom isomorphism we get $MP^0(BC_m) = MP^0[\![x]\!]/[m]_{MP}(x)$ and $MP^1(BC_m) = 0$ (and same for E, K).
- If $m = p^k m_1$ with $p \nmid m_1$ then $[m]_K(x)$ is a unit multiple of $[p^k]_K(x) = x^{p^{nk}}$ so $K^0(BC_m) = \mathbb{F}_p\{x^i \mid i < p^{nk}\}$.
- Similarly $E^0(BC_m) = E^0\{x^i \mid i < p^{nk}\}$ (free of finite rank over E^0).
- For A finite abelian: $A \simeq C_{m_1} \times \cdots \times C_{m_r}$ say and $E^0(BA) = E^0(BC_{m_1}) \otimes_{E^0} \cdots \otimes_{E^0} E^0(BC_{m_r})$.
  This is again free of finite rank, and $E^1BA = 0$.

Schematic interpretation

- For A finite abelian: $A \simeq C_{m_1} \times \cdots \times C_{m_r}$ say and $E^0(BA) = E^0(BC_{m_1}) \otimes_{E^0} \cdots \otimes_{E^0} E^0(BC_{m_r})$. This is again free of finite rank, and $E^1BA = 0$.
- Put $S = \mathrm{spf}(E^0)$ and $\mathbb{G} := \mathrm{spf}(E^0(\mathbb{C}P^\infty))$, so $\mathbb{G}$ is a formal group scheme over S.
- Put $A^* = \mathrm{Hom}(A, S^1)$ (written additively). For $\alpha \in A^*$ we get $B\alpha\colon BA \to BS^1 = \mathbb{C}P^\infty$ and $(B\alpha)^*\colon E^0(\mathbb{C}P^\infty) \to E^0(BA)$ and $\phi_\alpha = \mathrm{spf}((B\alpha)^*)\colon \mathrm{spf}(E^0(BA)) \to \mathbb{G}$ over S.
- As $\mathbb{G}$ is a group we can form $\phi_\alpha + \phi_\beta$, but we find this is the same as $\phi_{\alpha+\beta}$. We thus get $\phi\colon \mathrm{spf}(E^0(BA)) \to \mathrm{Hom}(A^*, \mathbb{G})$.
- The previous calculation implies that ϕ is iso.
- More concretely: $E^0(BA)$ is generated by $\{x_\alpha \mid \alpha \in A^*\}$ modulo $x_{\alpha+\beta} = x_\alpha +_F x_\beta$.

Characteristic classes

- Let E be an even periodic ring spectrum, so we can choose $x \in \widetilde{E}^0(\mathbb{C}P^\infty)$ with $E^0(\mathbb{C}P^\infty) = E^0[\![x]\!]$.
- Over $\mathbb{C}P^\infty = (\mathbb{C}[t] \setminus \{0\})/\mathbb{C}^\times$ we have the tautological bundle $T_{[f]} = \mathbb{C}f$.
- For any $\mathbb{C}$ line bundle $L \to X$ there exists $p\colon X \to \mathbb{C}P^\infty$, unique up to homotopy, with $L \simeq p^*(T)$. Put $e(L) =$ Euler class of $L = p^*(x) \in E^0(X)$.
- For the formal group law F_E we have $e(L \otimes M) = e(L) +_{F_E} e(M)$.
- Now consider a complex vector bundle $V \to X$ of dimension d.
- Put $PV = \{(x, L) \mid x \in X,\ L \leq V_x,\ \dim(L) = 1\}$.
- This has a tautological bundle $T_{(x,L)} = L$ and Euler class $e(T) \in E^0(PV)$.
- **Theorem:** $\{e(T)^i \mid 0 \leq i < d\}$ is a basis for $E^0(PV)$ as an $E^0(X)$-module.
- By expressing $e(T)^d$ in terms of this basis: there is a monic polynomial $f_V(x) = \sum_{i=0}^d c_i(V)x^{d-i}$ with $E^0(PV) = E^0(X)[x]/f_V(x)$ via $x \mapsto e(T)$.
- The elements $c_i(V) \in E^0(X)$ are the *Chern classes* of V.
- We have $f_{V\oplus W}(x) = f_V(x)f_W(x)$ or $c_k(V \oplus W) = \sum_{i=0}^k c_i(V)c_{k-i}(W)$.
- For a complex representation V of a finite group G we have a vector bundle $EG \times_G V$ over $BG = EG/G$ and thus Chern classes in $E^0(BG)$ which we just call $c_i(V)$.