Chromatic cohomology of finite groups 1

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Introduction

- Fix a prime p and integer $n > 0$. Everything will depend on these.
- For any space X there are graded rings $K^*(X)$ and $E^*(X)$, called the (2-periodic) *Morava K-theory* and *Morava E-theory* of X.
- For any finite group G there is an essentially unique space BG (the *classifying space* of G) with $\pi_1(BG) = G$ and $\pi_k(BG) = 0$ for $k \neq 1$.
- This course is about rings of the form $K^*(BG)$ and $E^*(BG)$.
- We will just discuss $K^*(X)$ for the moment.
- This has $K^{i+2}(X) \simeq K^i(X)$, and very often $K^1(X) = 0$, so we just need to consider $K^0(X)$.
- The ring $K^0(BG)$ is a finite algebra over the finite field $\mathbb{F}_p$, so it is very amenable to explicit calculation, sometimes by computer.
- Good answers are known for abelian groups, symmetric groups, finite general linear groups of characteristic different from p, and various groups that are not far from being abelian.
- Kriz and Lee produced examples of groups G with $|G| = p^6$ and $K^1(BG) \neq 0$ and $E^0(BG)$ not free. Probably generic groups are like that. But many of the most interesting examples have $K^1(BG) = 0$.

Generalised cohomology

- A *generalised cohomology theory* is a contravariant, homotopy invariant functor $E^*\colon \text{Spaces} \to \text{Rings}^*$ with properties similar to H^*, but $E^*(1)$ need not be $\mathbb{Z}$. It takes work to provide interesting examples.
- We often work with *even periodic theories* where $E^1(1) = 0$ and $E^{-2}(1)$ contains a unit. Here it is natural to focus on $E^0(X)$.
- Given an even periodic theory E we put $X_E = \text{spf}(E^0X)$.
- There is an even periodic theory KU with $KU^*(1) = \mathbb{Z}[u, u^{-1}]$ (where $|u| = -2$) and $KU^0(X)$ is the ring of virtual complex vector bundles on X.
- Put $MP(n) = \{(v, V) \mid v \in V \leq \mathbb{C}^{2n}\}_\infty$ and $\Sigma^m X = (\mathbb{R}^m \times X)_\infty$ and $MP^k(X) = \varinjlim_n [\Sigma^{2n-k}X, MP(n)]$.

  This gives an even periodic theory with $MP^0(1) = \mathbb{Z}[a_1, a_2, a_3, \ldots]$. This is called *periodic complex cobordism*.
- The Nilpotence (pre)Theorem of Hopkins-Devinatz-Smith: if $MP^*(u) = 0$ then $u^k = 0$ for large k. This is the most powerful known theorem of the type algebra $\Rightarrow$ topology.
- Fix a prime p and an integer $n > 0$. There is then an even periodic theory K with $K^*(1) = \mathbb{F}_p[u, u^{-1}]$. This is called *Morava K-theory*.
- The K's together (for all p and n) carry $\sim$ the same information as MP.

Formal groups — what are they good for?

- Every even periodic theory E gives a formal group $\mathbb{G} = P_E = \mathrm{spf}(E^0(\mathbb{C}P^\infty))$.
- The functor $E \mapsto P_E$ is not too far from being an equivalence.
- The most elementary examples of formal groups are the additive and multiplicative formal groups; these correspond to HP and KU. (Here $HP^i(X) = \prod_j H^{i+2j}(X)$.)
- Steenrod operations in $HP^0(X;\mathbb{F}_p)$ and Adams operations in $KU^0(X)$ are closely related to endomorphisms of the associated formal groups.
- The ring $MP^0(1)$ is naturally isomorphic to the Lazard ring, which plays a central role in formal group theory.
- The Morava K-theories $K(p,n)$ all have different formal groups.
- Together with $HP^0(X;\mathbb{F}_p)$ and $HP^0(X;\mathbb{Q})$ this gives all formal groups over fields up to Galois twisting.
- For many spaces X the scheme X_E can be described naturally in terms of $\mathbb{G}$. For example, if $X = BU(n) = \{n-\text{dimensional subspaces of } \mathbb{C}^\infty\}$ then $X_E = \mathbb{G}^n/\Sigma_n$.

Examples of formal groups

- For any ring R we define commutative groups as follows:
  - $G_a(R) = \{a \in R \mid a \text{ is nilpotent } \}$ (under addition)
  - $G_m(R) = \{u \in R \mid u - 1 \text{ is nilpotent } \}$ (under multiplication)
  - $G_r(R) = \{A = \left[\begin{smallmatrix} c & -s \\ s & c \end{smallmatrix}\right] \in M_2(R) \mid c^2 + s^2 = 1,\ c - 1 \text{ nilpotent } \}$
  - $G_e(R) = \{(u, v) \in \mathrm{Nil}(R)^2 \mid v - u^3 + uv^2 = 0\}$ (an elliptic curve)
- These are all functorial in R.
- We can define natural bijections $x_i \colon G_i(R) \to \mathrm{Nil}(R)$ by $x_a(a) = a$ and $x_m(u) = u - 1$ and $x_r(A) = s/c$ and $x_e(u, v) = u$.
- One can check that $x_i(a * b) = F_i(x_i(a), x_i(b))$ where $F_a(s, t) = s + t$ and $F_m(s, t) = s + t + st$ and $F_r(s, t) = (s + t)/(1 - st) = \sum_{i \geq 0} s^i t^i (s + t)$. (One cannot be so explicit for F_e.)
- The functors G_i are *formal groups*; the power series F_i are *formal group laws*.
- Axioms: $F(s, 0) = s$, $F(s, t) = F(t, s)$ and $F(F(s, t), u) = F(s, F(t, u))$.
- More general version: we have a ground ring k, and $G(R)$ is only functorial for k-algebras, and $F(s, t) \in k[\![s, t]\!]$.
- Example: for any $a \in k$ we have an FGL $F(s, t) = s + t + ast$ over k.

Formal groups from even periodic theories

- $P = (\mathbb{C}[t] \setminus \{0\})/\mathbb{C}^\times = \{1-\text{dim subspaces of } \mathbb{C}[t]\} = \mathbb{C}P^\infty$.
- This is a commutative topological monoid (with inverses up to homotopy).
- So P_E is a formal group scheme over $1_E = \text{spec}(E^0(1))$.
- We can calculate $E^*(\mathbb{C}P^n)$ by induction on n using Mayer-Vietoris. It follows that there exists x with $E^0(P) = E^0(1)[\![x]\!]$ (but there is no canonical choice of x).
- This gives $E^0(P \times P) = E^0(1)[\![x_1, x_2]\!]$. The multiplication map $\mu\colon P \times P \to P$ has $\mu^*(x) = F(x_1, x_2)$ for some formal group law F.
- Now fix a prime p and let $\phi_p\colon P \to P$ be the p'th power map and put $B = (\mathbb{C}[t] \setminus \{0\})/C_p = BC_p$.
- Suppose that $p = 0$ in $E^0(1)$. Under some conditions that are often satisfied, we have $E^0(B) = E^0(1)[\![x]\!]/\phi_p^*(x)$ and this is free of finite rank over $E^0(1)$. If so, then the rank is p^n for some $n > 0$, called the *height*.
- For $E = K(p, n)$ we have $\phi_p^*(x) = x^{p^n}$ and the height is n.
- Over an algebraically closed field of characteristic p, any two formal groups of the same height are isomorphic.