FORMAL SCHEMES AND FORMAL GROUPS

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Contents

1. Introduction 2
1.1. Notation and conventions 2
1.2. Even periodic ring spectra 2
2. Schemes 3
2.1. Points and sections 5
2.2. Colimits of schemes 6
2.3. Subschemes 7
2.4. Zariski spectra and geometric points 8
2.5. Nilpotents, idempotents and connectivity 9
2.6. Sheaves, modules and vector bundles 10
2.7. Faithful flatness and descent 13
2.8. Schemes of maps 18
2.9. Gradings 19
3. Non-affine schemes 19
4. Formal schemes 22
4.1. (Co)limits of formal schemes 23
4.2. Solid formal schemes 24
4.3. Formal schemes over a given base 26
4.4. Formal subschemes 28
4.5. Idempotents and formal schemes 29
4.6. Sheaves over formal schemes 30
4.7. Formal faithful flatness 31
4.8. Coalgebraic formal schemes 33
4.9. More mapping schemes 36
5. Formal curves 38
5.1. Divisors on formal curves 38
5.2. Weierstrass preparation 41
5.3. Formal differentials 43
5.4. Residues 44
6. Formal groups 45
6.1. Group objects in general categories 45
6.2. Free formal groups 48
6.3. Schemes of homomorphisms 50
6.4. Cartier duality 51
6.5. Torsors 51
7. Ordinary formal groups 53
7.1. Heights 54
7.2. Logarithms 55
7.3. Divisors 55
8. Formal schemes in algebraic topology 56
8.1. Even periodic ring spectra 56

Date: June 3, 2009.

8.2. Schemes associated to spaces 57
8.3. Vector bundles and divisors 62
8.4. Cohomology of Abelian groups 64
8.5. Schemes associated to ring spectra 65
8.6. Homology of Thom spectra 65
8.7. Homology operations 66
References 68

1. Introduction

In this paper we set up a framework for using algebraic geometry to study the generalised cohomology rings that occur in algebraic topology. This idea was probably first introduced by Quillen [21] and it implicitly or explicitly underlies much of our understanding of complex oriented cohomology theories, exemplified by the work of Morava. Most of the results presented here have close and well-known analogues in the algebro-geometric literature, but with different definitions or technical assumptions that are often inconvenient for topological applications. Our aim here is merely to put everything together in a systematic way that naturally incorporates the phenomena that we see in topology while discarding complications that never arise there. In more detail, in the classical situation one is often content to deal with finite dimensional, Noetherian schemes. Nilpotents are seen as a somewhat peripheral phenomenon, and formal schemes are only introduced at a late stage in the exposition. Schemes are defined as spaces with extra structure. The idea of a scheme as a functor occurs in advanced work (a nice example is [16]) but is usually absent from introductory treatments. For us, however, it is definitely most natural to think of schemes as functors. Our schemes are very often not Noetherian or finite dimensional, and nilpotents are of crucial importance. We make heavy use of formal schemes, and we need to define these in a more general way than is traditional. On the other hand, we can get a long way using only affine schemes, whereas the usual treatment devotes a great deal of attention to the non-affine case.

Section 2 is an exposition of the basic facts of algebraic geometry that is well adapted to the viewpoint discussed above, together with a number of useful examples.

In Section 3, we give a basic account of non-affine schemes from our point of view.

In Section 4, we give a very general definition of formal schemes which follows naturally from our description of ordinary (or "informal") schemes. We then work out the basic properties of the category of formal schemes, such as the existence of limits and colimits and the behaviour of regular monomorphisms (or "closed inclusions").

In Section 6, we discuss the Abelian monoid and group objects in the category of formal schemes. We then specialise in Section 7 to the case of smooth, commutative, one-dimensional formal groups, which we call "ordinary formal groups".

Finally, in Section 8, we construct functors from the homotopy category of spaces (or suitable subcategories) to the category of formal schemes. We use the work of Ravenel, Wilson and Yagita [24] to show that spaces whose Morava K-theory is concentrated in even degrees give formal schemes with good technical properties. We also discuss what happens to a number of popular spaces under our functors. Further applications of this point of view appear in [26, 27, 7, 11] and a number of other papers in preparation.

1.1. Notation and conventions.

We write Rings for the category of rings (by which we always mean commutative unital rings) and Sets for the category of sets. For any ring R, we write Mod_R for the category of R-modules, and Alg_R for the category of R-algebras. Given a category $\mathcal{C}$, we usually write $\mathcal{C}(X,Y)$ for the set of $\mathcal{C}$-morphisms from X to Y. We write $\mathcal{C}_X$ for the category of objects of $\mathcal{C}$ over X. More precisely, on object of $\mathcal{C}_X$ is a pair (Y,u) where $u\colon Y\to Z$, and $\mathcal{C}_X((Y,u),(Z,v))$ is the set of maps $f\colon Y\to Z$ in $\mathcal{C}$ such that $vf=u$.

We write $\mathcal{F}$ for the category of all functors Rings $\to$ Sets.

1.2. Even periodic ring spectra.

We now give a basic topological definition, as background for some motivating remarks to be made in subsequent sections. Details of topological applications will appear in

Section 8. The definition below will be slightly generalised there, to deal with unpleasantness at the prime 2.

Definition 1.1. An *even periodic ring spectrum* is a commutative and associative ring spectrum E such that

(1) $\pi_1 E = 0$
(2) $\pi_2 E$ contains a unit.

The example to bear in mind is the complex K-theory spectrum KU. Suitable versions of Morava E-theory and K-theory are also examples, as are periodised versions of MU and H; we write MP and HP for these. See Section 8 for more details.

2. Schemes

In this section we set up the basic categorical apparatus of schemes. We then discuss limits and colimits of schemes, and various kinds of subschemes. We compare our functorial approach with more classical accounts by discussing the Zariski space of a scheme. We then discuss various issues about nilpotent and idempotent functions. We define sheaves over functors, and show that our definition works as expected for schemes. We then define flatness and faithful flatness for maps of schemes, and prove descent theorems for schemes and sheaves over faithfully flat maps. Finally, we address the question of defining a "scheme of maps" $\mathrm{Map}(X, Y)$ between two given schemes X and Y.

Definition 2.1. An *affine scheme* is a covariant representable functor

$$X : \mathrm{Rings} \to \mathrm{Sets}.$$

We make little use of non-affine schemes, so we shall generally omit the word "affine". A map of schemes is just a natural transformation. We write $\mathcal{X}$ for the category of schemes, which is a full subcategory of $\mathcal{F}$. We write $\mathrm{spec}(A)$ for the functor represented by A, so $\mathrm{spec}(A)(R) = \mathrm{Rings}(A, R)$ and $\mathrm{spec}(A)$ is a scheme.

Remark 2.2. If E is an even periodic ring spectrum and Z is a finite spectrum we define $Z_E = \mathrm{spec}(E^0 Z)$. This gives a covariant functor $Z \mapsto Z_E$ from finite complexes to schemes. We also write $S_E = \mathrm{spec}(E^0)$.

Definition 2.3. We write $\mathbb{A}^1$ for the forgetful functor $\mathrm{Rings} \to \mathrm{Sets}$. This is isomorphic to $\mathrm{spec}(\mathbb{Z}[t])$ and thus is a scheme. Given any functor $X \in \mathcal{F}$, we write $\mathcal{O}_X$ for the set of natural maps $X \to \mathbb{A}^1$. (This can actually be a proper class for general X, but it will always be a set in the cases that we consider.) Note that $\mathcal{O}_X$ is a ring under pointwise operations.

Our category of schemes is equivalent to the algebraic geometer's category of affine schemes, which in turn is equivalent (by Yoneda's lemma) to the opposite of the category of rings.

We now describe the duality between schemes and rings in more detail. The Yoneda lemma tells us that $\mathcal{O}_{\mathrm{spec}(A)}$ is naturally isomorphic to A. For any functor $X \in \mathcal{F}$ we have a tautological map $\kappa \colon X \to \mathrm{spec}(\mathcal{O}_X)$. To define κ explicitly, suppose we have a ring R and an element $x \in X(R)$; we need to produce a map $\kappa_R(x) \colon \mathcal{O}_X \to R$. An element $f \in \mathcal{O}_X$ is a natural map $f \colon X \to \mathbb{A}^1$, so it has a component $f_R \colon X(R) \to R$, and we can define $\kappa_R(x)(f) = f_R(x)$. If $X = \mathrm{spec}(A)$ then κ is easily seen to be bijective. As schemes are by definition representable, any scheme X is equivalent to $\mathrm{spec}(A)$ for some A, so we see that the map $X \to \mathrm{spec}(\mathcal{O}_X)$ is always an isomorphism. Thus, the functor $X \to \mathcal{O}_X$ is inverse to the functor $\mathrm{spec} \colon \mathrm{Rings}^{\mathrm{op}} \to \mathcal{X}$.

We next give some examples of schemes.

Example 2.4. A basic example is the "multiplicative group" $\mathbb{G}_m$, which is defined by

$$\mathbb{G}_m(R) = R^\times = \text{ the group of units of } R.$$

This is a scheme because it is represented by $\mathbb{Z}[x^{\pm 1}]$.

Example 2.5. The affine n-space $\mathbb{A}^n$ is defined by $\mathbb{A}^n(R) = R^n$. This is a scheme because it is represented by $\mathbb{Z}[x_1, \ldots, x_n]$. If $f_1, \ldots, f_m$ are polynomials in n variables over $\mathbb{Z}$ then there is an obvious natural map $R^m \to R^n$ for all rings R, which sends $\underline{a} = (a_1, \ldots, a_m)$ to $(f_1(\underline{a}), \ldots, f_n(\underline{a}))$. Thus, this gives a map $\mathbb{A}^m \to \mathbb{A}^n$ of schemes. These are in fact all the maps between these schemes. The key point is of course that the set of ring maps $\mathbb{Z}[y_1, \ldots, y_m] \leftarrow \mathbb{Z}[x_1, \ldots, x_n]$ bijects naturally with the set of such tuples $(f_1, \ldots, f_m)$. It is a good exercise to work out all of the identifications going on here.

We next define the scheme FGL of formal group laws, which will play a central rôle in the applications of schemes to algebraic topology.

Example 2.6. A *formal group law* over a ring R is a formal power series

$$F(x, y) = \sum_{k,l \geq 0} a_{kl} x^k y^l \in R[\![x, y]\!]$$

satisfying

$$\begin{aligned} F(x, 0) &= x \\ F(x, y) &= F(y, x) \\ F(F(x, y), z) &= F(x, F(y, z)). \end{aligned}$$

We can define a scheme FGL as follows:

$$\mathrm{FGL}(R) = \{ \text{ formal group laws over } R\}.$$

To see that FGL is a scheme, we consider the ring $L_0 = \mathbb{Z}[a_{kl} \mid k, l > 0]$ and the formal power series $F_0(x, y) = x + y + \sum a_{kl} x^k y^l \in L_0[\![x, y]\!]$. We then let I be the ideal in L_0 generated by the coefficients of the power series $F_0(x, y) - F_0(y, x)$ and $F_0(F_0(x, y), z) - F_0(x, F_0(y, z))$. Finally, set $L = L_0/I$. It is easy to see that $\mathrm{FGL} = \mathrm{spec}(L)$. The ring L is called the *Lazard ring*. It is a polynomial ring in countably many variables; there is a nice exposition of the proof in [2, Part II]. Recall that MP denotes the 2-periodic version of MU; a fundamental theorem of Quillen [19, 20] (also proved in [2]) identifies the scheme $S_{MP} := \mathrm{spec}(MP^0)$ with FGL.

Example 2.7. Given any diagram of schemes $\{X_i\}$, we claim that the functor $X = \varprojlim_i X_i$ (which is defined by $(\varprojlim_i X_i)(R) = \varprojlim_i (X_i(R)))$ is also a scheme. Indeed, suppose that $X_i = \mathrm{spec}(A_i)$. As $\mathrm{spec}\colon \mathrm{Rings}^{\mathrm{op}} \to \mathcal{X}$ is an equivalence, we get a diagram of rings A_i with arrows reversed. It is well-known that the category of rings has colimits, and it is clear that $X = \mathrm{spec}(\varinjlim_i A_i)$.

In particular, if X and Y are schemes, we have a scheme $X \times Y$ with $(X \times Y)(R) = X(R) \times Y(R)$ and $\mathcal{O}_{X\times Y} = \mathcal{O}_X \otimes \mathcal{O}_Y$ (because coproducts of rings are tensor products). Similarly, if we have maps $X \xrightarrow{f} Z \xleftarrow{g} Y$ then we can form the pullback

$$(X \times_Z Y)(R) = X(R) \times_{Z(R)} Y(R) = \{(x, y) \in X(R) \times Y(R) \mid f(x) = g(y)\}.$$

This is represented by the tensor product $\mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathcal{O}_Y$.

We write 1 for any one-point set, and also for the constant functor $1(R) = 1$. Thus $1 = \mathrm{spec}(\mathbb{Z})$, and this is the terminal object in $\mathcal{X}$ or $\mathcal{F}$.

Example 2.8. Let Z and W be finite CW complexes, and let E be an even periodic ring spectrum. There is a natural map $(Z \times W)_E \to Z_E \times_{S_E} W_E$. This will be an isomorphism if $E^1 Z = 0 = E^1 W$ and we have a Künneth isomorphism $E^*(Z \times W) = E^*(Z) \otimes_{E^*} E^*(W)$. This holds in particular if $H_* Z$ is a free Abelian group, concentrated in even degrees.

Example 2.9. An *invertible power series* over a ring R is a formal power series $f \in R[\![x]\!]$ such that $f(x) = wx + O(x^2)$ for some $w \in R^\times$. This implies, of course, that f has a composition-inverse $g = f^{-1}$, so that $f(g(x)) = x = g(f(x))$. We write $\mathrm{IPS}(S)$ for the set of such f, which is easily seen to be a scheme. It is actually a group scheme, in that $\mathrm{IPS}(R)$ is a group (under composition), functorially in R.

The group IPS acts on FGL by

$$(f, F) \mapsto F_f \qquad\qquad F_f(x, y) = f(F(f^{-1}x, f^{-1}y)).$$

An *isomorphism* between formal group laws F and G is an invertible series f such that $f(F(a, b)) = G(f(a), f(b))$. Let FI be the following scheme:

$$\mathrm{FI}(R) = \{(F, f, G) \mid F, G \in \mathrm{FGL}(R) \text{ and } f\colon F \to G \text{ is an isomorphism } \}.$$

There is an evident composition map

$$\mathrm{FI} \times_{\mathrm{FGL}} \mathrm{FI} \to \mathrm{FI} \qquad ((F, f, G), (G, g, H)) \mapsto (F, gf, H).$$

Moreover, there is an isomorphism

$$\mathrm{IPS} \times \mathrm{FGL} \to \mathrm{FI} \qquad (F, f) \mapsto (F, f, F_f).$$

One can describe these maps by giving implicit formulae in the representing rings $\mathcal{O}_{\mathrm{IPS}}$, $\mathcal{O}_{\mathrm{FGL}}$ an $\mathcal{O}_{\mathrm{FI}}$, but this should be avoided where possible. Note that for each R we can regard $\mathrm{FGL}(R)$ as the set of objects of a groupoid, whose morphism set is $\mathrm{FI}(R)$. In other words, the schemes FGL and FI define a groupoid scheme. It is known that $\mathrm{FI} = \mathrm{spec}(MP_0MP)$ (this follows easily from the description of MU_*MU in [2]).

Example 2.10. We now give an example for which representability is less obvious. We say that *an effective divisor of degree n* on $\mathbb{A}^1$ over a scheme Y is a subscheme $D \subseteq Y \times \mathbb{A}^1 = \mathrm{spec}(\mathcal{O}_Y[x])$ such that $\mathcal{O}_D$ is a quotient of $\mathcal{O}_Y[x]$ and is free of rank n over $\mathcal{O}_Y$. We let $X(R) = \mathrm{Div}_n^+(\mathbb{A}^1)(R)$ denote the set of such divisors over $\mathrm{spec}(R)$, and we claim that $X = \mathrm{Div}_n^+(\mathbb{A}^1)$ is a scheme. Firstly, it is a functor of R: given a ring map $u\colon R \to R'$ and a divisor D over R we get a divisor $uD = \mathrm{spec}(R' \otimes_R \mathcal{O}_D) = \mathrm{spec}(R') \times_{\mathrm{spec}(R)} D$ over R'. Next, given a divisor D as above and an element $y \in R[x]$, we let $\lambda(y)$ be the map $u \mapsto uy$, which is an R-linear endomorphism of the module $\mathcal{O}_D \simeq R^n$. The map $\lambda(x)$ thus has a characteristic polynomial $f_D(t) = \sum_{i=0}^n a_i(D)t^{n-i} \in R[t]$. One checks that the map $a_i\colon X \to \mathbb{A}^1$ is natural, so we get an element a_i of $\mathcal{O}_X$. As $f_D(t)$ is monic, we have $a_0 = 1$. The remaining a_i's give us a map $X \to \mathbb{A}^n$.

The Cayley-Hamilton theorem tells us that $f_D(\lambda(x)) = 0$, but it is clear that $f_D(\lambda(x)) = \lambda(f_D(x))$ and $f_D(x) = \lambda(f_D(x))(1)$, so we find that $f_D(x) = 0$ in $\mathcal{O}_D$ and thus that $\mathcal{O}_D$ is a quotient of $R[x]/f_D(x)$. On the other hand, it is clear that $R[x]/f_D(x)$ is also free over R of rank n, and it follows that $\mathcal{O}_D = R[x]/f_D(x)$. Given this, we see that D is freely and uniquely determined by the coefficients $a_1, \dots, a_n$, so that our map $X \to \mathbb{A}^n$ is an isomorphism. This shows in particular that X is a scheme. (I learned this argument from [4].)

2.1. **Points and sections.** Let X be a scheme. A *point of* X means an element $x \in X(R)$ for some ring R. We write $\mathcal{O}_x$ for R, which conveniently allows us to mention x before giving R a name. Recall that points $x \in X(R)$ biject with maps $\mathrm{spec}(R) \to X$. We say that x is defined over R, or over $\mathrm{spec}(R)$.

We can also think of an element of R as a point of the scheme $\mathbb{A}^1$ over R. If $f \in \mathcal{O}_X$ then f is a natural map $X(S) \to S$ for all rings S, so in particular we have a map $X(R) \to R$. We thus have $f(x) \in \mathcal{O}_x = R$.

Example 2.11. Let F be a point of FGL, in other words a formal group law over some ring R. We can write

$$[3](x) = F(x, F(x, x)) = 3x + u(F)x^2 + v(F)x^3 + O(x^4)$$

for certain scalars $u(F)$ and $v(F)$. This construction associates to each point $F \in \mathrm{FGL}$ a point $v(F) \in \mathbb{A}^1$ in a natural way, thus giving an element $v \in \mathcal{O}_{\mathrm{FGL}}$. Of course, we know that $\mathcal{O}_{\mathrm{FGL}}$ is the Lazard ring L, which is generated by the coefficients a_{kl} of the universal formal group law

$$F_{\mathrm{univ}}(x, y) = \sum_{k,l} a_{kl} x^k y^l$$

Using this formal group law, we find that

$$[3](x) = 3x + 3a_{11}x^2 + (a_{11}^2 + 8a_{12})x^3 + O(x^4)$$

This means that

$$v(F_{\mathrm{univ}}) = a_{11}^2 + 8a_{12}$$

It follows that for any F over any ring R, the element $v(F)$ is the image of $a_{11}^2 + 8a_{12}$ under the map $L \to R$ classifying F.

Example 2.12. For any scalar a, we have a formal group law

$$H_a(x, y) = x + y + axy.$$

The construction $a \mapsto H_a$ gives a natural transformation $h\colon \mathbb{A}^1(R) \to \mathrm{FGL}(R)$, in other words a map of schemes $h\colon \mathbb{A}^1 \to \mathrm{FGL}$. This can be thought of as a family of formal group laws, parametrised by $a \in \mathbb{A}^1$. It can also be thought of as a single formal group law over $\mathbb{Z}[a] = \mathcal{O}_{\mathbb{A}^1}$.

Example 2.13. The point of view described above allows for some slightly schizophrenic constructions, such as regarding the two projections $\pi_0, \pi_1 \colon X \times X \to X$ as two points of X over X^2. Indeed, this is the universal example of a scheme Y equipped with two points of X defined over Y. Similarly, we can think of the identity map $X \to X$ as the universal example of a point of X. This is analogous to thinking of the identity map of $K(\mathbb{Z}, n)$ as a cohomology class $u \in H^n K(\mathbb{Z}, n)$; this is of course the universal example of a space with a given n-dimensional cohomology class.

Definition 2.14. For any functor $X \colon \text{Rings} \to \text{Sets}$, we define a category $\text{Points}(X)$, whose objects are pairs (R, x) with $x \in X(R)$. The maps $(R, x) \to (S, y)$ are ring maps $f \colon R \to S$ such that $X(f)(x) = y$.

Remark 2.15. Let X be a scheme. The following categories are equivalent:

(a) The category $\mathcal{X}_X$ of schemes Y equipped with a map $u \colon Y \to X$.
(b) The category of representable functors $Y' \colon \text{Points}(X) \to \text{Sets}$.
(c) The category of representable functors $Y'' \colon \mathcal{X}_X^{\text{op}} \to \text{Sets}$.
(d) The category $\text{Alg}_{\mathcal{O}_X}^{\text{op}}$ of algebras R over $\mathcal{O}_X$.
(e) The category $\text{Points}(X)^{\text{op}}$ of pairs (R, x) with $x \in X(R)$.

By Yoneda, an element $x \in X(R)$ corresponds to a map $x' \colon \text{spec}(R) \to X$. Similarly, a map $v \colon Z \to X$ gives a map $v^* \colon \mathcal{O}_X \to \mathcal{O}_Z$, making $\mathcal{O}_Z$ into an $\mathcal{O}_X$-algebra. This can also be regarded as an element of $\text{spec}(\mathcal{O}_X)(\mathcal{O}_Z) = X(\mathcal{O}_Z)$. With this notation, the equivalence is as follows.

$$\begin{aligned}
Y(S) &= \coprod_{z \in X(S)} Y'(S, z) \\
Y'(S, z) &= \text{ preimage of } z \in X(S) \text{ under } u \colon Y(S) \to X(S) \\
&= Y''(\text{spec}(S) \xrightarrow{z'} X) \\
Y''(Z \xrightarrow{v} X) &= Y'(\mathcal{O}_Z, v^*) \\
R &= \mathcal{O}_Y \\
Y &= \text{spec}(R).
\end{aligned}$$

For us, the most important part of this will be the equivalence (a)$\Leftrightarrow$(b).

Remark 2.16. If E is an even periodic ring spectrum and $S_E = \text{spec}(E^0)$ then we can regard the construction $Z \mapsto Z_E = \text{spec}(E^0 Z)$ as a functor from finite complexes to $\mathcal{X}_{S_E}$.

Definition 2.17. If X is a scheme over another scheme Y, and $y \in Y(R)$ is a point of Y, we write $X_y = \text{spec}(R) \times_Y X$, which is a scheme over $\text{spec}(R)$. Here we have used the map $\text{spec}(R) \to Y$ corresponding to the point $y \in Y(R)$ to form the pullback $\text{spec}(R) \times_Y X$. We call X_y the *fibre* of X over the point y.

2.2. **Colimits of schemes.** The category of rings has limits for small diagrams, and the category of schemes is dual to that of rings, so it has colimits for small diagrams. However, it seems that these colimits only interact well with our geometric point of view if they have some additional properties (this is also the reason for Mumford's geometric invariant theory, which is much more subtle than anything that we consider here.) One good property that often occurs (with $\mathcal{C} = \mathcal{X}$ or $\mathcal{C} = \mathcal{X}_Y$) is as follows.

Definition 2.18. Let $\mathcal{C}$ be a category with finite products, and let $\{X_i\}$ be a diagram in $\mathcal{C}$. We say that an object X with a compatible system of maps $X_i \to X$ is a *strong colimit* of the diagram if $W \times X$ is the colimit of $\{W \times X_i\}$ for each $W \in \mathcal{C}$. We define strong coproducts and strong coequalisers as special cases of this, in the obvious way.

Example 2.19. The categories $\mathcal{X}$ and $\mathcal{X}_Y$ have strong finite coproducts, and $\mathcal{O}_{\coprod_i X_i} = \prod_i \mathcal{O}_{X_i}$. Indeed, by the usual duality $\text{Rings}^{\text{op}} = \mathcal{X}$, we see that the coproduct exists and has $\mathcal{O}_{\coprod_i X_i} = \prod_i \mathcal{O}_{X_i}$. Thus, we need only check that $Z \times_Y \coprod_i X_i = \coprod_i Z \times_Y X_i$, or equivalently that $\mathcal{O}_Z \otimes_{\mathcal{O}_Y} \prod_i \mathcal{O}_{X_i} = \prod_i \mathcal{O}_Z \otimes_{\mathcal{O}_Y} \mathcal{O}_{X_i}$, which is clear because the indexing set is finite. Note that when $Y = 1$ is the terminal object, we have $\mathcal{X}_Y = \mathcal{X}$, so we have covered that case as well.

As a special case of the above, we can make the following definition.

Definition 2.20. Given a finite set A, we can define an associated *constant scheme* $\underline{A}$ by

$$\underline{A} = \coprod_{a \in A} 1$$

(where 1 is the terminal object in $\mathcal{X}$). This has the property that $X \times \underline{A} = \coprod_{a \in A} X$ for all X. We also have $\mathcal{O}_{\underline{A}} = F(A, \mathbb{Z})$, which denotes the ring of functions from the set A to $\mathbb{Z}$; this is a ring under pointwise operations.

Remark 2.21. It is not the case that $(X \amalg Y)(R) = X(R) \amalg Y(R)$ (unlike the case of products and pullbacks). Instead, we have

$$(X \amalg Y)(R) = \{(S, T, x, y) \mid S, T \leq R\ ,\ R = S \times T\ ,\ x \in X(S)\ ,\ y \in Y(T)\}.$$

To explain this, note that an element of $(X \amalg Y)(R)$ is (by Yoneda) a map $\operatorname{spec}(R) \to X \amalg Y$. This will be given by a decomposition $\operatorname{spec}(R) = \operatorname{spec}(S) \amalg \operatorname{spec}(T)$ and maps $\operatorname{spec}(S) \to X$ and $\operatorname{spec}(T) \to Y$. Clearly, if R does not split nontrivially as a product of smaller rings then we have the naive rule $(X \amalg Y)(R) = X(R) \amalg Y(R)$.

Similarly, the initial scheme $\emptyset = \operatorname{spec}(0)$ has $\emptyset(R) = \emptyset$ unless $R = 0$ in which case $\emptyset(R)$ has a single element.

Example 2.22. Let $f \colon X \to Y$ be a map of schemes. Let X^n_Y denote the fibre product of n copies of X over Y, so that the symmetric group Σ_n acts on X^n_Y, covering the trivial action on Y. Suppose that the resulting map $f^* \colon \mathcal{O}_Y \to \mathcal{O}_X$ makes $\mathcal{O}_X$ into a free module over $\mathcal{O}_Y$. We then claim that there is a strong colimit for the action of Σ_n on X^n_Y. To see this, write $A = \mathcal{O}_X$ and $B = \mathcal{O}_Y$ and $C = A^{\otimes_B n}$, so that $X^n_Y = \operatorname{spec}(C)$. Our claim reduces easily to the statement that $B' \otimes_B (C^{\Sigma_n}) = (B' \otimes_B C)^{\Sigma_n}$ for every algebra B' over B. To see that this holds, choose a basis for A over B. This gives an evident basis for C over B, which is permuted by the action of Σ_n. Clearly C^{Σ_n} is a free module over B, with one generator for each Σ_n-orbit in our basis for C. There is a similar description for $(B' \otimes_B C)^{\Sigma_n}$, which quickly implies our claim.

Some more circumstances in which colimits have unexpectedly good behaviour are discussed in [7], which mostly follows ideas of Quillen [21].

2.3. **Subschemes.**

Recall that an element of $\mathcal{O}_X$ is a natural map $X \to \mathbb{A}^1$. Thus, if x is a point of X then $f(x)$ is a scalar (more precisely, if $x \in X(R)$ then $f(x) \in R$) and we can ask whether $f(x) = 0$, or whether $f(x)$ is invertible.

Definition 2.23. Given a scheme X and an ideal $I \leq \mathcal{O}_X$, we define a scheme $V(I)$ by

$$V(I)(R) = \{x \in X(R) \mid f(x) = 0 \text{ for all } f \in I\}.$$

One checks that $V(I) = \operatorname{spec}(\mathcal{O}_X/I)$, so this really is a scheme. Schemes of this form are called *closed subschemes* of X.

Given an element $f \in \mathcal{O}_X$, we define a scheme $D(f)$ by

$$D(f)(R) = \{x \in X(R) \mid f(x) \in R^\times\}.$$

One checks that $D(f) = \operatorname{spec}(\mathcal{O}_X[1/f])$, so this really is a scheme. Schemes of this form are called *basic open subschemes* of X.

A *locally closed subscheme* is a basic open subscheme of a closed subscheme. Such a thing has the form $D(f) \cap V(I) = \operatorname{spec}(\mathcal{O}_X[1/f]/I)$.

Remark 2.24. Recall that a map $f \colon R \to S$ of rings is said to be a *regular epimorphism* if and only if it is the coequaliser of some pair of maps $T \rightrightarrows R$, which happens if and only if it is the coequaliser of the obvious maps $R \times_S R \rightrightarrows R$. It is easy to check that this holds if and only if f is surjective. Given this, we see that the regular monomorphisms of schemes are precisely the closed inclusions, and that composites and pushouts of regular monomorphisms are regular monomorphisms.

Example 2.25. The map h in Example 2.12 gives an isomorphism between $\mathbb{A}^1$ and the closed subscheme $V((a_{ij} \mid i + j > 2))$ of FGL. The multiplicative group $\mathbb{G}_m$ is an open subscheme of $\mathbb{A}^1$.

Example 2.26. If X is a scheme and $e \in \mathcal{O}_X$ satisfies $e^2 = e$ then it is easy to check that $D(e) = V(1-e)$, so this subscheme is both open and closed. Moreover, we have $X = D(e) \amalg D(1-e)$. More generally, if we have idempotents $e_1, \ldots, e_m \in \mathcal{O}_X$ with $\sum_i e_i = 1$ and $e_i e_j = \delta_{ij} e_i$ then $X = \coprod_i D(e_i)$, and every splitting of X as a finite coproduct occurs in this way.

Example 2.27. Suppose $X = \mathrm{spec}(k[x])$ is the affine line over a field k, and $\lambda, \mu \in k$. The closed subscheme $V(x-\lambda) = \mathrm{spec}(k[x]/(x-\lambda)) \simeq \mathrm{spec}(k)$ corresponds to the point λ of the affine line; it is natural to refer to it as $\{\lambda\}$. The closed subscheme $V((x-\lambda)(x-\mu))$ corresponds to the pair of points $\{\lambda, \mu\}$. If $\lambda = \mu$, this is to be thought of as the point λ with multiplicity two, or as an infinitesimal thickening of the point λ.

We can easily form the intersection of locally closed subschemes:

$$D(a) \cap V(I) \cap D(b) \cap V(J) = D(ab) \cap V(I+J).$$

We cannot usually form the union of basic open subschemes and still have an affine scheme. Again, it would be easy enough to consider non-affine schemes, but it rarely seems to be necessary. Moreover, a closed subscheme $V(a)$ determines the complementary open subscheme $D(a)$ but not conversely; $D(a) = D(a^2)$ but $V(a) \neq V(a^2)$ in general.

We say that a scheme X is *reduced* if $\mathcal{O}_X$ has no nonzero nilpotents, and write $X_{\mathrm{red}} = \mathrm{spec}(\mathcal{O}_X/\sqrt{0})$, which is the largest reduced closed subscheme of X. Moreover, if $Y \subseteq X$ is closed then $Y_{\mathrm{red}} = X_{\mathrm{red}}$ if and only if $X(k) = Y(k)$ for every field k (we leave the proof as an exercise).

We define the union of closed subschemes by $V(I) \cup V(J) = V(I \cap J)$. We also define the *schematic union* by $V(I) + V(J) = V(IJ)$. This is a sort of "union with multiplicity" — in particular, $V(I) + V(I) \neq V(I)$ in general. In the previous example, we have

$$\{\lambda\} \cup \{\lambda\} = V((x-\lambda)^2)$$

which is a thickening of $\{\lambda\}$. Note that $V(IJ)_{\mathrm{red}} = V(I \cap J)_{\mathrm{red}}$, because $(I \cap J)^2 \leq IJ \leq I \cap J$.

We shall say that X is *connected* if it cannot be split nontrivially as $Y \amalg Z$, if and only if there are no idempotents in $\mathcal{O}_X$ other than 0 and 1.

We shall say that a scheme X is *integral* if and only if $\mathcal{O}_X$ is an integral domain, and that X is *irreducible* if and only if X_{red} is integral. We also say that X is *Noetherian* if and only if the ring $\mathcal{O}_X$ is Noetherian. If so, then X_{red} can be written in a unique way as a finite union $\bigcup_i Y_i$ with Y_i an integral closed subscheme. The schemes Y_i are called the *irreducible components* of X_{red}; they are precisely the schemes $V(\mathfrak{p}_i)$ for $\mathfrak{p}_i$ a minimal prime ideal of $\mathcal{O}_X$. See [18, section 6] for this material.

Suppose that X is Noetherian and reduced, say $X = \bigcup_{i \in S} Y_i$ as above for some finite set S. Suppose that $S = S' \amalg S''$. Write $X' = \bigcup_{S'} Y_i = V(I')$, where $I' = \bigcap_{S'} \mathfrak{p}_i$, and similarly for X'' and I''. If we then write

$$\Gamma(I') = \{a \in \mathcal{O}_X \mid a(I')^N = 0 \text{ for } N \gg 0\},$$

we find that $\Gamma(I') = I''$ and thus $V(\Gamma(I')) = X''$.

Example 2.28. Take $Z = \mathrm{spec}(k[x,y]/(xy^2))$ and set

$$\begin{aligned} X &= V(y) = \mathrm{spec}(k[x]) \\ X' &= V(y^2) = \mathrm{spec}(k[x,y]/(y^2)) \\ Y &= V(x) = \mathrm{spec}(k[y]) \end{aligned}$$

Then X is the x-axis, Y is the y-axis and X' is an infinitesimal thickening of X. The schemes X and Y are integral, and X' is irreducible because $X'_{\mathrm{red}} = X$. The scheme Z is reducible, and its irreducible components are X and Y.

2.4. **Zariski spectra and geometric points.** If R is a ring, we define the *Zariski space* to be

$$\mathrm{zar}(R) = \{ \text{ prime ideals } \mathfrak{p} < R \ \}.$$

If X is a scheme, we write $X_{\mathrm{zar}} = \mathrm{zar}(\mathcal{O}_X)$. Note that

$$\begin{aligned} V(I)_{\mathrm{zar}} &= \mathrm{zar}(\mathcal{O}_X/I) = \{\mathfrak{p} \in X_{\mathrm{zar}} \mid I \leq \mathfrak{p}\} \\ D(f)_{\mathrm{zar}} &= \mathrm{zar}(\mathcal{O}_X[1/f]) = \{\mathfrak{p} \in X_{\mathrm{zar}} \mid f \notin \mathfrak{p}\} \\ (X \amalg Y)_{\mathrm{zar}} &= X_{\mathrm{zar}} \amalg Y_{\mathrm{zar}} \end{aligned}$$

There is a map
$$(X \times Y)_{\mathrm{zar}} \to X_{\mathrm{zar}} \times Y_{\mathrm{zar}},$$
but it is almost never a bijection.

Suppose that $Y, Z \leq X$ are locally closed; then
$$(Y \cap Z)_{\mathrm{zar}} = Y_{\mathrm{zar}} \cap Z_{\mathrm{zar}}.$$
If Y and Z are closed then
$$(Y \cup Z)_{\mathrm{zar}} = (Y + Z)_{\mathrm{zar}} = Y_{\mathrm{zar}} \cup Z_{\mathrm{zar}}.$$
We give X_{zar} the topology with closed sets $V(I)_{\mathrm{zar}}$. A map of schemes $X \to Y$ then induces a continuous map $X_{\mathrm{zar}} \to Y_{\mathrm{zar}}$.

Suppose that R is an integral domain, and that $x \in X(R)$. Then x gives a map $x^* \colon \mathcal{O}_X \to R$, whose kernel $\mathfrak{p}_x$ is prime. We thus have a map $X(R) \to X_{\mathrm{zar}}$, which is natural for monomorphisms of R and arbitrary morphisms of X.

A *geometric point* of X is an element of $X(k)$, for some algebraically closed field k. Suppose that either $\mathcal{O}_X$ is a $\mathbb{Q}$-algebra, or that some prime p is nilpotent in $\mathcal{O}_X$. Let k be an algebraically closed field of the appropriate characteristic, with transcendence degree at least the cardinality of $\mathcal{O}_X$. Then it is easy to see that $X(k) \to X_{\mathrm{zar}}$ is epi.

A useful feature of the Zariski space is that it behaves quite well under colimits [21, 7]. The following proposition is another example of this.

Proposition 2.29. Suppose that a finite group G acts on a scheme X. Then $(X/G)_{\mathrm{zar}} = X_{\mathrm{zar}}/G$.

Proof. Write $S = \mathcal{O}_X$ and $R = S^G = \mathcal{O}_{X/G}$. Given a prime $\mathfrak{p} \in \mathrm{zar}(R) = (X/G)_{\mathrm{zar}}$, the fibre F over $\mathfrak{p}$ in $\mathrm{zar}(S) = X_{\mathrm{zar}}$ is just $\mathrm{zar}(S_{\mathfrak{p}}/\mathfrak{p}S\mathfrak{p})$ (see [18, Section 7]). We need to prove that F is nonempty, and that G acts transitively on F.

As localisation is exact, we have $(S_{\mathfrak{p}})^G = R_{\mathfrak{p}}$, so we can replace R by $R_{\mathfrak{p}}$ and thus assume that R is local at $\mathfrak{p}$. With this assumption, we have $F = \mathrm{zar}(S/\mathfrak{p}S)$. For $a \in S$ we write $f_a(t) = \prod_{g \in G}(t - ga) \in S[t]^G = R[t]$, so that f_a is a monic polynomial with $f_a(a) = 0$. This shows that S is an integral extension over R, so $F \neq \emptyset$ and there are no inclusions between the elements of F [18, Theorem 9.3].

Let $\mathfrak{q}$ and $\mathfrak{r}$ be two points of F, so they are prime ideals in S with $\mathfrak{q} \cap R = \mathfrak{q}^G = \mathfrak{p}$ and $\mathfrak{r} \cap R = \mathfrak{r}^G = \mathfrak{p}$. Write $I = \bigcap_{g \in G} g.\mathfrak{q} \leq S$. If $a \in I$ then $g.a \in \mathfrak{q}$ for all g so $f_a(t) \in t^{|G|} + \mathfrak{q}[t]$ but also $f_a(t)$ is G-invariant so $f_a(t) \in t^{|G|} + \mathfrak{q}^G[t] \subseteq t^{|G|} + \mathfrak{r}[t]$. As $f_a(a) = 0$ we conclude that a is nilpotent mod $\mathfrak{r}$ but $\mathfrak{r}$ is prime so $a \in \mathfrak{r}$. Thus $\bigcap_{g \in G} g.\mathfrak{q} \leq \mathfrak{r}$. As $\mathfrak{r}$ is prime, we deduce that $g.\mathfrak{q} \leq \mathfrak{r}$ for some $g \in G$. As there are no inclusions between the elements of F, we conclude that $g.\mathfrak{q} = \mathfrak{r}$. Thus G acts transitively on F, which proves that $(X/G)_{\mathrm{zar}} = X_{\mathrm{zar}}/G$. ☐

A number of interesting things can be detected by looking at Zariski spaces. For example, X_{zar} splits as a disjoint union if and only if X does — see Corollary 2.40.

We also use the space X_{zar} to define the *Krull dimension* of X.

Definition 2.30. If there is a chain $\mathfrak{p}_0 < \ldots < \mathfrak{p}_n$ in X_{zar}, but no longer chain, then we say that $\dim(X) = n$. If there are arbitrarily long chains then $\dim(X) = \infty$.

Example 2.31. The terminal object 1 has dimension one (because there are chains $(0) < (p)$ of prime ideals in $\mathbb{Z}$). If $\mathcal{O}_X$ is a field then $\dim(X) = 0$. If $\mathcal{O}_X$ is Noetherian then $\dim(\mathbb{G}_m \times X) = 1 + \dim(X)$ and $\dim(\mathbb{A}^n \times X) = n + \dim(X)$ [18, Section 15]. In particular, we have $\dim(\mathbb{A}^n) = \dim(1 \times \mathbb{A}^n) = n + 1$.

Example 2.32. The schemes FGL, IPS and FI all have infinite dimension.

2.5. **Nilpotents, idempotents and connectivity.**

Proposition 2.33. Suppose that $e \in R$ is idempotent, and $f = 1 - e$. Then
$$eR = R/f = R[e^{-1}] = \{a \in R \mid fa = 0\}.$$
Moreover, this is a ring with unit e, and we have $R = eR \times fR$ as rings. ☐

Proposition 2.34. If X is a scheme, then splittings $X = \coprod_{i=1}^n X_i$ biject with systems of idempotents $\{e_1, \ldots, e_n\}$ with $\sum_i e_i = 1$ and $e_i e_j = \delta_{ij} e_j$. ☐

Example 2.35. Let $\mathrm{Mult}(n)$ be the scheme of polynomials $\phi(u)$ of degree at most n such that $\phi(1) = 1$ and $\phi(uv) = \phi(u)\phi(v)$. Such a series can be written as $\phi(u) = \sum_{i=0}^{n} e_i u^i$, and the conditions on ϕ are equivalent to $\sum_i e_i = 1$ and $e_i e_j = \delta_{ij} e_j$. In other words, the elements e_i are orthogonal idempotents. Using this, we see easily that $\mathrm{Mult}(n) = \coprod_{i=0}^{n} 1$.

Example 2.36. Now let $E(n)$ be the scheme of $n \times n$ matrices A over R such that $A^2 = A$. Define $\alpha_A(u) = uA + (1-A) = (u-1)A + 1 \in M_n(R[u])$ and $\phi_A(u) = \det(\alpha_A(u)) \in R[u]$. We find easily that $\alpha_A(1) = 1$ and $\alpha_A(uv) = \alpha_A(u)\alpha_A(v)$, so $\phi_A(u) \in \mathrm{Mult}(n)(R)$. This construction gives a map $E(n) \to \mathrm{Mult}(n) = \coprod_{i=0}^{n} 1$, which gives a splitting $E(n) = \coprod_{i=0}^{n} E(n,i)$, where $E(n,i)$ is the scheme of $n \times n$ matrices A such that $A^2 = A$ and $\phi_A(u) = u^i$.

Note that the function $A \mapsto \mathrm{trace}(A)$ lies in $\mathcal{O}_{E(n)}$ and that $E(n,i)$ is contained in the closed subscheme $E'(n,i) = \{A \mid \mathrm{trace}(A) = i\}$. However, if $n > 0$ but $n = 0$ in R then $E'(n,0)(R)$ and $E'(n,n)(R)$ are not disjoint, which shows that $E'(n,i) \neq E(n,i)$ in general.

For any ring R, we let $\mathrm{Nil}(R)$ denote the set of nilpotents in R.

Proposition 2.37. $\mathrm{Nil}(R)$ is the intersection of all prime ideals in R.

Proof. [18, Section 1] □

Proposition 2.38 (Idempotent Lifting)**.** Suppose that $e \in R/\mathrm{Nil}(R)$ is idempotent. Then there is a unique idempotent $\tilde{e} \in R$ lifting e.

Proof. Choose a (not necessarily idempotent) lift of e to R, call it e, and write $f = 1 - e$. We know that ef is nilpotent, say $e^n f^n = 0$. Define

$$c = e^n + f^n - 1 = e^n + f^n - (e+f)^n$$

This is visibly divisible by ef, hence nilpotent; thus $e^n + f^n = 1 + c$ is invertible. Define

$$\tilde{e} = e^n/(1+c) \qquad \tilde{f} = f^n/(1+c) = 1 - \tilde{e}$$

Then $\tilde{e}$ is an idempotent lifting e. If $\tilde{e}_1$ is another such then $\tilde{e}_1\tilde{f}$ is idempotent. It lifts $ef = 0$, so it is also nilpotent. It follows that $\tilde{e}_1\tilde{f} = 0$ and $\tilde{e}_1 = \tilde{e}\tilde{e}_1$. Similarly, $\tilde{e} = \tilde{e}\tilde{e}_1$, so $\tilde{e} = \tilde{e}_1$. □

Theorem 2.39 (Chinese Remainder Theorem)**.** Suppose that $\{I_\alpha\}$ is a finite family of ideals in R, which are pairwise coprime (i.e. $I_\alpha + I_\beta = R$ when $\alpha \neq \beta$). Then

$$R/\bigcap_\alpha I_\alpha = \prod_\alpha R/I_\alpha$$

Proof. [18, Theorems 1.3,1.4] □

Corollary 2.40. Suppose that $\mathrm{zar}(R) = \coprod_\alpha \mathrm{zar}(R/I_\alpha)$ (a finite coproduct). Then there are unique ideals $J_\alpha \leq I_\alpha \leq \sqrt{J_\alpha}$ such that $R \simeq \prod_\alpha R/J_\alpha$.

Proof. Proposition 2.37 implies that $\bigcap_\alpha I_\alpha$ is nilpotent. If $\alpha \neq \beta$ then no prime ideal contains $I_\alpha + I_\beta$, so $I_\alpha + I_\beta = R$. Now use the Chinese remainder theorem, followed by idempotent lifting. □

Remark 2.41. There are nice topological applications of these ideas in [15, 7], for example.

2.6. **Sheaves, modules and vector bundles.** The simplest definition of a sheaf over a scheme X is just as a module over the ring $\mathcal{O}_X$. (It would be more accurate to refer to this as a quasi-coherent sheaf of $\mathcal{O}$-modules over X, but we shall just call it a sheaf.) However, we shall give a different (but equivalent) definition which fits more neatly with our emphasis on schemes as functors, and which generalises more easily to formal schemes.

Definition 2.42. A *sheaf* over a functor $X \in \mathcal{F}$ consists of the following data:

(a) For each $(R,x) \in \mathrm{Points}(X)$, a module M_x over R.
(b) For each map $f\colon (R,x) \to (S,y)$ in $\mathrm{Points}(X)$, an isomorphism $\theta(f) = \theta(f,x)\colon S \otimes_R M_x \to M_y$ of S-modules.

The maps $\theta(f,x)$ are required to satisfy the functorality conditions

(i) In the case $f = 1\colon (R,x) \to (R,x)$ we have $\theta(1,x) = 1\colon M_x \to M_x$.

(ii) Given maps $(R,x) \xrightarrow{f} (S,y) \xrightarrow{g} (T,z)$, the map $\theta(gf,x)$ is just the composite

$$T \otimes_R M_x = T \otimes_S S \otimes_R M_x \xrightarrow{1\otimes\theta(f,x)} T \otimes_S M_y \xrightarrow{\theta(g,y)} M_z.$$

We write $\mathrm{Sheaves}_X$ for the category of sheaves over X. This has direct sums (with $(M \oplus N)_x = M_x \oplus N_x$) and tensor products (with $(M \otimes N)_x = M_x \otimes_R N_x$ when $x \in X(R)$). The unit for the tensor product is the sheaf $\mathcal{O}$, which is defined by $\mathcal{O}_x = R$ for all $x \in X(R)$.

Remark 2.43. If M and N are sheaves over a sufficiently bad functor X, it can happen that $\mathrm{Sheaves}_X(M,N)$ is a proper class. This will not be the case if X is a scheme or a formal scheme, however.

Example 2.44. Let x be a point of $\mathbb{A}^1(R)$, or in other words an element of R. Define $M_x = R/x$; this gives a sheaf over $\mathbb{A}^1$. Note that $M_x = 0$ if x is invertible, but $M_x = R$ if $x = 0$. Thus, M is concentrated at the origin of $\mathbb{A}^1$.

Definition 2.45. (1) Let X be a functor in $\mathcal{F}$. If N is a module over the ring $\mathcal{O}_X = \mathcal{F}(X,\mathbb{A}^1)$, we define a sheaf $\tilde{N}$ over X by $\tilde{N}_x = R \otimes_{\mathcal{O}_X} N$, where we use x to make R into an algebra over $\mathcal{O}_X$.

(2) If M is a sheaf over X and R is a ring, we write $\mathbb{A}(M)(R) = \coprod_{x\in X(R)} M_x$. If $f\colon R \to S$ is a homomorphism, we define a map $\mathbb{A}(M)(R) \to \mathbb{A}(M)(S)$, which sends M_x to $M_{f(x)}$ by $m \mapsto \theta(f,x)(1 \otimes m)$. This gives a functor $\mathbb{A}(M) \in \mathcal{F}_X$.

(3) If M is a sheaf over X, we define $\Gamma(X,M) = \mathcal{F}_X(X,\mathbb{A}(M))$. Thus, an element $u \in \Gamma(X,M)$ is a system of elements $u_x \in M_x$ for all rings R and points $x \in X(R)$, which behave in the obvious way under maps of rings. If $M = \mathcal{O}$ then $\mathbb{A}(\mathcal{O}) = \mathbb{A}^1 \times X$ and $\Gamma(X,\mathcal{O}) = \mathcal{O}_X$. It follows that $\Gamma(X,M)$ is a module over $\mathcal{O}_X$ for all M.

(4) If Y is a scheme over X, we also define $\Gamma(Y,M) = \mathcal{F}_X(Y,\mathbb{A}(M))$.

Proposition 2.46. For any functor $X \in \mathcal{F}$, the functor $\Gamma(X,-)\colon \mathrm{Sheaves}_X \to \mathrm{Mod}_{\mathcal{O}_X}$ is right adjoint to the functor $N \mapsto \tilde{N}$.

Proof. For typographical convenience, we will write TN for $\tilde{N}$ and GM for $\Gamma(X,M)$. We define maps $\eta\colon N \to GTN$ and $\epsilon\colon TGM \to M$ as follows. Let n be an element of N; for each point $x \in X(R)$, we define $\eta(n)_x = 1 \otimes n \in R \otimes_{\mathcal{O}_X} N = (TN)_x$, giving a map η as required. Next, we define $\epsilon_x\colon (TGM)_x = R \otimes_{\mathcal{O}_X} \Gamma(X,M) \to M_x$ by $\epsilon_x(a \otimes u) = au_x$. We leave it to the reader to check the triangular identities $(\epsilon_T)(T\eta) = 1_T$ and $(G\epsilon)(\eta_G) = 1_G$, which show that we have an adjunction. □

Proposition 2.47. Let X be a scheme, and let $x_0 \in X(\mathcal{O}_X)$ be the tautological point, which corresponds to the identity map of $\mathcal{O}_X$ under the isomorphism $X = \mathrm{spec}(\mathcal{O}_X)$. Then there is a natural isomorphism $\Gamma(X,M) = M_{x_0}$, and $\Gamma(X,-)\colon \mathrm{Sheaves}_X \to \mathrm{Mod}_{\mathcal{O}_X}$ is an equivalence of categories.

Proof. First, we define a map $\alpha\colon \Gamma(X,M) \to M_{x_0}$ by $u \mapsto u_{x_0}$. Next, suppose that $m \in M_{x_0}$. If $x \in X(R)$ for some ring R then we have a corresponding ring map $\hat{x}\colon f \mapsto f(x)$ from $(\mathcal{O}_X, x_0)$ to (R,x). We define $\beta(m)_x = \theta(\hat{x},x_0)(m) \in M_x$. One can check that this gives an element $\beta(m) \in \Gamma(X,M)$, and that $\beta\colon M_{x_0} \to \Gamma(X,M)$ is inverse to α. It follows that $\Gamma(X,\tilde{N}) = \tilde{N}_{x_0}$, which is easily seen to be the same as N. Also, if $N = M_{x_0}$ then $\tilde{N}_x = R \otimes_{\mathcal{O}_X} M_{x_0}$, and $\theta(\hat{x},x_0)$ gives an isomorphism of this with M_x, so $\tilde{N} = M$. It follows that the functor $N \mapsto \tilde{N}$ is inverse to $\Gamma(X,-)$. □

It follows that when X is a scheme, the category $\mathrm{Sheaves}_X$ is Abelian. Because tensor products preserve colimits and finite products, we see that the functors $M \mapsto M_x$ preserve colimits and finite products.

We next need some recollections about finitely generated projective modules. If M is such a module over a ring R and $\mathfrak{p} \in \mathrm{zar}(R)$ then $M_\mathfrak{p}$ is a finitely generated module over the local ring $R_\mathfrak{p}$ and thus is free [18, Theorem 2.5], of rank $r_\mathfrak{p}(M)$ say. Note that $r_\mathfrak{p}(M)$ is also the dimension of $\kappa(\mathfrak{p}) \otimes_R M$ over the field $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. If this is independent of $\mathfrak{p}$ then we call it $r(M)$ and say that M has constant rank. Clearly, if any two of M, N and $M \oplus N$ have constant rank then so does the third and $r(M \oplus N) = r(M) + r(N)$. Also, if $r(M) = 0$ then $M = 0$.

Definition 2.48. Let M be a sheaf over a functor X. If M_x is a finitely generated projective module over $\mathcal{O}_x$ for all $x \in X$, we say that M is a *vector bundle* or *locally free sheaf* over X. If in addition M_x has rank one for all x, we say that M is a *line bundle* or *invertible sheaf*.

If X is a scheme, a sheaf M is a vector bundle if and only if $\Gamma(X, M)$ is a finitely generated projective module over $\mathcal{O}_X$. However, this does not generalise easily to formal schemes, so we do not take it as the definition. It is not hard to check that M_x has constant rank r for all R and all $x \in X(R)$ if and only if M_x has dimension r over K for all algebraically closed fields K and all $x \in X(K)$.

Remark 2.49. In algebraic topology, it is very common that the naturally occurring projective modules are free, and thus that the corresponding vector bundles and line bundles are trivialisable. However, they are typically not equivariantly trivial for important groups of automorphisms, so it is conceptually convenient to avoid choosing bases. The main example is that if Z is a finite complex and V is a complex vector bundle over Z with Thom complex Z^V then $\widetilde{E}^0 Z^V$ gives a line bundle over Z_E. A choice of complex orientation on E gives a Thom class and thus a trivialisation, but this is not invariant under automorphisms of E.

Example 2.50. Recall the scheme $E(n) = \coprod_{i=0}^n E(n, i)$ of Example 2.36. A point of $E(n)(R)$ is an $n \times n$ matrix A over R with $A^2 = A$. This means that $M_A = A.R^n$ is a finitely generated projective R-module, so this construction defines a vector bundle M over $E(n)$. If A is a point of $E(n, i)$ (so that $\det((u-1)A+1) = u^i \in R[u]$) and R is an algebraically closed field, then elementary linear algebra shows that A has rank i. It follows that the restriction of M to $E(n, i)$ has rank i.

Let N be a vector bundle over an arbitrary scheme X. The associated projective $\mathcal{O}_X$-module is then a retract of a finitely generated free module, so there is a matrix $A \in E(n)(\mathcal{O}_X)$ such that $\Gamma(X, N) = A.\mathcal{O}_X^n$ for some n. The point $A \in E(n)(\mathcal{O}_X)$ corresponds to a map $\alpha\colon X \to E(n)$, and we find that $\alpha^* M = N$. If X_i denotes the preimage of $E(n, i)$ under α, then $X = \coprod_i X_i$ and the restriction of N to X_i has rank i.

Let X be a scheme. Using equivalence $\mathrm{Sheaves}_X \simeq \mathrm{Mod}_{\mathcal{O}_X}$ again, we see that there are sheaves $\mathrm{Hom}(M, N)$ such that

$$\mathrm{Sheaves}_X(L, \mathrm{Hom}(M, N)) = \mathrm{Sheaves}_X(L \otimes M, N).$$

In particular, we define $M^\vee = \mathrm{Hom}(M, \mathcal{O})$. If M is a vector bundle then we have $\mathrm{Hom}(M, N)_x = \mathrm{Hom}_R(M_x, N_x)$ and thus $(M^\vee)_x = \mathrm{Hom}(M_x, R)$. In that case $M^\vee$ is again a vector bundle and $M^{\vee\vee} = M$. If M is a line bundle then we also have $M \otimes M^\vee = \mathcal{O}$.

Example 2.51. Let Y be a closed subscheme of X, with inclusion map $j\colon Y \to X$. Then $I_Y = \{f \in \mathcal{O}_X \mid f(y) = 0 \text{ for all points } y \in Y\}$ is an ideal in $\mathcal{O}_X$ and $\mathcal{O}_Y = \mathcal{O}_X / I_Y$. We define $j_* \mathcal{O}$ to be the sheaf over X corresponding to the $\mathcal{O}_X$-module $\mathcal{O}_Y$. More explicitly, we have

$$(j_* \mathcal{O})_x = \mathcal{O}_x / (f(x) \mid f \in J_Y \subseteq \mathcal{O}_X).$$

We also let $\mathcal{I}_Y$ be the sheaf associated to the $\mathcal{O}_X$-module I_Y, so that $(\mathcal{I}_Y)_x = \mathcal{O}_x \otimes_{\mathcal{O}_X} I_Y$ for all points x of X. Note that the sequence $\mathcal{I}_Y \rightarrowtail \mathcal{O} \twoheadrightarrow j_* \mathcal{O}$ is short exact in $\mathrm{Sheaves}_X$, even though the sequences $(\mathcal{I}_Y)_x \to \mathcal{O}_X \twoheadrightarrow (j_* \mathcal{O})_x$ need only be right exact.

Example 2.52. Given a sheaf N over a functor Y and a map $f\colon X \to Y$, we can define a sheaf $f^* N$ over X by $(f^* N)_x = N_{f(x)}$. The functor $f^*\colon \mathrm{Sheaves}_Y \to \mathrm{Sheaves}_X$ clearly preserves colimits and tensor products. If N is a vector bundle then so is $f^* N$ and we have $f^* \mathrm{Hom}(N, M) = \mathrm{Hom}(f^* N, f^* M)$ for all M. If X and Y are schemes, we find that $\Gamma(X, f^* N) = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \Gamma(Y, N)$.

Example 2.53. If the functor f^* defined above has a right adjoint, we call it f_*. If X and Y are schemes then we know from Proposition 2.47 that there is an essentially unique functor $f_*\colon \mathrm{Sheaves}_X \to \mathrm{Sheaves}_Y$ such that $\Gamma(Y, f_* M) = \Gamma(X, M)$ (where the right hand side is regarded as an $\mathcal{O}_Y$-module using the map $\mathcal{O}_X \to \mathcal{O}_Y$ induced by f). Using the fact that $\Gamma(X, f^* N) = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \Gamma(Y, N)$ one checks that f_* is right adjoint to f^* as required.

Proposition 2.54. If M is a vector bundle over a scheme X, then $\mathbb{A}(M)$ is a scheme.

Proof. Write $N = \mathrm{Mod}_{\mathcal{O}_X}(\Gamma(X, M), \mathcal{O}_X)$. Then for any map $(x\colon \mathcal{O}_X \to R) \in X(R)$ we have $M_x = \mathrm{Mod}_{\mathcal{O}_X}(N, R)$, where R is considered as an $\mathcal{O}_X$-module via x. If we let S be the symmetric algebra $\mathrm{Sym}_{\mathcal{O}_X}[N]$ then we have $M_x = \mathrm{Alg}_{\mathcal{O}_X}(S, R)$. It follows easily that $\mathrm{Rings}(S, R) = \coprod_x \mathrm{Alg}_{\mathcal{O}_X, x}(S, R) = \coprod_x M_x = \mathbb{A}(M)(R)$, so $\mathbb{A}(M)$ is representable as required. $\square$

Definition 2.55. Given a line bundle L over a functor X, we define a functor $\mathbb{A}(L)^{\times}$ over X by

$$\mathbb{A}(L)^{\times}(R) = \coprod_{x \in X(R)} \{ \text{ isomorphisms } u\colon R \to L_x \text{ of } R\text{-modules } \}.$$

If X is a scheme, an argument similar to the one for $\mathbb{A}(M)$ shows that $\mathbb{A}(L)^{\times} = \operatorname{spec}(\bigoplus_{n\in\mathbb{Z}} N^{\otimes n})$, where $N = \operatorname{Mod}_{\mathcal{O}_X}(\Gamma(X,L), \mathcal{O}_X)$ and $N^{\otimes(-n)}$ means the dual of $N^{\otimes n}$. In particular, $\mathbb{A}(L)^{\times}$ is a scheme in this case.

2.7. **Faithful flatness and descent.**

Definition 2.56. Let $f\colon X \to Y$ be a map of schemes, and $f^*\colon \mathcal{X}_Y \to \mathcal{X}_X$ the associated pullback functor. We say that f is *flat* if f^* preserves finite colimits. By Example 2.19, it is equivalent to say that f^* preserves coequalisers. We say that f is *faithfully flat* if f^* preserves finite colimits and reflects isomorphisms.

Remark 2.57. Let $f\colon X \to Y$ be faithfully flat. We claim that f^* reflects finite colimits, so that $f^*Z = \varinjlim_i f^*Z_i$ if and only if $Z = \varinjlim_i Z_i$. More precisely, if $\{Z_i\}$ is a finite diagram in $\mathcal{X}_Y$ and $\{Z_i \to Z\}$ is a cone under the diagram, then $\{f^*Z_i \to f^*Z\}$ is a colimit cone in $\mathcal{X}_X$ if and only if $\{Z_i \to Z\}$ is a colimit cone in $\mathcal{X}_Y$. The "if" part is clear. For the "only if" part, write $Z' = \varinjlim_i Z_i$, so we have a canonical map $u\colon Z' \to Z$. As f is flat we have $f^*Z' = \varinjlim_i f^*Z_i = f^*Z$. As f^* reflects isomorphisms, we see that u is an isomorphism if f^*u is an isomorphism. The claim follows.

Remark 2.58. Classically, a module M over a ring A is said to be flat if the functor $M \otimes_A (-)$ is exact. It is said to be faithfully flat if in addition, whenever $M \otimes_A L = 0$ we have $L = 0$. It turns out that f is (faithfully) flat if and only if the associated ring map $\mathcal{O}_Y \to \mathcal{O}_X$ makes $\mathcal{O}_X$ into a (faithfully) flat module over $\mathcal{O}_Y$. We leave this as an exercise (consider schemes of the form $\operatorname{spec}(\mathcal{O}_X \oplus L)$, where L is an $\mathcal{O}_X$ module and the ring structure is such that $L.L = 0$).

Remark 2.59. The idea of faithful flatness was probably first used in topology by Quillen [21]. He observed that if V is a complex vector bundle over a finite complex Z and F is the bundle of complete flags in V, then the projection map $F_E \to Z_E$ is faithfully flat. This idea was extended and used to great effect in [12].

We next define some other useful properties of maps, which do not seem to fit anywhere else.

Definition 2.60. We say that a map $f\colon X \to Y$ is *very flat* if it makes $\mathcal{O}_X$ into a free module over $\mathcal{O}_Y$. A very flat map is flat, and even faithfully flat provided that $X \neq \emptyset$.

Definition 2.61. We say that a map $f\colon X \to Y$ is *finite* if it makes $\mathcal{O}_X$ into a finitely generated module over $\mathcal{O}_Y$.

Remark 2.62. A flat map $f\colon X \to Y$ is faithfully flat if and only if the resulting map $f_{\mathrm{zar}}\colon X_{\mathrm{zar}} \to Y_{\mathrm{zar}}$ is surjective [18, Theorem 7.3].

Example 2.63. An open inclusion $D(a) \to X$ (where $a \in \mathcal{O}_X$) is always flat. If $a_1, \dots, a_m \in \mathcal{O}_X$ generate the unit ideal then $\coprod_k D(a_k) \to X$ is faithfully flat.

Example 2.64. If D is a divisor on $\mathbb{A}^1$ over Y (as in Example 2.10) then $D \to Y$ is very flat and thus faithfully flat.

Definition 2.65. Given a ring R and an R-algebra S, we write I for the kernel of the multiplication map $S \otimes_R S \to S$, and $\Omega^1_{S/R} = I/I^2$, which is a module over S. Given a map of schemes $X \to Y$, we define $\Omega^1_{X/Y} = \Omega^1_{\mathcal{O}_X/\mathcal{O}_Y}$, which we think of as a sheaf over X. We say that X is *smooth* over Y of relative dimension n if the map $X \to Y$ is flat and $\Omega^1_{X/Y}$ is a vector bundle of rank n over X (we allow the case $n = \infty$). In that case, we write $\Omega^k_{X/Y}$ for the k'th exterior power of $\Omega^1_{X/Y}$ over $\mathcal{O}_X$, which is a vector bundle over X of rank $\left(\begin{array}{c} n \\ k \end{array} \right)$.

Remark 2.66. If X and Y are reduced affine algebraic varieties over $\mathbb{C}$, and X is smooth over Y then the preimage of each point $y \in Y$ is a smooth variety of dimension independent of y. The converse is probably not true but at least that is roughly the right idea. It has nothing to do with the question of whether the map $X \to Y$ is a smooth map of manifolds. The latter only makes sense if X and Y are both smooth varieties (in other words, smooth over $\mathrm{spec}(\mathbb{C})$), and in that case it holds automatically for any algebraic map $X \to Y$.

The following two propositions summarise the basic properties of (faithfully) flat maps.

Proposition 2.67. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps of schemes. Then:

(a) If f and g are flat then gf is flat.
(b) If f and g are faithfully flat then gf is faithfully flat.
(c) If f is faithfully flat and gf is flat then g is flat.
(d) If f and gf are faithfully flat then g is faithfully flat.

Proof. All this follows easily from the definitions. □

Proposition 2.68. Suppose we have a pullback diagram of schemes

$$\begin{tikzcd} W \arrow[r, "r"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Y \arrow[r, "s"'] & Z. \end{tikzcd}$$

Then:

(a) If s is flat then r is flat.
(b) If s is faithfully flat then r is faithfully flat.
(c) If g is faithfully flat and r is flat then s is flat.
(d) If g and r are faithfully flat so s is faithfully flat.

Proof. Consider the functor $f_* \colon \mathcal{X}_W \to \mathcal{X}_Y$, which sends a scheme $U \xrightarrow{u} W$ over W to the scheme $U \xrightarrow{fu} Y$ over Y. Colimits in $\mathcal{X}_W$ are constructed by forming the colimit in $\mathcal{X}$ and equipping it with the obvious map to W. This means that f_* preserves and reflects colimits, as does g_*. For any scheme V over X, we have $W \times_X V = (Y \times_Z X) \times_X V = Y \times_Z V$, or in other words $f_* r^* V = s^* g_* V$ in $\mathcal{X}_Y$. It follows that if s^* preserves or reflects finite colimits then so does r^*, which gives (a) and (b).

For part (c), suppose that g is faithfully flat and r is flat. This implies that $sf = gr$ is flat. Also, part (b) says that any pullback of a faithfully flat map is faithfully flat, and f is a pullback of g so f is faithfully flat. As sf is flat, part (c) of the previous proposition tells us that s is flat, as required. A similar argument proves (d). □

Proposition 2.69. Let $f \colon X \to Y$ be a faithfully flat map, and let $\{V_i\}$ be a finite diagram in $\mathcal{X}_Y$. If $\{f^* V_i\}$ has a strong colimit in $\mathcal{X}_X$, then $\{V_i\}$ has a strong colimit in $\mathcal{X}_X$. In other words, f^* reflects strong finite colimits.

Proof. Write $V = \varinjlim_i V_i$. Given a map $g \colon X' \to X$, we need to show that $g^* V = \varinjlim_i g^* V_i$. To see this, form the pullback square

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

We know from Proposition 2.68 that f' is faithfully flat. Because f is flat, we have $f^* V = \varinjlim_i f^* V_i$. By hypothesis, this colimit is strong, so $(g')^* f^* V = \varinjlim_i (g')^* f^* V_i$. As $gf' = fg'$, we have $(f')^* g^* V = \varinjlim_i (f')^* g^* V_i$. As f' is faithfully flat, the functor $(f')^*$ reflects colimits, so $g^* V = \varinjlim_i g^* V_i$ as required. □

Proposition 2.70. If $f\colon X \to Y$ is faithfully flat and $Y \to Z$ is arbitrary then the diagram

$$X \times_Y X \rightrightarrows X \xrightarrow{f} Y$$

is a strong coequaliser in $\mathcal{X}_Z$.

Proof. As $f^*\colon \mathcal{X}_Y \to \mathcal{X}_X$ reflects strong coequalisers, it is enough to show that the above diagram becomes a strong coequaliser after applying f^*. Explicitly, we need to show that the following is a strong coequaliser:

$$X \times_Y X \times_Y X \underset{d_1}{\overset{d_0}{\rightrightarrows}} X \times_Y X \xrightarrow{d} X,$$

where

$$\begin{aligned} d_0(a,b,c) &= (b,c) \\ d_1(a,b,c) &= (a,c) \\ d(a,b) &= b. \end{aligned}$$

In fact, one can check that this is a split coequaliser, with splitting given by the maps

$$X \times_Y X \times_Y X \xleftarrow{s} X \times_Y X \xleftarrow{t} X,$$

where

$$\begin{aligned} s(a,b) &= (a,b,b) \\ t(a) &= (a,a). \end{aligned}$$

As split coequalisers are preserved by all functors, they are certainly strong coequalisers. $\square$

Now suppose that $f\colon X \to Y$ is faithfully flat, and that U is a scheme over X. We will need to know when U descends to Y, in other words when there is a scheme V over Y such that $U = V \times_Y X$. Given a point $a \in X(R)$, we regard a as a map $\mathrm{spec}(R) \to X$ and write U_a for the pullback of U along this map, which is a scheme over $\mathrm{spec}(R)$.

Definition 2.71. Let $f\colon X \to Y$ be a map of schemes, and let U be a scheme over X. A *system of descent data* for U consists of a collection of maps $\theta_{a,b}\colon U_a \to U_b$ of schemes over $\mathrm{spec}(R)$, for any ring R and any pair of points $a, b \in X(R)$ with $f(a) = f(b)$. These maps are required to be natural in (a,b), and to satisfy the cocycle conditions $\theta_{a,a} = 1$ and $\theta_{a,c} = \theta_{b,c} \circ \theta_{a,b}$.

We write $\mathcal{X}_f$ for the category of pairs (U,θ), where U is a scheme over X and θ is a system of descent data.

Remark 2.72. The naturality condition for the maps $\theta_{a,b}$ just means that they give rise to a map $\pi_0^*U \to \pi_1^*U$ of schemes over $X \times_Y X$.

Remark 2.73. Note also that the cocycle conditions imply that $\theta_{a,b} \circ \theta_{b,a} = 1$, so $\theta_{a,b}$ is an isomorphism.

Definition 2.74. If V is a scheme over Y and $f\colon X \to Y$ then there is an obvious system of descent data for $U = f^*V$, in which $\theta_{a,b}$ is the identity map of $U_a = V_{f(a)} = V_{f(b)} = U_b$. We can thus consider f^* as a functor $\mathcal{X}_Y \to \mathcal{X}_f$. We say that a system of descent data θ on U is *effective* if (U,θ) is equivalent to an object in the image of f^*. It is equivalent to say that there is a scheme V over Y and an isomorphism $\phi\colon U \simeq f^*V$ such that

$$\theta_{a,b} = (U_a \xrightarrow{\phi} V_{f(a)} = V_{f(b)} \xrightarrow{\phi^{-1}} U_b)$$

for all (a,b).

Definition 2.75. Given a map $f\colon X \to Y$, a scheme $U \xrightarrow{g} X$ over X, and a system of descent data θ for U, we define $U \xrightarrow{q} QU$ to be the coequaliser of the maps $d_0, d_1\colon U \times_Y X \to U$ defined by

$$\begin{aligned} d_0(u,a) &= u \\ d_1(u,a) &= \theta_{g(u),a}(u). \end{aligned}$$

We note that d_0 and d_1 have a common splitting $s\colon u \mapsto (u, g(u))$, so we have a reflexive coequaliser. We also note that there is a unique map $r\colon QU \to Y$ such that $rq = fg$, so we can think of QU as a scheme over Y.

Proposition 2.76 (Faithfully flat descent)**.** If $f\colon X \to Y$ is faithfully flat, then the functor $f^*\colon \mathcal{X}_Y \to \mathcal{X}_f$ is an equivalence, with inverse given by Q. Moreover, the coequaliser in $\mathcal{X}_Y$ that defines QU is a strong coequaliser.

Proof. Firstly, it is entirely formal to check that Q is left adjoint to f^*. Next, we claim that $Qf^* = 1$, or in other words that the projection map $f^*V = V \times_Y X \to V$ is a coequaliser of the maps $d_0, d_1\colon V \times_Y X \times_Y X \to V \times_Y X$. Explicitly, we need to show that $(v,a) \mapsto v$ is the coequaliser of $(v,b,a) \mapsto (v,b)$ and $(v,b,a) \mapsto (v,a)$. This is just the same as Proposition 2.70. Thus $Qf^* = 1$ as claimed.

We now show that $f^*QU = U$. As f^* preserves coequalisers, it will be enough to show that the projection $f^*U = U \times_Y X \to U$ is the coequaliser of the fork $U \times_Y X \times_Y X \underset{f^*d_1}{\overset{f^*d_0}{\rightrightarrows}} U \times_Y X$. More explicitly, we need to show that the map $(u,a) \mapsto u$ is the coequaliser of the maps $(u,a,b) \mapsto (u,b)$ and $(u,a,b) \mapsto (\theta_{g(u),a}(u), b)$. In fact, it is a split coequaliser, with splitting given by the maps $u \mapsto (u, g(u))$ and $(u,a) \mapsto (u,a,a)$. Thus, $f^*Q = 1$ as claimed. We also see that the coequaliser defining QU becomes split and thus strong after applying f^*. It follows from Proposition 2.69 that it was a strong coequaliser in the first place. $\square$

Corollary 2.77. If $f\colon X \to Y$ is faithfully flat, then the functor $f^*\colon \mathcal{X}_Y \to \mathcal{X}_X$ is faithful. $\square$

We also have a similar result for sheaves.

Definition 2.78. Let $f\colon X \to Y$ be a map of schemes, and let M be a sheaf over X. A *system of decent data* for M consists of a collection of maps $\theta_{a,b}\colon M_a \to M_b$ of R-modules, for every ring R and every pair of points $a, b \in X(R)$ with $f(a) = f(b)$. These are supposed to be natural in (a,b) and to satisfy the conditions $\theta_{a,a} = 1$ and $\theta_{b,c} \circ \theta_{a,b} = \theta_{a,c}$. We write Sheaves_f for the category of sheaves over X equipped with descent data. The pullback functor f^* can be regarded as a functor from Sheaves_Y to Sheaves_f.

Proposition 2.79. If f is faithfully flat, then the functor $f^*\colon \text{Sheaves}_X \to \text{Sheaves}_f$ is an equivalence of categories.

The proof is similar to that of Proposition 2.76, and is omitted.

We shall say that a statement holds *locally in the flat topology* or *fpqc locally* if it is true after pulling back along a faithfully flat map. (fpqc stands for fidèlement plat et quasi-compact; the compactness condition is automatic for affine schemes). Suppose that a certain statement S is true whenever it holds fpqc-locally. We then say that S is an fpqc-local statement.

Remark 2.80. Let X be a topological space. We say that a statement S holds locally on X if and only if there is an open covering $X = \bigcup_i U_i$ such that S holds on each U_i. Write $Y = \coprod_i U_i$, so $Y \to X$ is a coproduct of open inclusions and is surjective. We could call such a map an "disjoint covering map". We would then say that S holds locally if and only if it holds after pulling back along a disjoint covering map. One can get many analogous concepts varying the class of maps in question. For example, we could use covering maps in the ordinary sense. In the category of compact smooth manifolds, we could use submersions. This is the conceptual framework in which the above definition is supposed to fit.

Example 2.81. Suppose that N is a sheaf on Y which vanishes fpqc-locally. This means that there is a faithfully flat map $f\colon X \to Y$ such that $\Gamma(X, f^*N) = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \Gamma(Y,N) = 0$. By the classical definition of faithful flatness, this implies that $N = 0$. In other words, the vanishing of N is an fpqc-local condition.

Example 2.82. Let N be a sheaf over Y, and let n be an element of $\Gamma(Y,N)$ that vanishes fpqc-locally. This means that there is a faithfully flat map $f\colon X \to Y$ such that the image of n in $\Gamma(X, f^*N) = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \Gamma(Y,N)$ is zero. Let g be the projection $X \times_Y X \to Y$. One can show that the diagram

$$\Gamma(Y,N) \xrightarrow{f^*} \Gamma(X, f^*N) \rightrightarrows \Gamma(X \times_Y X, g^*N)$$

is an equaliser. Indeed, it becomes split after tensoring with $\mathcal{O}_X$ over $\mathcal{O}_Y$, and that functor reflects equalisers by the classical definition of faithful flatness. In particular, the map marked f^* is injective, so $n = 0$. Thus, the vanishing of n is an fpqc-local condition.

Example 2.83. Suppose that M is a vector bundle of rank r over a scheme X. We claim that M is fpqc-locally free of rank r, in other words that there is a faithfully flat map $f\colon W \to X$ such that $f^*M \simeq \mathcal{O}^r$. To prove this, choose a matrix $A \in M_n(\mathcal{O}_X)$ such that $\Gamma(X,M) = A.\mathcal{O}_X^n$. If R is a ring and $x \in X(R)$ then

$A(x) \in M_n(R)$ and $M_x = A(x).R^n$. Let $W(R)$ be the set of triples (x, P, Q) such that $x \in X(R)$ and P and Q are matrices over R of shape $r \times n$ and $n \times r$ such that $\det(PA(x)Q)$ is invertible. This is easily seen to be a scheme over X. In fact, it is an open subscheme of the scheme of all triples (x, P, Q), which can be identified with $\mathbb{A}^{2nr} \times X$. It follows that W is flat over X. Moreover, if R is a field then elementary linear algebra tells us that the map $W(R) \to X(R)$ is surjective, so that W is faithfully flat over R. If (x, P, Q) is a point of W then $A(x)Q\colon R^r \to M_x$ is a split monomorphism. By comparison of ranks, it is an isomorphism. It follows that M becomes free after pulling back to W.

Example 2.84. Proposition 2.68 tells us that flatness and faithful flatness are themselves fpqc-local properties.

Example 2.85. Let M be a vector bundle of rank r over a scheme X, as in Example 2.83. Let $\mathrm{Bases}(M)$ be the functor of pairs (x, B) where x is a point of X and $B\colon \mathcal{O}_x^r \to M_x$ is an isomorphism. Note that $\mathrm{Bases}(M)(R)$ can be identified with the set of tuples $(x, b_1, \ldots, b_r, \beta_1, \ldots, \beta_r)$ such that $b_i \in M_x$ and $\beta_j \in M_x^\vee$ and $\beta_j(b_i) = \delta_{ij}$, so $\mathrm{Bases}(M)$ is a closed subscheme of $\mathbb{A}(M)_X^r \times_X \mathbb{A}(M^\vee)_X^r$.

It is clear that M becomes free after pulling back along the projection

$$f\colon \mathrm{Bases}(M) \to X.$$

If $M = \mathcal{O}^r$ is free, then $\mathrm{Bases}(M)$ is just the scheme $\mathrm{GL}_r \times X$, where GL_r is the scheme of invertible $r \times r$ matrices. It's not hard to see that $\mathcal{O}_{\mathrm{GL}_r} = \mathbb{Z}[x_{i,j} \mid 0 \leq i, j < r][\det(x_{ij})^{-1}]$ is torsion-free, and clearly $\mathrm{GL}_r(k) \neq \emptyset$ for all fields k, and one can conclude that the map $\mathrm{GL}_r \to 1 = \mathrm{spec}(\mathbb{Z})$ is faithfully flat. It follows that $\mathrm{Bases}(M)$ is faithfully flat over X when M is free. Even if M is not free, we see from Example 2.83 that it is fpqc-locally free, so the map $\mathrm{Bases}(M) \to X$ is fpqc-locally faithfully flat. As remarked in Example 2.84, faithful flatness is itself a local condition, so $\mathrm{Bases}(M) \to X$ is faithfully flat.

Example 2.86. Any monic polynomial $f \in R[x]$ can be factored as a product of linear terms, locally in the flat topology. Indeed, suppose

$$f = \sum_0^m (-1)^{m-k} a_{m-k} x^k$$

with $a_0 = 1$. It is well known that $S = \mathbb{Z}[x_1, \ldots x_m]$ is free of rank $m!$ over $T = S^{\Sigma_m} = \mathbb{Z}[\sigma_1, \ldots \sigma_m]$, where σ_k is the k'th elementary symmetric function in the x's. A basis is given by the monomials $x^\alpha = \prod x_k^{\alpha_k}$ for which $\alpha_k < k$. We can map T to R by sending σ_k to a_k, and then observe that $U = S \otimes_T R$ is free and thus faithfully flat over R. Clearly $f(x) = \prod_k (x - x_k)$ in $U[x]$, as required.

We conclude this section with some remarks about open mappings. We have to make a slightly twisted definition, because in our affine context we do not have enough open subschemes. Suppose that $f\colon X \to Y$ is a map of spaces, and that $W \subseteq X$ is closed. We can then define $W' = \{y \in Y \mid f^{-1}y \subseteq W\} = f(W^c)^c$. Clearly f is open iff (W closed implies W' closed). We will define openness for maps of schemes by analogy with this.

Definition 2.87. Let $f\colon X \to Y$ be a map of schemes. For any closed subscheme $W \subseteq X$, we define a subfunctor W' of Y by

$$W'(R) = \{y \in Y(R) \mid W_y = X_y\}.$$

We say that f is *open* if for every W, the corresponding subfunctor $W' \subseteq Y$ is actually a closed subscheme.

Proposition 2.88. A very flat map is open.

Proof. Let $f\colon X \to Y$ be very flat. Write $A = \mathcal{O}_X$ and $B = \mathcal{O}_Y$, and choose a basis $A = B\{e_\alpha\}$. Suppose that $W = V(I)$ is a closed subscheme of X. Let $\{g_\beta\}$ be a system of generators of I, so we can write $g_\beta = \sum_\alpha g_{\beta\alpha} e_\alpha$ for suitable elements $g_{\alpha\beta} \in A$. Consider a point $y \in Y(R)$, corresponding to a map $y^*\colon B \to R$. This will lie in $W'(R)$ iff $R \otimes_B A = R \otimes_B (A/I)$, iff the image of I in $R \otimes_B A = R\{e_\alpha\}$ is zero. This image is generated by the elements $h_\beta = \sum_\alpha y^*(g_{\beta\alpha}) e_\alpha$. Thus, it vanishes iff $y^*(g_{\beta\alpha}) = 0$ for all α and β. This shows that $W' = V(I')$, where $I' = (g_{\beta\alpha})$, so W' is a closed subscheme as required. □

2.8. **Schemes of maps.**

Definition 2.89. Let Z be a functor Rings $\to$ Sets, and let X and Y be functors over Z. For any ring R, we let $\mathrm{Map}_Z(X,Y)(R)$ be the class of pairs (z,u), where $z \in Z(R)$ and $u\colon X_z \to Y_z$ is a map of functors over $\mathrm{spec}(R)$. If this is a set (rather than a proper class) for all R, then we get a functor $\mathrm{Map}_Z(X,Y) \in \mathcal{F}$. This is clearly the case whenever X, Y and Z are all schemes. However, the functor $\mathrm{Map}_Z(X,Y)$ need not itself be a scheme.

When $Z = 1$ is the terminal scheme we will usually write $\mathrm{Map}(X,Y)$ rather than $\mathrm{Map}_1(X,Y)$.

Remark 2.90. It is formal to check that

$$\mathcal{F}_Z(W, \mathrm{Map}_Z(X,Y)) = \mathcal{F}_Z(W \times_Z X, Y) = \mathcal{F}_W(W \times_Z X, W \times_Z Y).$$

In particular, if X, Y, Z and $\mathrm{Map}_Z(X,Y)$ are all schemes then we have

$$\mathcal{X}_Z(W, \mathrm{Map}_Z(X,Y)) = \mathcal{X}_Z(W \times_Z X, Y) = \mathcal{X}_W(W \times_Z X, W \times_Z Y).$$

Example 2.91. It is not hard to see that maps $\mathbb{A}^n \times \mathrm{spec}(R) \to \mathbb{A}^m \times \mathrm{spec}(R)$ over $\mathrm{spec}(R)$ biject with m-tuples of polynomials over R in n variables, so $\mathrm{Map}(\mathbb{A}^n, \mathbb{A}^m)(R) = R[x_1, \ldots, x_n]^m$, which is isomorphic to $\bigoplus_{n\in\mathbb{N}} R$ (naturally in R). This functor is not a representable (it does not preserve infinite products, for example) so $\mathrm{Map}(\mathbb{A}^n, \mathbb{A}^m)$ is not a scheme. It is a formal scheme, however.

Example 2.92. Write $D(n)(R) = \{a \in R \mid a^{n+1} = 0\}$, so

$$D(n) = \mathrm{spec}(\mathbb{Z}[x]/x^{n+1})$$

is a scheme. We find that $\mathrm{Map}(D(n), \mathbb{A}^1)(R) = R[x]/x^{n+1} \simeq \prod_{i=0}^n R$, so that $\mathrm{Map}(D(n), \mathbb{A}^1) \simeq \mathbb{A}^{n+1}$ is a scheme.

Example 2.93. Let E be an even periodic ring spectrum. As $\Omega U(n)$ is a commutative H-space, we see that $E_0(\Omega U(n))$ is a ring, so we can define a scheme $\mathrm{spec}(E_0(\Omega U(n)))$. We will see later that there is a canonical isomorphism

$$\mathrm{spec}(E_0(\Omega U(n))) \simeq \mathrm{Map}_{S_E}((\mathbb{C}P^{n-1})_E, \mathbb{G}_m).$$

We now give a proposition which generalises the last two examples.

Proposition 2.94. Let Z be a scheme and let X and Y be schemes over Z, and suppose that X is finite and very flat over Z. Then $\mathrm{Map}_Z(X,Y)$ is a scheme.

Proof. Let R be a ring, and z a point of $Z(R)$, giving a map $\hat{z}\colon \mathcal{O}_Z \to R$. We need to produce an algebra B over $\mathcal{O}_Z$ such that the maps $B \to R$ of $\mathcal{O}_Z$-algebras biject with maps $X_z \to Y_z$ of schemes over $\mathrm{spec}(R)$, or equivalently with maps $R \otimes_{\mathcal{O}_Z} \mathcal{O}_Y \to R \otimes_{\mathcal{O}_Z} \mathcal{O}_X$ of R-algebras, or equivalently with maps $\mathcal{O}_Y \to R \otimes_{\mathcal{O}_Z} \mathcal{O}_X$ of $\mathcal{O}_Z$-algebras.

Write $\mathcal{O}_X^\vee = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{O}_Z)$ and $A = \mathrm{Sym}_{\mathcal{O}_Z}[\mathcal{O}_X^\vee \otimes_{\mathcal{O}_Z} \mathcal{O}_Y]$. Then

$$\mathrm{Alg}_{\mathcal{O}_Z}(A, R) = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{O}_X^\vee \otimes_{\mathcal{O}_Z} \mathcal{O}_Y, R) = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{O}_Y, R \otimes_{\mathcal{O}_Z} \mathcal{O}_X).$$

A suitable quotient B of A will pick out the algebra maps from $\mathcal{O}_Y$ to $\mathcal{O}_W \otimes_{\mathcal{O}_Z} \mathcal{O}_X$. To be more explicit, let $\{e_1, \ldots, e_n\}$ be a basis for $\mathcal{O}_X$ over $\mathcal{O}_Z$, with $1 = \sum_i d_i e_i$ and $e_i e_j = \sum_k c_{ijk} e_k$. Let $\{\epsilon_i\}$ be the dual basis for $\mathcal{O}_X^\vee$. Then B is A mod the relations

$$\begin{aligned} \epsilon_k \otimes ab &= \sum_{i,j} c_{ijk} (\epsilon_i \otimes a)(\epsilon_j \otimes b) \\ \epsilon_i \otimes 1 &= d_i. \end{aligned}$$

More abstractly, if we write q for the projection $A \to B$ and j for the inclusion $\mathcal{O}_X^\vee \otimes \mathcal{O}_Y \to A$, then B is the largest quotient of A such that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{O}_Y \otimes \mathcal{O}_Y \otimes \mathcal{O}_X^\vee \arrow[r, "\mu_Y \otimes 1"] \arrow[d, "1\otimes\mu_X^\vee"'] & \mathcal{O}_Y \otimes \mathcal{O}_X^\vee \arrow[dddd, "qj"] \\
\mathcal{O}_Y \otimes \mathcal{O}_Y \otimes \mathcal{O}_X^\vee \otimes \mathcal{O}_X^\vee \arrow[d, "\mathrm{twist}"'] & \\
\mathcal{O}_Y \otimes \mathcal{O}_X^\vee \otimes \mathcal{O}_Y \otimes \mathcal{O}_X^\vee \arrow[d, "qj\otimes qj"'] & \\
B \otimes B \arrow[r, "\mu_B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}_X^\vee \arrow[r, "\eta^\vee"] \arrow[d, "1\otimes\eta"'] & \mathcal{O}_Z \arrow[d, "\eta"] \\
\mathcal{O}_X^\vee \otimes \mathcal{O}_Y \arrow[r, "qj"'] & B
\end{tikzcd}$$

We conclude that $\mathrm{spec}(B)$ has the defining property of $\mathrm{Map}_Z(X, Y)$. $\square$

2.9. **Gradings.** In this section, we show that graded rings are essentially the same as schemes with an action of the multiplicative group $\mathbb{G}_m$.

Definition 2.95. A *grading* of a ring R is a system of additive subgroups $R_k \leq R$ for $k \in \mathbb{Z}$ such that $R = \bigoplus_k R_k$ and $1 \in R_0$ and $R_j R_k \subseteq R_{j+k}$ for all j, k. We say that a map $g\colon R \to S$ between graded rings is *homogeneous* if $g(R_k) \subseteq S_k$ for all k.

Proposition 2.96. Let X be a scheme. Then gradings of $\mathcal{O}_X$ biject with actions of the group scheme $\mathbb{G}_m$ on X. Given such actions on X and Y, a map $f\colon X \to Y$ is $\mathbb{G}_m$-equivariant if and only if the corresponding map $\mathcal{O}_Y \to \mathcal{O}_X$ is homogeneous.

Proof. Given an action of $\mathbb{G}_m$ on X, we define $(\mathcal{O}_X)_k$ to be the set of maps $f\colon X \to \mathbb{A}^1$ such that $f(u.x) = u^k f(x)$ for all rings R and points $u \in \mathbb{G}_m(R)$, $x \in X(R)$. It is clear that $1 \in (\mathcal{O}_X)_0$ and that $(\mathcal{O}_X)_j(\mathcal{O}_X)_k \subseteq (\mathcal{O}_X)_{j+k}$. We need to check that $\mathcal{O}_X = \bigoplus_k (\mathcal{O}_X)_k$. For this, we consider the map $\alpha^*\colon \mathcal{O}_X \to \mathcal{O}_{\mathbb{G}_m \times X} = \mathcal{O}_X[u^{\pm 1}]$. If $\alpha^*(f) = \sum_k u^k f_k$ (so $f_k = 0$ for almost all k), then we find that the f_k are the unique functions $X \to \mathbb{A}^1$ such that $f(u.x) = \sum_k u^k f_k(x)$ for all u and x. By taking $u = 1$, we see that $f = \sum_k f_k$. We also find that

$$\sum_k u^k v^k f_k(x) = f((uv).x) = f(u.(v.x)) = \sum_{j,k} u^j v^k f_{kj}(x).$$

By working in the universal case $R = \mathcal{O}_X[u^{\pm 1}, v^{\pm 1}]$ and comparing coefficients, we see that $f_{kj} = \delta_{jk} f_k$ so that $f_k \in (\mathcal{O}_X)_k$. It follows easily that the addition map $\bigoplus_k (\mathcal{O}_X)_k \to \mathcal{O}_X$ is an isomorphism, with inverse $f \mapsto (f_k)_{k\in\mathbb{Z}}$. Thus, we have a grading of $\mathcal{O}_X$.

Conversely, suppose we have a grading $(\mathcal{O}_X)_*$. We can then write any element $f \in \mathcal{O}_X$ as $\sum_k f_k$ with $f_k \in (\mathcal{O}_X)_k$ and $f_k = 0$ for almost all k. We define $\alpha^*(f) = \sum_k u^k f_k$, and check that this gives a ring map $\mathcal{O}_X \to \mathcal{O}_X[u^{\pm 1}]$. One can also check that $\alpha = \mathrm{spec}(\alpha^*)\colon \mathbb{G}_m \times X \to X$ is an action, and that this construction is inverse to the previous one. $\square$

Example 2.97. Recall the scheme FGL from Example 2.6. We can let $\mathbb{G}_m$ act on FGL by $(u.F)(x, y) = uF(x/u, y/u)$; this gives a grading of $\mathcal{O}_{\mathrm{FGL}}$. Write $F(x, y) = \sum_{i,j} a_{ij}(F) x^i y^j$, and recall that the elements a_{ij} generate $\mathcal{O}_{\mathrm{FGL}}$. It is clear that $(u.F)(x, y) = \sum_{i,j} u^{1-i-j} a_{ij}(F) x^i y^j$, so that $a_{ij}(u.F) = u^{1-i-j} a_{ij}(F)$, so a_{ij} is homogeneous of degree $1 - i - j$. This is of course the same as the grading coming from the isomorphisms $\mathcal{O}_{\mathrm{FGL}} = \pi_0 MP = \pi_* MU$, except that all degrees are halved.

3. Non-affine schemes

Let $\mathcal{E}$ be the category of (not necessarily affine) schemes in the classical sense, as discussed in [9] for example. In this section we show that $\mathcal{E}$ can be embedded as a full subcategory of $\mathcal{F}$, containing our category $\mathcal{X}$ of affine schemes. We show that our definition of sheaves over functors gives the right answer for functors coming from non-affine schemes, and we investigate the schemes $\mathbb{P}^n$ from this point of view. This theory is useful in topology when one wants to study elliptic cohomology, for example [11]. The results here are surely known to algebraic geometers, but I do not know a reference.

Given a ring A, we write $\mathrm{zar}(A)$ for the Zariski spectrum of A, considered as an object of $\mathcal{E}$ in the usual way. The results of this section will allow us to identify $\mathrm{zar}(A)$ with $\mathrm{spec}(A)$. Of course, in most treatments, $\mathrm{spec}(A)$ is defined to be what we call $\mathrm{zar}(A)$.

Definition 3.1. Given a scheme $X \in \mathcal{E}$, we define a functor $FX \in \mathcal{F}$ by

$$FX(R) = \mathcal{E}(\mathrm{zar}(R), X).$$

It is well-known that

$$\mathcal{E}(\mathrm{zar}(R), \mathrm{zar}(A)) = \mathrm{Rings}(A, R),$$

so that $F(\mathrm{zar}(A)) = \mathrm{spec}(A)$.

Proposition 3.2. The functor $F\colon \mathcal{E} \to \mathcal{F}$ is full and faithful.

Proof. Let $X, Y \in \mathcal{E}$ be schemes; we need to show that the map $F\colon \mathcal{E}(X,Y) \to \mathcal{F}(FX, FY)$ is an isomorphism. First suppose that X is affine, say $X = \mathrm{zar}(A)$. Then the Yoneda lemma tells us that

$$\mathcal{F}(FX, FY) = \mathcal{F}(\mathrm{spec}(A), FY) = FY(A) = \mathcal{E}(\mathrm{zar}(A), Y) = \mathcal{E}(X, Y)$$

as required.

Now let X be an arbitrary scheme. We can cover X by open affine subschemes X_i, and for each i and j we can cover $X_i \cap X_j$ by open affine subschemes X_{ijk}. This gives rise to a diagram as follows.

$$\begin{tikzcd}
\mathcal{E}(X,Y) \arrow[r, tail] \arrow[d, "F"'] & \prod_i \mathcal{E}(X_i, Y) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "F"', "\simeq"] & \prod_{ijk} \mathcal{E}(X_{ijk}, Y) \arrow[d, "F"', "\simeq"] \\
\mathcal{F}(FX, FY) \arrow[r, "J"'] & \prod_i \mathcal{F}(FX_i, FY) \arrow[r, shift left] \arrow[r, shift right] & \prod_{ijk} \mathcal{F}(FX_{ijk}, FY).
\end{tikzcd}$$

Standard facts about the category $\mathcal{E}$ show that the top line is an equaliser. The affine case of our proposition shows that the middle and right-hand vertical arrows are isomorphisms. If we can prove that the map J is injective, then a diagram chase will show that the left-hand vertical map is an isomorphism, as required.

Suppose we have two maps $f, g\colon FX \to FY$ and that $Jf = Jg$, or in other words $f|_{FX_i} = g|_{FX_i}$ for all i. We need to show that $f = g$. Consider a ring R and a point $x \in FX(R)$, or equivalently a map $W = \mathrm{zar}(R) \xrightarrow{x} X$. We need to show that $f(x) = g(x)$ as maps from W to Y. We can cover W by open affine subschemes W_s such that $x\colon W_s \to X$ factors through X_i for some i. As $f|_{FX_i} = g|_{FX_i}$, we see that $f(x) \circ j_s = g(x) \circ j_s$, where $j_s\colon W_s \to W$ is the inclusion. As the schemes W_s cover W, we see that $f(x) = g(x)$ as required. □

Proposition 3.3. Let $X \in \mathcal{E}$ be a scheme. Then the category of quasicoherent sheaves of $\mathcal{O}$-modules over X is equivalent to the category of sheaves over FX.

Proof. Let M be a quasicoherent sheaf of $\mathcal{O}$-modules over X. Consider a ring R and a point $x \in FX(R)$, corresponding to a map $x\colon \mathrm{zar}(R) \to X$. We can pull M back along this map to get a quasicoherent sheaf of $\mathcal{O}$-modules over $\mathrm{zar}(R)$, whose global sections form a module $G(M)_x = \Gamma(\mathrm{zar}(R), x^*M)$ over R. It is not hard to see that this construction gives a sheaf GM over the functor FX. If X is affine then we know from Proposition 2.47 that sheaves over FX are the same as modules over $\mathcal{O}_X$, and it is classical that these are the same as quasicoherent sheaves of $\mathcal{O}$-modules over X, so the functor G is an equivalence in this case.

Now let $X \in \mathcal{E}$ be an arbitrary scheme, and let N be a sheaf over FX. We can cover X by open affine subschemes X_i, and we can cover $X_i \cap X_j$ by open affine subschemes X_{ijk}. By the affine case of the proposition, we can identify $N_i = N|_{FX_i}$ with a quasicoherent sheaf M_i of $\mathcal{O}$-modules over X_i. The obvious isomorphism $N_i|_{FX_{ijk}} = N_j|_{FX_{ijk}}$ gives an isomorphism $M_i|_{X_{ijk}} = M_j|_{X_{ijk}}$ (because our functor G is an equivalence for the affine scheme X_{ijk}). One checks that these isomorphisms satisfy the relevant cocycle condition, so we can glue together the sheaves M_i to get a quasicoherent sheaf M over X. One can also check that this construction is inverse to our previous one, which implies that G is an equivalence of categories. □

From now on we will not usually distinguish between X and FX.

We next examine how projective spaces fit into our framework. Let $\mathbb{P}^n$ be the scheme obtained by gluing together $n+1$ copies of $\mathbb{A}^n$ in the usual way. In more detail, we consider the scheme $\mathbb{A}^{n+1} = \prod_{i=0}^n \mathbb{A}^1$, and let U_i be the closed subscheme where $x_i = 1$, so $U_i \simeq \mathbb{A}^n$. If $j \neq i$ we let V_{ij} be the open subscheme of U_i where x_j is invertible. We define $\phi_{ij} \colon V_{ij} \to V_{ji}$ by

$$\phi_{ij}(x_0, \dots, x_n) = (x_0, \dots, x_n)/x_j.$$

We use these maps to glue the U_i's together to get a scheme $\mathbb{P}^n$.

We define a sheaf L_i over U_i by $L_{i,\underline{a}} = R\underline{a} \leq R^{n+1}$ for $\underline{a} \in U_i(R)$. Note that if $\pi_i \colon R^{n+1} \to R$ is the i'th projection then π_i induces an isomorphism $L_{i,\underline{a}} \to R$, so $L_{i,\underline{a}}$ is a line bundle over U_i. If $\underline{a} \in V_{ij}(R)$ then it is clear that $L_{i,\underline{a}} = L_{j,\phi_{ij}(\underline{a})}$. It follows that the bundles L_i glue together to give a line bundle L over $\mathbb{P}^n$. From the construction, we see that there is a short exact sequence $L \rightarrowtail \mathcal{O}^{n+1} \twoheadrightarrow V$, in which V is a vector bundle of rank n. We also write $\mathcal{O}(k)$ for the $(-k)$'th tensor power of L, which is again a line bundle over $\mathbb{P}^n$.

Proposition 3.4. For any ring R, we can identify $\mathbb{P}^n(R) = \mathcal{E}(\mathrm{zar}(R), \mathbb{P}^n)$ with the set of submodules $M \leq R^{n+1}$ such that M is a summand and has rank one.

This will be proved after a lemma.

Definition 3.5. Write $Q^n(R)$ for the set of submodules $M \leq R^{n+1}$ such that L is a rank-one projective module and a summand, or equivalently R^{n+1}/M is a projective module of rank n. Given a map $R \to R'$ we have a map $Q^n(R) \to Q^n(R')$ sending M to $R' \otimes_R M$, which makes Q^n into a functor.

We now define a map $\gamma \colon \mathbb{P}^n \to Q^n$, which will turn out to be an isomorphism. Consider a ring R and a point $x \in \mathbb{P}^n(R)$, corresponding to a map $x \colon \mathrm{spec}(R) \to \mathbb{P}^n$. By pulling back the sequence $L \rightarrowtail \mathcal{O}^{n+1} \twoheadrightarrow V$ and identifying sheaves over $\mathrm{spec}(R)$ with R-modules, we get a short exact sequence $x^*L \rightarrowtail R^{n+1} \twoheadrightarrow x^*V$. Here x^*L and x^*V are projective, with ranks one and n respectively, so $x^*L \in Q^n(R)$. We define $\gamma(x) = x^*L$.

Lemma 3.6. Let W be an affine scheme, and let $W_1, \dots, W_m$ be a finite cover of W by basic affine open subschemes $W_i = D(a_i)$. Then there is an equaliser diagram

$$\mathcal{F}(W, Q^n) \to \prod_i \mathcal{F}(W_i, Q^n) \rightrightarrows \prod_{ij} \mathcal{F}(W_i \cap W_j, Q^n).$$

Proof. Write $W' = \coprod_i W_i$ and $W'' = \coprod_{ij} W_i \cap W_j$, so that the evident map $f \colon W' \to W$ is faithfully flat and $W'' = W' \times_W W'$. We can thus use Proposition 2.79 to identify $\mathrm{Sheaves}_W$ with the category $\mathrm{Sheaves}_f$ of sheaves on W' equipped with descent data. It follows that for any sheaf F on W, the subsheaves of F biject with subsheaves $K \leq f^*F$ that are preserved by the descent data for f^*F. This condition is equivalent to the condition $\pi_0^*K = \pi_1^*K \leq (f\pi_0)^*F = (f\pi_1)^*F$. Now take $F = \mathcal{O}^{n+1}$, and the lemma follows easily. $\square$

Proof of Proposition 3.4. Suppose we have two points $x \in U_i(R) \subset \mathbb{P}^n(R)$ and $y \in U_j(R) \subset \mathbb{P}^n(R)$, and that $\gamma(x) = \gamma(y)$. It then follows easily from the definitions that $x = y$.

Now suppose we have two points $x, y \in \mathbb{P}^n(R)$ such that $\gamma(x) = \gamma(y)$. We write $W = \mathrm{spec}(R)$, so $x \colon W \to \mathbb{P}^n$. We can cover W by basic affine open subsets $W_1, \dots, W_m$ with the property that each $x(W_k)$ is contained in some U_i, and each $y(W_k)$ is contained in some U_j. This implies (by the previous paragraph) that $x = y$ as maps $W_k \to \mathbb{P}^n$. We can now deduce from Lemma 3.6 that $x = y$. Thus, $\gamma \colon \mathbb{P}^n(R) \to Q^n(R)$ is always injective.

Now consider a point $M \in Q^n(R)$, so M is a sheaf over $W = \mathrm{spec}(R)$. We claim that we can cover W by basic open subschemes V such that $M|_V$ lies in the image of $\gamma \colon \mathcal{F}(V, \mathbb{P}^n) \to \mathcal{F}(V, Q^n)$. Indeed, as M is projective, we can start by covering W with basic open subschemes on which M is free. It is easy to see that over such a subscheme, there exist maps $\mathcal{O} \xrightarrow{u} \mathcal{O}^{n+1} \xrightarrow{v} \mathcal{O}$ such that the image of u is M and $vu = 1$. If we write u and v in terms of bases in the obvious way then $\sum_i u_i v_i = 1$, so the elements u_i generate the unit ideal, so the basic open subschemes $D(u_i)$ form a covering. On $D(u_i)$ we can define $x = (u_0, \dots, u_n)/u_i \in U_i$, and it is clear that $\gamma(x) = M$.

We can thus choose a basic open covering $W = W_1 \cup \ldots \cup W_m$ and maps $x_k \colon W_k \to \mathbb{P}^n$ such that $\gamma(x_k) = M|_{W_k}$. Let x_{jk} be the restriction of x_j to $W_{jk} = W_j \cap W_k$. We then have $\gamma(x_{jk}) = M|_{W_{jk}} = \gamma(x_{kj})$

and γ is injective so $x_{jk} = x_{kj}$. We also have a diagram

$$\begin{tikzcd}
\mathcal{F}(W,\mathbb{P}^n) \arrow[r, tail] \arrow[d, "\gamma"'] & \prod_i \mathcal{F}(W_i,\mathbb{P}^n) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\gamma"'] & \prod_{ij} \mathcal{F}(W_{ij},\mathbb{P}^n) \arrow[d, "\gamma"] \\
\mathcal{F}(W,Q^n) \arrow[r] & \prod_i \mathcal{F}(W_i,Q^n) \arrow[r, shift left] \arrow[r, shift right] & \prod_{ij} \mathcal{F}(W_{ij},Q^n).
\end{tikzcd}$$

The top row is unchanged if we replace $\mathcal{F}$ by $\mathcal{E}$, and this makes it clear that it is an equaliser diagram. The bottom row is an equaliser diagram by Lemma 3.6. We have already seen that the vertical maps are injective. The elements x_i give an element of $\prod_i \mathcal{F}(W_i,\mathbb{P}^n)$, whose image in $\prod_i \mathcal{F}(W_i,Q^n)$ is the same as that of $M \in \mathcal{F}(W,Q^n)$. We conclude by diagram chasing that there is an element $x \in \mathcal{F}(W,\mathbb{P}^n)$ such that $\gamma(x) = M$. Thus γ is also surjective, as required. □

Definition 3.7. Suppose that we have elements $a_0,\dots,a_n \in R$, which generate the unit ideal, say $\sum_i b_i a_i = 1$. Let M be the submodule of R^{n+1} generated by $\underline{a} = (a_0,\dots,a_n)$. The elements b_j define a map $R^{n+1} \to R$ which carries L isomorphically to R. It follows that $M \in Q^n(R)$; the submodules M that occur in this way are precisely those that are free over R. We write $[a_0 : \dots : a_n]$ for the corresponding point of $\mathbb{P}^n(R)$. Most of the time, when working with points of $\mathbb{P}^n$, we can assume that they have this form, and handle the general case by localising.

We finish this section with a useful lemma.

Lemma 3.8. We have $[a_0 : \dots : a_n] = [a'_0 : \dots : a'_n]$ if and only if there is a unit $u \in R^\times$ such that $ua'_j = a_j$ for all j, if and only if $a_i a'_j = a_j a'_i$ for all i and j.

Proof. The first equivalence is clear if we think in terms of $Q^n(R)$. For the second, suppose that $ua'_j = a_j$ for all j. Then $a_i a'_j = u^{-1} a_i a_j = a'_i a_j$ as required. Conversely, suppose that $a_i a'_j = a_j a'_i$ for all i and j. We can choose sequences $b_0,\dots,b_n$ and $b'_0,\dots,b'_n$ such that $\sum_i a_i b_i = 1$ and $\sum_i a'_i b'_i = 1$. Now define $u = \sum_i a_i b'_i$ and $v = \sum_j a'_j b'_j$. Then

$$ua'_j = \sum_i b'_i a_i a'_j = \sum_i b'_i a'_i a_j = a_j.$$

Moreover, we have

$$u \sum_j b_j a'_j = \sum_j b_j a_j = 1,$$

so u is a unit as required. □

4. Formal schemes

In this section we define formal schemes, and set up an extensive categorical apparatus for dealing with them, and generalise our results for schemes to formal schemes as far as possible. We define the subcategory of solid formal schemes, which is convenient for some purposes. We also define functors from various categories of coalgebras to the category of formal schemes, which are useful technical tools. Finally, we study the question of when $\mathrm{Map}_Z(X,Y)$ is a formal scheme.

Definition 4.1. A *formal scheme* is a functor $X\colon \text{Rings} \to \text{Sets}$ that is a small filtered colimit of schemes. More precisely, there must be a small filtered category $\mathcal{I}$ and a functor $i \mapsto X_i$ from $\mathcal{I}$ to $\mathcal{X} \subseteq \mathcal{F} = [\text{Rings}, \text{Sets}]$ such that $X = \varinjlim_i X_i$ in $\mathcal{F}$, or equivalently $X(R) = \varinjlim_i X_i(R)$ for all R. We call such a diagram $\{X_i\}$ a *presentation* of X. We write $\widehat{\mathcal{X}}$ for the category of formal schemes.

Example 4.2. The most basic example is the functor $\widehat{\mathbb{A}}^1$ defined by $\widehat{\mathbb{A}}(R) = \text{Nil}(R)$. This is clearly the colimit over N of the functors $D(N) = \text{spec}(\mathbb{Z}[x]/x^{N+1})$. We also define $\widehat{\mathbb{A}}^n(R) = \text{Nil}(R)^n$.

Example 4.3. More generally, given a scheme X and a closed subscheme $Y = V(I)$, we define a formal scheme $X^\wedge_Y = \varinjlim_N V(I^N)$.

Example 4.4. For a common example not of the above type, consider the functor $\widehat{\mathbb{A}}^{(\infty)}(R) = \bigoplus_{n\in\mathbb{N}} \mathrm{Nil}(R)$, so $X = \varinjlim_n \widehat{\mathbb{A}}^n$, which is again a formal scheme.

Example 4.5. If Z is an infinite CW complex and $\{Z_\alpha\}$ is the collection of finite subcomplexes and E is an even periodic ring spectrum, we define $Z_E = \varinjlim_\alpha (Z_\alpha)_E$. This is clearly a formal scheme.

We can connect this with the framework of [8, Section 8] by taking $\mathcal{C}$ to be the category $\mathrm{Rings}^{\mathrm{op}}$. From this point of view, a formal scheme is an ind-representable contravariant functor from $\mathrm{Rings}^{\mathrm{op}}$ to Sets. We shall omit any mention of universes here, leaving the set-theoretically cautious reader to lift the necessary details from [8, Appendice], or to avoid the problem in some other way.

Given two filtered diagrams $X\colon \mathcal{I} \to \mathcal{X}$ and $Y\colon \mathcal{J} \to \mathcal{X}$ we know from [8, 8.2.5.1] that

$$\widehat{\mathcal{X}}(\varinjlim_i X_i, \varinjlim_j Y_j) = \varprojlim_i \varinjlim_j \mathcal{X}(X_i, Y_j).$$

It follows that $\widehat{\mathcal{X}}$ is equivalent to the category whose objects are pairs $(\mathcal{I}, X)$ and whose morphisms are given by the above formula. We will feel free to use either model for $\widehat{\mathcal{X}}$ where convenient.

Proposition 4.6. A functor $X\colon \mathrm{Rings} \to \mathrm{Sets}$ is a formal scheme if and only if

(a) X preserves finite limits, and
(b) There is a set of schemes X_i and natural maps $X_i \to X$ such that the resulting map $\coprod_i X_i(R) \to X(R)$ is surjective for all R.

Proof. This is essentially [8, Théorème 8.3.3]. To see this, let $\mathcal{D}$ be the category of schemes over X. A map $\mathrm{spec}(R) \to X$ is the same (by Yoneda) as an element of $X(R)$, so $\mathcal{D}^{\mathrm{op}}$ is equivalent to the category $\mathrm{Points}(X)$. This category corresponds to the category $\mathcal{C}_{/F}$ of the cited theorem. Thus, by the equivalence (i)$\Leftrightarrow$(iii) of that theorem, we see that X is a formal scheme if and only if X preserves finite limits, and $\mathcal{D}$ has a small cofinal subcategory. (Grothendieck actually talks about finite colimits, but in our case that implicitly refers to colimits in $\mathrm{Rings}^{\mathrm{op}}$ and thus limits in Rings.) It is shown in the proof of the theorem that if X preserves finite limits, then $\mathcal{D}$ is a filtered category, so we can use [8, Proposition 8.1.3(c)] to recognise cofinal subcategories. This means that a small collection $\{X_i\}$ of schemes over X gives a cofinal subcategory if and only if each map from a scheme Y to X factors through some X_i. By writing $Y = \mathrm{spec}(R)$ and using the Yoneda lemma, it is equivalent to say that the map $\coprod_i X_i(R) \to X(R)$ is surjective for all R. □

4.1. **(Co)limits of formal schemes.**

Proposition 4.7. The category $\widehat{\mathcal{X}}$ has all small colimits. The inclusion $\mathcal{X} \to \widehat{\mathcal{X}}$ preserves finite colimits, and the inclusion $\widehat{\mathcal{X}} \to \mathcal{F} = [\mathrm{Rings}, \mathrm{Sets}]$ preserves filtered colimits. Moreover, if $X \in \mathcal{X}$ then the functor $\widehat{\mathcal{X}}(X, -)\colon \widehat{\mathcal{X}} \to \mathrm{Sets}$ also preserves colimits.

Proof. Apart from the last sentence, the proof is the same as that of [14, Theorem VI.1.6]. Johnstone assumes that $\mathcal{C}$ (which is $\mathrm{Rings}^{\mathrm{op}}$ in our case) is small, but he does not really use this. The last sentence is [14, Lemma VI.1.8]. □

Example 4.8. It is not hard to see that the functor $Z \mapsto Z_E$ of example 4.5 converts filtered homotopy colimits to colimits of formal schemes.

Suppose we have a diagram of formal schemes $X : \mathcal{I} \to \widehat{\mathcal{X}}$. For each $i \in \mathcal{I}$ we then have a filtered category $\mathcal{J}(i)$ and a functor $X(i, -)\colon \mathcal{J}(i) \to \mathcal{X}$ such that $X(i) = \varinjlim_{\mathcal{J}(i)} X(i, j)$. For many purposes, it is convenient if we can take all the categories $\mathcal{J}(i)$ to be the same. This motivates the following definition.

Definition 4.9. A category $\mathcal{I}$ is *rectifiable* if for every functor $X : \mathcal{I} \to \widehat{\mathcal{X}}$ there is a filtered category $\mathcal{J}$ and a functor $Y : \mathcal{I} \times \mathcal{J} \to \mathcal{X}$ such that $X(i) = \varinjlim_{\mathcal{J}} Y(i, j)$ as functors of i.

Proposition 4.10. If $\mathcal{I}$ is a finite category such that $\mathcal{I}(i, i) = \{1\}$ for all $i \in \mathcal{I}$, then $\mathcal{I}$ is rectifiable.

Proof. See [8, Proposition 8.8.5]. □

Proposition 4.11. If $\mathcal{I}$ is a discrete small category (in other words, a set), then $\mathcal{I}$ is rectifiable.

Proof. As $X(i)$ is a formal scheme, there is a filtered category $\mathcal{J}(i)$ and a functor $Z(i,-)\colon \mathcal{J}(i) \to \mathcal{X}$ such that $X(i) = \varinjlim_{\mathcal{J}(i)} Z(i,j)$. Write $\mathcal{J} = \prod_i \mathcal{J}(i)$, let $\pi_i \colon \mathcal{J} \to \mathcal{J}(i)$ be the projection, and let $Y(i,-)$ be the composite functor $\mathcal{J} \xrightarrow{\pi_i} \mathcal{J}(i) \xrightarrow{Z(i,-)} \mathcal{X}$. It is easy to check that $\mathcal{J}$ is filtered and that π_i is cofinal, so $X(i) = \varinjlim_{\mathcal{J}} Y(i,j)$, as required. $\square$

Proposition 4.12. The category $\widehat{\mathcal{X}}$ has finite limits, and the inclusions $\mathcal{X} \to \widehat{\mathcal{X}} \to \mathcal{F}$ preserve all limits that exist. Moreover, finite limits in $\widehat{\mathcal{X}}$ commute with filtered colimits.

Proof. First consider a diagram $X\colon \mathcal{I} \to \widehat{\mathcal{X}}$ indexed by a finite rectifiable category. We define $U(R) = \varprojlim_{\mathcal{I}} X(i)(R)$, which gives a functor Rings $\to$ Sets. It is well-known that this is the inverse limit of the diagram X in the functor category $\mathcal{F}$, so it will suffice to show that U is a formal scheme. As $\mathcal{I}$ is rectifiable, we can choose a diagram $Y\colon \mathcal{I} \times \mathcal{J} \to \mathcal{X}$ as in Definition 4.9. As $\mathcal{X}$ has limits, we can define $Z(j) = \varprojlim_i Y(i,j) \in \mathcal{X}$, and then define $W = \varinjlim_j Z(j) \in \widehat{\mathcal{X}}$. Then $W(R) = \varinjlim_j \varprojlim_i Y(i,j)(R)$. As filtered colimits commute with finite limits in the category of sets, this is the same as $\varprojlim_i \varinjlim_j Y(i,j)(R) = \varprojlim_i X(i)(R) = V(R)$. Thus $V = W$ is a formal scheme, as required.

Both finite products and equalisers can be considered as limits indexed by rectifiable categories, and we can write any finite limit as the equaliser of two maps between finite products. This shows that $\widehat{\mathcal{X}}$ has finite limits.

Now let $\{X_i\}$ be a diagram of formal schemes, let X be a formal scheme, and let $\{f_i\colon X \to X_i\}$ be a cone. If this is a limit cone in $\widehat{\mathcal{X}}$ then we must have $X(R) = \widehat{\mathcal{X}}(\mathrm{spec}(R), X) = \varprojlim_i \widehat{\mathcal{X}}(\mathrm{spec}(R), X_i) = \varprojlim_i X_i(R)$, which means that it is a limit cone in $\mathcal{F}$ (because limits in functor categories are computed pointwise). The converse is equally easy, so the inclusion $\widehat{\mathcal{X}} \to \mathcal{F}$ preserves and reflects limits. Similarly, the inclusion $\mathcal{X} \to \mathcal{F}$ preserves and reflects limits, and it follows that the same is true of the inclusion $\mathcal{X} \to \widehat{\mathcal{X}}$. $\square$

4.2. **Solid formal schemes.**

Definition 4.13. A *linear topology* on a ring R is a topology such that the cosets of open ideals are open and form a basis of open sets. One can check that such a topology makes R into a topological ring. We write LRings for the category of rings with a given linear topology, and continuous homomorphisms. For any ring S, the discrete topology is a linear topology on S, so we can think of Rings as a full subcategory of LRings. Given a linearly topologised ring R, we define $\mathrm{spf}(R)\colon \mathrm{Rings} \to \mathrm{Sets}$ by

$$\mathrm{spf}(R)(S) = \mathrm{LRings}(R,S) = \varinjlim_J \mathrm{Rings}(R/J, S),$$

where J runs over the directed set of open ideals. Clearly this defines a functor $\mathrm{spf}\colon \mathrm{LRings}^{\mathrm{op}} \to \widehat{\mathcal{X}}$.

Definition 4.14. Let R be a linearly topologised ring. The *completion* of R is the ring $\widehat{R} = \varprojlim_I R/I$, where I runs over the open ideals in R. There is an evident map $R \to \widehat{R}$, and the composite $R \to \widehat{R} \to R/I$ is surjective so we have $R/I = \widehat{R}/\overline{I}$ for some ideal $\overline{I} \le \widehat{R}$. These ideals form a filtered system, so we can give $\widehat{R}$ the linear topology for which they are a base of neighbourhoods of zero. It is easy to check that $\widehat{\widehat{R}} = \widehat{R}$ and that $\mathrm{spf}(\widehat{R}) = \mathrm{spf}(R)$. We say that R is *complete*, or that it is a *formal ring*, if $R = \widehat{R}$. Thus $\widehat{R}$ is always a formal ring. We write FRings for the category of formal rings.

Definition 4.15. Given a formal scheme X, we recall that $\mathcal{O}_X = \widehat{\mathcal{X}}(X, \mathbb{A}^1)$. This is again a ring under pointwise operations. If $\{X_i\}$ is a presentation of X then $\mathcal{O}_X = \varprojlim_i \mathcal{O}_{X_i}$.

For any point x of X we define $I_x = \{f \in \mathcal{O}_X \mid f(x) = 0 \in \mathcal{O}_x\}$. From a slightly different point of view, we can think of x as a map $Y = \mathrm{spec}(\mathcal{O}_x) \to X$ and I_x as the kernel of the resulting map $\mathcal{O}_X \to \mathcal{O}_Y$. As the informal schemes over X form a filtered category, we see that the ideals I_x form a directed system. Thus, there is a unique linear topology on $\mathcal{O}_X$, such that the ideals I_x form a base of neighbourhoods of zero. With this topology, if $\{X_i\}$ is a presentation of X, then $\mathcal{O}_X = \varprojlim_i \mathcal{O}_{X_i}$ as topological rings.

Note that

$$\widehat{\mathcal{X}}(X, \mathrm{spf}(R)) = \varprojlim_i \widehat{\mathcal{X}}(X_i, \mathrm{spf}(R)) = \varprojlim_i \mathrm{LRings}(R, \mathcal{O}_{X_i}) = \mathrm{LRings}(R, \mathcal{O}_X),$$

so that $\mathcal{O}\colon \widehat{\mathcal{X}} \to \mathrm{LRings}^{\mathrm{op}}$ is left adjoint to $\mathrm{spf}\colon \mathrm{LRings}^{\mathrm{op}} \to \widehat{\mathcal{X}}$. In particular, we have a unit map $X \to \mathrm{spf}(\mathcal{O}_X)$ in $\widehat{\mathcal{X}}$, and a counit map $R \to \mathcal{O}_{\mathrm{spf}(R)}$ in LRings. The latter is just the completion map $R \to \widehat{R}$.

Definition 4.16. We say that a formal scheme X is *solid* if it is isomorphic to $\mathrm{spf}(R)$ for some linearly topologised ring R. We write $\widehat{\mathcal{X}}_{\mathrm{sol}}$ for the category of solid formal schemes.

In the earlier incarnation of this paper [25] we defined formal schemes to be what we now call solid formal schemes. While only solid formal schemes seem to occur in the cases of interest, the category of all formal schemes has rather better categorical properties, so we use it instead.

Example 4.17. Any informal scheme X is a solid formal scheme (because the zero ideal is open).

Example 4.18. The formal scheme $\widehat{\mathbb{A}}^n$ is solid. To see this, consider the formal power series ring $R = \mathbb{Z}[\![x_1, \dots, x_n]\!]$, with the usual linear topology defined by the ideals I^k, where $I = (x_1, \dots, x_k)$. This is clearly a formal ring, and $\widehat{\mathbb{A}}^n = \mathrm{spf}(R)$.

Example 4.19. If R is a complete Noetherian semilocal ring with Jacobson radical I (for example, a complete Noetherian local ring with maximal ideal I) then it is natural to give R the linear topology defined by the ideals I^k, and to define $\mathrm{spf}(R)$ using this. With this convention, the set $\widehat{\mathcal{X}}(\mathrm{spf}(R), \mathrm{spf}(S))$ (where S is another ring of the same type) is just the set of local homomorphisms $S \to R$. Thus, the categories of formal schemes used in [26] and [7] embed as full subcategories of our category $\widehat{\mathcal{X}}$.

Example 4.20. Let Z be an infinite CW complex with finite subcomplexes $\{Z_\alpha\}$, and let E be an even periodic ring spectrum. Let J_α be the kernel of the map $E^0 Z \to E^0 Z_\alpha$. These ideals define a linear topology on $E^0 Z$. In good cases $E^0 Z$ will be complete and we will have $Z_E = \mathrm{spf}(E^0 Z)$, so this is a solid formal scheme. See Section 8 for technical results that guarantee this.

Proposition 4.21.

(a) If X is a solid formal scheme then $\mathcal{O}_X$ is a formal ring.
(b) A formal scheme X is solid if and only if it is isomorphic to $\mathrm{spf}(R)$ for some formal ring R, if and only if the natural map $X \to \mathrm{spf}(\mathcal{O}_X)$ is an isomorphism.
(c) The functor $X \mapsto X_{\mathrm{sol}} = \mathrm{spf}(\mathcal{O}_X)$ is left adjoint to the inclusion of $\widehat{\mathcal{X}}_{\mathrm{sol}}$ in $\widehat{\mathcal{X}}$.
(d) The functor $R \mapsto \widehat{R}$ is left adjoint to the inclusion of FRings in LRings.
(e) The functors $R \mapsto \mathrm{spf}(R)$ and $X \mapsto \mathcal{O}_X$ give an equivalence between $\widehat{\mathcal{X}}_{\mathrm{sol}}$ and $\mathrm{FRings}^{\mathrm{op}}$.

Proof. (a): If X is solid then $X = \mathrm{spf}(R)$ for some linearly topologised ring R, so $\mathcal{O}_X = \mathcal{O}_{\mathrm{spf}(R)} = \widehat{R}$ which is a formal ring.

(b): If X is solid then $X = \mathrm{spf}(R)$ as above, but $\mathrm{spf}(R) = \mathrm{spf}(\widehat{R})$ so we may assume that R is formal. We find as in (a) that $\mathcal{O}_X = R$ and thus that the map $X \to \mathrm{spf}(\mathcal{O}_X) = \mathrm{spf}(R)$ is an isomorphism. The converse is easy.

(c): Let T denote the functor $X \mapsto X_{\mathrm{sol}}$. This arises from an adjunction, so it is a monad. On the other hand, if $R = \mathcal{O}_X$ then R is formal by (a), so $R = \mathcal{O}_{\mathrm{spf}(R)} = \mathcal{O}_{X_{\mathrm{sol}}}$. By applying $\mathrm{spf}(-)$, we see that $(X_{\mathrm{sol}})_{\mathrm{sol}} = X_{\mathrm{sol}}$, so $T^2 = T$ and T is an idempotent monad. Moreover, $\widehat{\mathcal{X}}_{\mathrm{sol}}$ is the subcategory of formal schemes for which the unit map $\eta_X\colon X \to TX$ is an isomorphism. It is well-known that this is automatically a reflective subcategory. In outline, if Y is solid and X is arbitrary and $f\colon X \to Y$, then $f' = \eta_Y^{-1} \circ Tf\colon X_{\mathrm{sol}} \to Y$ is the unique map such that $f' \circ \eta_X = f$.

(d): The proof is similar.

(e): If R is formal then $\mathrm{spf}(R)$ is solid and $\mathcal{O}_{\mathrm{spf}(R)} = \widehat{R} = R$. If X is solid then $\mathcal{O}_X$ is formal (by (a)) and $X = \mathrm{spf}(\mathcal{O}_X)$ (by (b)). $\square$

Definition 4.22. Let R, S and T be linearly topologised rings, and let $R \to S$ and $R \to T$ be continuous homomorphisms. We then give $S \otimes_R T$ the linear topology defined by the ideals $I \otimes T + S \otimes J$, where I runs over open ideals in S and J runs over open ideals in T. This is easily seen to be the pushout of S and T under R in LRings. We also define $S \widehat{\otimes}_R T$ to be the completion of $S \otimes_R T$. If R, S and T are formal then $S \widehat{\otimes}_R T$ is the pushout in FRings (because completion is left adjoint to the inclusion FRings $\to$ LRings).

Proposition 4.23. The subcategory $\widehat{\mathcal{X}}_{\mathrm{sol}} \subseteq \widehat{\mathcal{X}}$ is closed under finite products and arbitrary coproducts. It also has its own colimits for arbitrary diagrams, which need not be preserved by the inclusion $\widehat{\mathcal{X}}_{\mathrm{sol}} \to \widehat{\mathcal{X}}$.

Proof. One can check that $\mathrm{spf}(R \otimes S) = \mathrm{spf}(R \widehat{\otimes} S) = \mathrm{spf}(R) \times \mathrm{spf}(S)$, which gives finite products. Let $\{R_i \mid i \in \mathcal{I}\}$ be a family of formal rings, and write $R = \prod_i R_i$. We give this ring the product topology, which is the same as the linear topology defined by the ideals of the form $\prod_i J_i$, where J_i is open in R_i and $J_i = R_i$ for almost all i. We claim that $\mathrm{spf}(R) = \coprod_i \mathrm{spf}(R_i)$.

To see this, let $\mathcal{J}$ denote the set ideals $J = \prod_i J_i$ as above. This is easily seen to be a directed set. For $J \in \mathcal{J}$ we see that $R/J = \prod_i R_i/J_i$, where almost all terms in the product are zero. Thus $\mathrm{spec}(R/J) = \coprod_{i \in \mathcal{I}} \mathrm{spec}(R_i/J_i)$, where almost all terms in the coproduct are empty. As colimits commute with coproducts, we see that $\mathrm{spf}(R) = \coprod_{\mathcal{I}} \varinjlim_{\mathcal{J}} \mathrm{spec}(R_i/J_i)$. As the projection from $\mathcal{J}$ to the set of open ideals in R_i is cofinal, we see that $\varinjlim_{\mathcal{J}} \mathrm{spec}(R_i/J_i) = \mathrm{spf}(R_i)$, so that $\mathrm{spf}(R) = \coprod_{\mathcal{I}} \mathrm{spf}(R_i)$ as claimed.

Now let $\{X_i\}$ be an arbitrary diagram of solid formal schemes, and let X be its colimit in $\widehat{\mathcal{X}}$. As the functor $Y \mapsto Y_{\mathrm{sol}}$ is left adjoint to the inclusion $\widehat{\mathcal{X}}_{\mathrm{sol}} \to \widehat{\mathcal{X}}$, we see that X_{sol} is the colimit of our diagram in $\widehat{\mathcal{X}}_{\mathrm{sol}}$. $\square$

Remark 4.24. We will see in Corollary 4.40 that $\widehat{\mathcal{X}}_{\mathrm{sol}}$ is actually closed under finite limits.

Example 4.25. As a special case of the preceeding proposition, consider an infinite set A. Let R be the ring of functions $u\colon A \to \mathbb{Z}$ with the product topology, so that $\underline{A} = \mathrm{spf}(R) = \coprod_{a \in A} 1$. We call formal schemes of this type *constant formal schemes*. More generally, given a formal scheme X we write $\underline{A}_X = \coprod_{a \in A} X$. If X is solid then $\underline{A}_X = \mathrm{spf}(C(A, \mathcal{O}_X))$, where $C(A, \mathcal{O}_X)$ is the ring of functions $A \to \mathcal{O}_X$, under the evident product topology. Clearly, if E is an even periodic ring spectrum and we regard A as a discrete space then $A_E = \underline{A} \times S_E$.

4.3. **Formal schemes over a given base.** Let X be a formal scheme. Write $\widehat{\mathcal{X}}_X$ for the category of formal schemes over X, and $\mathcal{X}_X$ for the full subcategory of informal schemes over X. We also write $\mathrm{Points}(X)$ for the category of pairs (R, x), where R is a ring and $x \in X(R)$; the maps are as in Definition 2.14. Again, the Yoneda isomorphism $X(R) = \widehat{\mathcal{X}}(\mathrm{spec}(R), X)$ gives an equivalence $\mathrm{Points}(X) = \mathcal{X}_X^{\mathrm{op}}$. Moreover, formal schemes Y over X biject with ind-representable functors $Y'\colon \mathrm{Points}(X) \to \mathrm{Sets}$ by the rules

$$Y'(R, x) = \text{ preimage of } x \text{ under the map } Y(R) \to X(R)$$
$$Y(R) = \coprod_{x \in X(R)} Y'(R, x).$$

Now consider a formal scheme X with presentation $\{X_i\}$, indexed by a filtered category $\mathcal{I}$. We next investigate the relationship between the categories $\widehat{\mathcal{X}}_X$ and $\widehat{\mathcal{X}}_{X_i}$, which we now define.

Definition 4.26. Given a diagram $\{X_i\}$ as above, we write $\mathcal{D}_{\{X_i\}}$ for the category of diagrams $\{Y_i\}\colon \mathcal{I} \to \widehat{\mathcal{X}}$ equipped with a map of diagrams $\{Y_i\} \to \{X_i\}$. For any such diagram $\{Y_i\}$ and any map $u\colon i \to j$ in $\mathcal{I}$, we have a commutative square

$$\begin{tikzcd} Y_i \arrow[r, "Y_u"] \arrow[d] & Y_j \arrow[d] \\ X_i \arrow[r, "X_u"'] & X_j. \end{tikzcd}$$

We write $\widehat{\mathcal{X}}_{\{X_i\}}$ for the full subcategory of $\mathcal{D}_{\{X_i\}}$ consisting of diagrams $\{Y_i\}$ for which all such squares are pullbacks.

We define functors $F\colon \mathcal{D}_{\{X_i\}} \to \widehat{\mathcal{X}}_X$ and $G\colon \widehat{\mathcal{X}}_X \to \mathcal{D}_{\{X_i\}}$ by

$$F\{Y_i\} = \varinjlim_i Y_i$$
$$GY = \{Y \times_X X_i\}.$$

Proposition 4.27. The functor F is left adjoint to G, and it preserves finite limits. The functor G is full and faithful, and its image is $\widehat{\mathcal{X}}_{\{X_i\}}$. The functors F and G give an equivalence between $\widehat{\mathcal{X}}_X$ and $\widehat{\mathcal{X}}_{\{X_i\}}$.

Moreover, if W is an informal scheme over X and $\{Y_i\} \in \widehat{\mathcal{X}}_{\{X_i\}}$, then any factorisation $W \to X_i \to X$ of the given map $W \to X$ gives an isomorphism $W \times_X F\{Y_i\} = W \times_{X_i} Y_i$.

Proof. A map $F\{Y_i\} \to Z$ is the same as a compatible system of maps $Y_i \to Z$ over X. As the map $Y_i \to X$ has a given factorisation through X_i, this is the same as a compatible system of maps $Y_i \to Z \times_X X_i = G(Z)_i$ over X_i, or in other words a map $\{Y_i\} \to G(Z)$. Thus F is left adjoint to G.

As filtered colimits commute with finite limits, we see that $FG(Y) = \varinjlim_i (Y \times_X X_i) = Y \times_X \varinjlim_i X_i = Y$. This means that

$$\mathcal{D}_{\{X_i\}}(GY, GZ) = \widehat{\mathcal{X}}_X(Y, FGZ) = \widehat{\mathcal{X}}_X(Y, Z),$$

so G is full and faithful. This means that G is an equivalence of $\widehat{\mathcal{X}}_X$ with its image, and it is clear that the image is contained in $\widehat{\mathcal{X}}_{\{X_i\}}$. The commutation of finite limits and filtered colimits also implies that F preserves finite limits.

We now prove the last part of the proposition; afterwards we will deduce that the image of G is precisely $\widehat{\mathcal{X}}_{\{x_i\}}$. Consider an informal scheme W and a map $f\colon W \to X$, and an object $\{Y_i\}$ of $\widehat{\mathcal{X}}_{\{X_i\}}$. Let $\mathcal{J}$ be the category of pairs (i, g), where $i \in \mathcal{I}$ and $g\colon W \to X_i$ and the composite $W \xrightarrow{g} X_i \to X$ is the same as f. It is not hard to check that $\mathcal{J}$ is filtered and that the projection functor $\mathcal{J} \to \mathcal{I}$ is cofinal. For each $(i, g) \in \mathcal{J}$ we have a pullback diagram

$$\begin{tikzcd} W \times_{X_i} Y_i \arrow[r] \arrow[d] & Y_i \arrow[d] \\ W \arrow[r] & X_i. \end{tikzcd}$$

By taking the colimit over $\mathcal{J}$ we get a pullback diagram

$$\begin{tikzcd} \varinjlim W \times_{X_i} Y_i \arrow[r] \arrow[d] & F\{Y_i\} \arrow[d] \\ W \arrow[r] & X. \end{tikzcd}$$

On the other hand, for each map $u\colon (i, g) \to (j, h)$ in $\mathcal{J}$ we have $Y_i = X_i \times_{X_j} Y_j$ (by the definition of $\widehat{\mathcal{X}}_{\{X_i\}}$) and thus $W \times_{X_i} Y_i = W \times_{X_j} Y_j$. It follows easily that for each (i, g) the map $W \times_{X_i} Y_i \to \varinjlim W \times_{X_j} Y_j$ is an isomorphism, and thus (by the diagram) that $W \times_X F\{Y_i\} = W \times_{X_i} Y_i$.

Now take $W = X_i$ and $g = 1$ in the above. We find that $X_i \times_X F\{Y_i\} = Y_i$, and thus that $FG\{Y_i\} = \{Y_i\}$, and thus that $\{Y_i\}$ is in the image of G. This shows that the image of G is precisely $\widehat{\mathcal{X}}_{\{X_i\}}$, as required. $\square$

Definition 4.28. Let Y be a formal scheme over a formal scheme X. We say that Y is *relatively informal* over X if for all informal schemes X' over X, the pullback $Y \times_X X'$ is informal.

Proposition 4.29. The category of relatively informal schemes over X has limits, which are preserved by the inclusion into $\widehat{\mathcal{X}}_X$.

Proof. We can write X as the colimit of a filtered diagram of informal schemes X_i. It is clear that the category of relatively informal schemes is equivalent to the subcategory $\mathcal{C}$ of $\widehat{\mathcal{X}}_{\{X_i\}}$ consisting of systems $\{Y_i\}$ of informal schemes. As the category of informal schemes has limits, we see that the category of informal schemes over X_i has limits. Moreover, for each map $X_i \to X_j$, the functor $X_i \times_{X_j} (-) \colon \mathcal{X}_{X_j} \to \mathcal{X}_{X_i}$ preserves limits. Given this, it is easy to check that $\mathcal{C}$ has limits, as required. As the inclusion $\mathcal{X} \to \widehat{\mathcal{X}}$ preserves limits, one can check that the same is true of the inclusions $\mathcal{X}_{X_i} \to \widehat{\mathcal{X}}_{X_i}$ and $\mathcal{C} \to \widehat{\mathcal{X}}_{\{X_i\}} = \widehat{\mathcal{X}}_X$. □

4.4. **Formal subschemes.**

Definition 4.30. We say that a map $f \colon X \to Y$ of formal schemes is a *closed inclusion* if it is a regular monomorphism in $\widehat{\mathcal{X}}$. (This means that it is the equaliser of some pair of arrows $Y \rightrightarrows Z$, or equivalently that it is the equaliser of the pair $Y \rightrightarrows Y \amalg_X Y$.) A *closed formal subscheme* of a formal scheme Y is a subfunctor X of Y such that X is a formal scheme and the inclusion $X \to Y$ is a closed inclusion.

Remark 4.31. The functor $Z \mapsto Z(R)$ is representable (by $\operatorname{spec}(R)$). It follows that if $f \colon V \to W$ is a monomorphism in $\widehat{\mathcal{X}}$ then $V(R) \to W(R)$ is injective for all R, so V is isomorphic to a subfunctor of W. If f is a regular monomorphism, then the corresponding subfunctor is a closed subscheme.

Example 4.32. Let J be an ideal in $\mathcal{O}_X$, generated by elements $\{f_i \mid i \in I\}$ say. We define

$$V(J)(R) = \{x \in X(R) \mid f(x) = 0 \text{ for all } f \in J\} = \{x \mid f_i(x) = 0 \text{ for all } i\}.$$

Define a scheme $\mathbb{A}^I$ by $\mathbb{A}^I(R) = \prod_{i \in I} R$ (this is represented by the polynomial algebra $\mathbb{Z}[x_i \mid i \in I]$). This is just the product $\prod_{i \in I} \mathbb{A}^1$; by Proposition 4.12, it does not matter whether we interpret this in $\mathcal{X}$ or $\widehat{\mathcal{X}}$. It follows that there is a map $f \colon X \to \mathbb{A}^I$ with components f_i, and another map $g \colon X \to \mathbb{A}^I$ with components 0. Clearly $V(J)$ is the equaliser of f and g, and thus it is a closed formal subscheme of X. There is a natural map $\mathcal{O}_X/J \to \mathcal{O}_{V(J)}$ which is an isomorphism in most cases of interest, but I suspect that this is not true in general (compare Remark 4.39).

Example 4.33. If X is an informal scheme and Y is a closed informal subscheme of X then the evident map $X^\wedge_Y \to X$ is a closed inclusion.

Proposition 4.34. A map $f \colon X \to Y$ of informal schemes is a closed inclusion in $\widehat{\mathcal{X}}$ if and only if it is a closed inclusion in $\mathcal{X}$.

Proof. It follows from Proposition 4.7 that the pushout $Y \amalg_X Y$ is the same whether constructed in $\mathcal{X}$ or $\widehat{\mathcal{X}}$. It follows in turn from Proposition 4.12 that the equaliser of the two maps $Y \rightrightarrows Y \amalg_X Y$ is the same whether constructed in $\mathcal{X}$ or $\widehat{\mathcal{X}}$. The map f is a closed inclusion if and only if X maps isomorphically to this equaliser, so the proposition follows. □

Proposition 4.35. If $X \in \widehat{\mathcal{X}}$ and $Y \in \mathcal{X}$, then a map $f \colon X \to Y$ is a closed inclusion if and only if there is a directed set of closed informal subschemes Y_i of Y such that $X = \varinjlim_i Y_i$.

Proof. First suppose that f is a closed inclusion. We can write X as a colimit of informal schemes, say $X = \varinjlim_{i \in \mathcal{I}} X_i$. Write $Z_i = Y \amalg_{X_i} Y$. One checks that these schemes give a functor $\mathcal{I} \to \mathcal{X}$, and that $\varinjlim_i Z_i = Y \amalg_X Y$. Let Y_i be the equaliser of the two maps $X \rightrightarrows Z_i$, so that Y_i is a closed informal subscheme of X, and again the schemes Y_i give a functor $\mathcal{I} \to \mathcal{X}$. As finite limits commute with filtered colimits in $\widehat{\mathcal{X}}$, we see that $\varinjlim_i Y_i$ is the equaliser of the maps $Y \rightrightarrows \varinjlim_i Z_i = Y \amalg_X Y$. This is just X, because f is assumed to be a regular monomorphism.

Conversely, suppose that $\{Y_i\}$ is a directed family of closed subschemes of an informal scheme Y. Write $Z_i = Y \amalg_{Y_i} Y$ and $Z = \varinjlim_i Z_i$. By much the same logic as above, we see that there is a pair of maps $Y \rightrightarrows Z$ whose equaliser in $X = \varinjlim_i Y_i$, so that X is a closed formal subscheme of Y. □

Proposition 4.36. A map $f \colon X \to Y$ in $\widehat{\mathcal{X}}$ is a closed inclusion if and only if for all informal schemes Y' and all maps $Y' \to Y$, the pulled-back map $f' \colon X' \to Y'$ is a closed inclusion.

Proof. It is clear that the condition is necessary, because in any category a pullback of a regular monomorphism is a regular monomorphism. For sufficiency, suppose that $f \colon X \to Y$ is such that all maps of the form $f' \colon X' \to Y'$ are closed inclusions. Write Y as a colimit of informal schemes Y_i in the usual way, and let $f_i \colon X_i \to Y_i$ be the pullback of f along the map $Y_i \to Y$. As finite limits in $\widehat{\mathcal{X}}$ commute with filtered colimits, we see that $X = \varinjlim_i X_i$. By assumption, f_i is a closed inclusion. Write $Z_i = Y_i \amalg_{X_i} Y_i$, so X_i is the equaliser of the fork $Y_i \rightrightarrows Z_i$. Write $Z = \varinjlim_i Z_i$. As finite limits in $\widehat{\mathcal{X}}$ commute with filtered colimits, we see that X is the equaliser of the maps $Y \rightrightarrows Z$, and thus that f is a closed inclusion. □

Proposition 4.37. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps of formal schemes. If f and g are closed inclusions, then so is gf. Conversely, if gf is a closed inclusion and g is a monomorphism then f is a closed inclusion.

Proof. The second part is a formal statement which holds in any category: if we have maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that gf is the equaliser of a pair $Z \underset{q}{\overset{p}{\rightrightarrows}} W$, then a diagram chase shows that $X \xrightarrow{f} Y$ is the equaliser of pg and qg and thus is a regular monomorphism.

For the first part, we can assume by Proposition 4.36 that Z is an informal scheme. We then know from Proposition 4.35 that there is a filtered system of closed subschemes Z_i of Z such that Y is the colimit of the Z_i. The maps $Y \to Z$ and $Z_i \to Y \to Z$ are closed inclusions, so the second part tells us that $Y_i \to Y$ is a closed monomorphism. Let X_i be the preimage of $Z_i \subseteq Y$ under the map $f\colon X \to Y$. The maps $X_i \to Z_i$ and $Z_i \to Z$ are closed inclusions of informal schemes, so the composite $X_i \to Z$ is easily seen to be a closed inclusion (because closed inclusions in the informal category are just dual to surjections of rings). As filtered colimits commute with pullbacks, we see that $X = \varinjlim_i X_i$. It follows from Proposition 4.35 that $X \to Z$ is a closed inclusion. □

Proposition 4.38. Any closed formal subscheme of a solid formal scheme is again solid.

Proof. Let $W \xrightarrow{f} X \underset{h}{\overset{g}{\rightrightarrows}} Y$ be an equaliser diagram, and suppose that X is solid. We need to show that W is solid. Choose a presentation $Y = \varinjlim_{i \in \mathcal{I}} Y_i$ for Y. Let $\mathcal{J}$ be the set of tuples $j = (J, i, g', h')$, where J is an open ideal in $\mathcal{O}_X$ and $i \in \mathcal{I}$ and $g', h'\colon V(J) \to Y_i$ and the following diagram commutes.

$$\begin{tikzcd}
V(J) \arrow[r, shift left, "g'"] \arrow[r, shift right, "h'"'] \arrow[d, tail] & Y_i \arrow[d] \\
X \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & Y
\end{tikzcd}$$

One can make $\mathcal{J}$ into a filtered category so that $j \mapsto J$ is a cofinal functor to the directed set of open ideals of $\mathcal{O}_X$, and $j \mapsto i$ is a cofinal functor to $\mathcal{I}$ (see the proof of [8, Proposition 8.8.5]). The equaliser of g' and h' is a closed subscheme of $V(J)$, so it has the form $V(I_j)$ for some ideal $I_j \geq J$. As equalisers commute with filtered colimits, we see that $W = \varinjlim_{\mathcal{J}} V(I_j)$. Let $\mathcal{K}$ be the set of ideals of the form I_j for some j. The functor $j \mapsto I_j$ from $\mathcal{J}$ to $\mathcal{K}$ is cofinal, so we have $W = \varinjlim_{I \in \mathcal{K}} V(I)$. We can define a new linear topology on $R = \mathcal{O}_X$ by letting the ideals $I \in \mathcal{K}$ be a base of neighbourhoods of zero, and we conclude that $W = \operatorname{spf}(R)$. Thus, W is solid. □

Remark 4.39. In the above proof, suppose that Y is also solid, and let K be the ideal in $\mathcal{O}_X$ generated by elements of the form $g^*u - h^*u$ with $u \in \mathcal{O}_Y$. One can then check that $\mathcal{O}_W = \varprojlim_J \mathcal{O}_X/(K + J)$, where J runs over the open ideals in $\mathcal{O}_X$. The kernel of the map $\pi\colon \mathcal{O}_X \to \mathcal{O}_W$ is $\bigcap_J (J + K)$, which is just the closure of K. One would like to say that π was surjective, but in fact its cokernel is $\varprojlim^1_J (J + K)$, which can presumably be nonzero.

Corollary 4.40. The subcategory $\widehat{\mathcal{X}}_{\mathrm{sol}} \subseteq \widehat{\mathcal{X}}$ of solid formal schemes is closed under finite limits.

Proof. We know from Proposition 4.23 that a finite product of solid schemes is solid, and a finite limit is a closed formal subscheme of a finite product. □

4.5. **Idempotents and formal schemes.**

Proposition 4.41. Let X be a formal scheme. Then systems of formal subschemes X_i such that $X = \coprod_i X_i$ biject with systems of idempotents $e_i \in \mathcal{O}_X$ such that $e_i e_j = \delta_{ij} e_i$ and $\sum_i e_i$ converges to 1 in the natural topology in $\mathcal{O}_X$. More explicitly, we require that for every open ideal $J \leq \mathcal{O}_X$ the set $S = \{i \mid e_i \notin J\}$ is finite, and $\sum_S e_i = 1 \pmod{J}$.

Proof. Suppose that $X = \coprod_{i \in \mathcal{I}} X_i$. Then $\mathcal{O}_X = \widehat{\mathcal{X}}(X, \mathbb{A}^1) = \prod_i \widehat{\mathcal{X}}(X_i, \mathbb{A}^1) = \prod_i \mathcal{O}_{X_i}$ as rings. If K is a finite subset of $\mathcal{I}$, we write $X_K = \coprod_{i \in K} X_i$. We then have $X = \varinjlim_K X_K$, and this is a filtered colimit, so $X(R) = \varinjlim_K X_K(R)$ for all R. Using this, it is not hard to check that $\mathcal{O}_X = \prod_i \mathcal{O}_{X_i}$ as topological rings, where the right hand side is given the product topology. Note that the product topology is defined by the ideals of the form $\prod_i J_i$, where J_i is an open ideal in $\mathcal{O}_{X_i}$ and $J_i = \mathcal{O}_{X_i}$ for almost all i.

For each i there is an evident idempotent e_i in $\mathcal{O}_X = \prod_i \mathcal{O}_{X_i}$, whose j'th component is δ_{ij}. This gives a system of idempotents as described in the proposition.

Conversely, suppose we start with such a system of idempotents. For any idempotent $e \in \mathcal{O}_X$ it is easy to check that $D(e) = V(1-e)$, so we can define $X_i = D(e_i) = V(1-e_i)$. We need to check that $X = \coprod_i X_i$. We can write $X = \varinjlim_{\mathcal{J}} Y_j$ for some filtered system of informal schemes Y_j. Let e_{ij} be the image of e_i in $\mathcal{O}_{Y_j}$ and write $Z_{ij} = D(e_{ij}) = V(1-e_{ij}) \subseteq Y_j$. As Y_j is informal we know that the kernel of the map $\mathcal{O}_X \to \mathcal{O}_{Y_j}$ is open and thus that $e_{ij} = 0$ for almost all i. We thus have a decomposition $Y_j = \coprod_i Z_{ij}$, in which only finitely many factors are nonempty. If we fix i, it is easy to check that the schemes Z_{ij} are functors of j, and that $\varinjlim_j Z_{ij} = X_i$. As colimits commute with coproducts, we find that $X = \coprod_i X_i$ as claimed. $\square$

Corollary 4.42. Coproducts in $\widehat{\mathcal{X}}$ or $\widehat{\mathcal{X}}_X$ are strong.

Proof. Let $\{Y_i\}$ be a family of schemes over X, and write $Y = \coprod_i Y_i$. Let Z be another scheme over X, and write $Z_i = Z \times_X Y_i$. We need to show that $Z \times_X Y = \coprod_i Z_i$. To see this, take idempotents $e_i \in \mathcal{O}_Y$ as in the proposition, so that $Y_i = D(e_i) = V(1-e_i)$. Let e'_i be the image of e_i under the evident map $\mathcal{O}_Z \to \mathcal{O}_Y$; it is easy to check that $Z_i = D(e'_i)$. As the idempotents e_i are orthogonal and sum to 1 and the map $\mathcal{O}_{Z \times_X Y} \to \mathcal{O}_Y$ is a continuous map of topological rings, we see that the e'_i are also orthogonal idempotents whose sum is 1. This shows that $Z \times_X Y = \coprod_i Z_i$ as claimed. $\square$

4.6. **Sheaves over formal schemes.** In Section 2.6, we defined sheaves and vector bundles over all functors, and in particular over formal schemes.

Remark 4.43. If M is a vector bundle and L is a line bundle over a formal scheme X, we can define functors $\mathbb{A}(M)(R)$ and $\mathbb{A}(L)^\times(R)$ just as in Definitions 2.45 and 2.55. We claim that these are formal schemes. Given a map $f\colon W \to X$, it is easy to check that $f^*\mathbb{A}(M) = \mathbb{A}(f^*M)$ (where the pullback on the left hand side is computed in the functor category $\mathcal{F}$). In particular, if W is informal then Proposition 2.54 shows that $f^*\mathbb{A}(M)$ is a scheme. Now write $X = \varinjlim_i X_i$ in the usual way, and let M_i be the pullback of M over X_i. We find easily that $\mathbb{A}(M) = \varinjlim_i \mathbb{A}(M_i)$, so $\mathbb{A}(M)$ is a formal scheme. Similarly, $\mathbb{A}(L)^\times$ is a formal scheme.

Remark 4.44. If M is a sheaf such that M_x is an infinitely generated free module for all x, we find that $\mathbb{A}(M)$ is a formal scheme over X. Unlike the case of a vector bundle, it is not relatively informal over X. We leave the proof as an exercise.

Remark 4.45. Let $\{X_i\}$ be a presentation of a formal scheme X. If M is a sheaf over X then one can check that $\Gamma(X, M) = \varprojlim_i \Gamma(X_i, M)$. In particular, if X is solid and $M_J = \Gamma(V(J), M)$ for all open ideals $J \leq \mathcal{O}_X$ we find that $\Gamma(X, M) = \varprojlim_J M_J$. Moreover, if $J \leq K$ we find that $M_K = M_J / K M_J$.

In particular, if N is an $\mathcal{O}_X$-module we find that $\Gamma(X, \tilde{N}) = \varprojlim_J N/JN$. We say that N is *complete* if $N = \varprojlim_J N/JN$. It follows that the functor $N \mapsto \tilde{N}$ embeds the category of complete modules as a full subcategory of $\mathrm{Sheaves}_X$. **Warning:** it seems that the functor $N \mapsto \varprojlim_J N/JN$ need not be idempotent in bad cases, so $\varprojlim_J N/JN$ need not be complete.

We next consider the problem of constructing sheaves over filtered colimits.

Definition 4.46. Let $\{X_i\}$ be a filtered diagram of functors, with colimit X. Let $\mathrm{Sheaves}_{\{X_i\}}$ denote the category of systems $(\{M_i\}, \phi)$ of the following type:

(a) For each i we have a sheaf M_i over X_i.
(b) For each $u\colon i \to j$ (with associated map $X_u\colon X_i \to X_j$) we have an isomorphism $\phi(u)\colon M_i \simeq X_u^* M_j$.
(c) In the case $u = 1\colon i \to i$ we have $\phi(1) = 1$.
(d) Given $i \xrightarrow{u} j \xrightarrow{v} k$ we have $\phi(vu) = (X_u^*\phi(v)) \circ \phi(u)$.

Proposition 4.47. Let $\{X_i \mid i \in \mathcal{I}\}$ be a filtered diagram of functors, with colimit X. The category $\mathrm{Sheaves}_{\{X_i\}}$ is equivalent to $\mathrm{Sheaves}_X$.

Proof. Given a sheaf M over X, we define a system of sheaves $M_i = v_i^* M$, where $v_i\colon X_i \to X$ is the given map. If $u\colon i \to j$ then $v_j \circ X_u = v_i$ so we have a canonical identification $M_i = X_u^* M_j$, which we take as $\phi(u)$. This gives an object of $\mathrm{Sheaves}_{\{X_i\}}$.

On the other hand, suppose we start with an object $\{M_i\}$ of $\mathrm{Sheaves}_{\{X_i\}}$, and we want to construct a sheaf M over X. Given a ring R and a point $x \in X(R)$, we need to define a module M_x over R. As $X = \varinjlim_i X_i(R)$, we can choose $i \in \mathcal{I}$ and $y \in X_i(R)$ such that $v_i(y) = x$. We would like to define $M_x = M_{i,y}$, but we need to check that this is canonically independent of the choices made. We thus let $\mathcal{J}$ be the category of all such pairs (i, y). Because $X(R) = \varinjlim_i X_i(R)$, we see that $\mathcal{J}$ is filtered. For each $(i, y) \in \mathcal{J}$ we have an R-module $M_{i,y}$, and the maps $\phi(u)$ make this a functor $\mathcal{J} \to \mathrm{Mod}_R$. We define $M_x = \varinjlim_{\mathcal{J}} M_{i,y}$. Because this is a filtered diagram of isomorphisms, each of the canonical maps $M_{i,y} \to M_x$ is an isomorphism. We leave it to the reader to check that this construction produces a sheaf, and that it is inverse to our previous construction. □

Corollary 4.48. Let $X \to Y$ be a map of formal schemes. To construct a sheaf over X, it suffices to construct sheaves over $W \times_Y X$ in a sufficiently natural way, for all informal schemes W over Y. It also suffices to construct sheaves over X_y in a sufficiently natural way, for all points y of Y.

Proof. The two claims are really the same, as points of Y biject with informal schemes over Y by sending a point $y \in Y(R)$ to the usual map $\mathrm{spec}(R) \xrightarrow{y} Y$.

For the first claim, we choose a presentation $Y = \varinjlim_i Y_i$ and write $X_i = Y_i \times_Y X$, and note that $X = \varinjlim_i X_i$. By assumption, we have sheaves M_i over X_i. "Sufficiently natural" means that we have maps $\phi(u)$ making $\{M_i\}$ into an object of $\mathrm{Sheaves}_{\{X_i\}}$, so the proposition gives us a sheaf over X. □

4.7. **Formal faithful flatness.**

Definition 4.49. Let $f\colon X \to Y$ be a map of formal schemes. We say that f is *flat* if the pullback functor $f^*\colon \widehat{\mathcal{X}}_Y \to \widehat{\mathcal{X}}_X$ preserves finite colimits. We say that f is *faithfully flat* if f^* preserves and reflects finite colimits.

Remark 4.50. For any map $f\colon X \to Y$ of formal schemes, we know that f^* preserves all small coproducts. Thus f is flat if and only if f^* preserves coequalisers, if and only if f^* preserves all small colimits.

Definition 4.49 could in principle conflict with Definition 2.56; the following proposition shows that this is not the case.

Proposition 4.51. A map $f\colon X \to Y$ of informal schemes is flat (resp. faithfully flat) as a map of informal schemes if and only if it is flat (resp. faithfully flat) as a map of formal schemes.

Proof. Recall that the inclusion $\mathcal{X} \to \widehat{\mathcal{X}}$ preserves finite colimits. Given this, we see easily that a map that is formally flat (resp. faithfully flat) flat is also informally flat (resp. faithfully flat).

Now suppose that f is informally flat. Let $U \rightrightarrows V \to W$ be a coequaliser in $\widehat{\mathcal{X}}_Y$. By Proposition 4.10, we can find a filtered system of diagrams $U_i \rightrightarrows V_i$ (with U_i and V_i in $\mathcal{X}$) whose colimit is the diagram $U \rightrightarrows V$. We define W_i to be the coequaliser of $U_i \rightrightarrows V_i$. As colimits commute, we have $W = \varinjlim_i W_i$. Clearly all this can be thought of as happening over W and thus over Y. By assumption, the diagram $f^*U_i \rightrightarrows f^*V_i \to f^*W_i$ is a coequaliser. We now take the colimit over i, noting that f^* commutes with filtered colimits and that colimits of coequalisers are coequalisers. This shows that $f^*U \rightrightarrows f^*V \to f^*W$ is a coequaliser. Thus, f is flat.

Now suppose that f is informally faithfully flat, and let $u\colon U \to V$ be a map of formal schemes over Y such that f^*u is an isomorphism. Choose a presentation $V = \varinjlim_i V_i$ and write $U_i = U \times_V V_i$, so that $U = \varinjlim_i U_i$. As f^* preserves pullbacks, we see that the map $f^*U_i \to f^*V_i$ is the pullback of the isomorphism $f^*U \to f^*V$ along the map $f^*V_i \to f^*V$, and thus that the map $f^*U_i \to f^*V_i$ is itself an isomorphism. As f is informally faithfully flat, we conclude that $U_i \simeq V_i$. By passing to colimits, we see that $U \simeq V$ as claimed. $\square$

Remark 4.52. Propositions 2.67, 2.68, 2.70 and 2.76 are general nonsense, valid in any category with finite limits and colimits. They therefore carry over directly to formal schemes.

Lemma 4.53. Let $f\colon X \to Y$ be a map of formal schemes. Let $\mathcal{X}_Y$ be the category of informal schemes with a map to Y, and let $f_0^*\colon \mathcal{X}_Y \to \widehat{\mathcal{X}}_X$ be the restriction of f^* to $\mathcal{X}_Y$. If f_0^* preserves coequalisers, then f is flat.

Proof. Suppose that f_0^* preserves coequalisers. Let $U \rightrightarrows V \to W$ be a coequaliser in $\widehat{\mathcal{X}}_Y$. By Proposition 4.10, we can find a filtered system of diagrams $U_i \rightrightarrows V_i$ (with U_i and V_i in $\mathcal{X}$) whose colimit is the diagram $U \rightrightarrows V$. We define W_i to be the coequaliser of $U_i \rightrightarrows V_i$. As colimits commute, we have $W = \varinjlim_i W_i$. Clearly all this can be thought of as happening over W and thus over Y. By assumption, the diagram $f^*U_i \rightrightarrows f^*V_i \to f^*W_i$ is a coequaliser. We now take the colimit over i, noting that f^* commutes with filtered colimits and that colimits of coequalisers are coequalisers. This shows that $f^*U \rightrightarrows f^*V \to f^*W$ is a coequaliser. Thus, f is flat. $\square$

Proposition 4.54. Let $f\colon X \to Y$ be a map of formal schemes. Suppose that Y has a presentation $Y = \varinjlim_i Y_i$ for which the maps $f_i\colon X_i = f^*Y_i \to Y_i$ are (faithfully) flat. Then f is (faithfully) flat.

Proof. First suppose that each f_i is flat. Let $U \rightrightarrows V \to W$ be a coequaliser of informal schemes over Y. By Lemma 4.53, it is enough to check that $f^*U \rightrightarrows f^*V \to f^*W$ is a coequaliser. We know from Proposition 4.7 that $\widehat{\mathcal{X}}(W, Y) = \varinjlim_i \widehat{\mathcal{X}}(W, Y_i)$, so we can choose a factorisation $W \to Y_i \to Y$ of the given map $W \to Y$, for some i. We then have $f^*W = W \times_Y X = W \times_{Y_i} Y_i \times_Y X = W \times_{Y_i} X_i = f_i^*W$. Similarly, we have $f^*V = f_i^*V$ and $f^*U = f_i^*U$. As f_i is flat, we see that $f^*U \rightrightarrows f^*V \to f^*W$ is a coequaliser, as required.

Now suppose that each f_i is faithfully flat. Let $s\colon U \to V$ be a morphism in $\widehat{\mathcal{X}}_Y$ such that f^*s is an isomorphism. We need to show that s is an isomorphism. We have a pullback square of the following form.

$$\begin{tikzcd}
X_i \arrow[r, "f_i"] \arrow[d, "u_i"'] & Y_i \arrow[d, "v_i"] \\
X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

As f^*s is an isomorphism, we see that $f_i^*v_i^*s = u_i^*f^*s$ is an isomorphism. As f_i is faithfully flat, we conclude that $v_i^*s\colon v_i^*U \to v_i^*V$ is an isomorphism for all i. We also know that $U = \varinjlim_i v_i^*U$ and $V = \varinjlim_i v_i^*V$, and it follows easily that s is an isomorphism. $\square$

Proposition 4.55. Let M be a vector bundle of rank r over a formal scheme X. Then there is a faithfully flat map $f\colon \mathrm{Bases}(M) \to X$ such that $f^*M \simeq \mathcal{O}^r$.

Proof. Let $\mathrm{Bases}(M)(R)$ be the set of pairs (x, B), where $x \in X(R)$ and $B\colon R^r \to M_x$ is an isomorphism. Define $f\colon \mathrm{Bases}(M) \to X$ by $f(x, B) = x$. As in the informal case (Example 2.85) we see that $\mathrm{Bases}(M)$ is a formal scheme over X, and that $f^*M \simeq \mathcal{O}^r$. If X_i is an informal scheme and $u\colon X_i \to X$ then one checks that $u^*\mathrm{Bases}(M) = \mathrm{Bases}(u^*M)$, which is faithfully flat over X_i by Example 2.85. It follows from Proposition 4.54 that $\mathrm{Bases}(M)$ is faithfully flat over X. $\square$

Definition 4.56. A map $f\colon X \to Y$ of formal schemes is *very flat* if for all informal schemes Y' over Y, the scheme $X' = f^*Y'$ is informal and the map $X' \to Y'$ is very flat (in other words, $\mathcal{O}_{X'}$ is a free module over $\mathcal{O}_{Y'}$). Similarly, we say that f is *finite* if for all such Y', the scheme X' is informal and the map $X' \to Y'$ is finite.

4.8. **Coalgebraic formal schemes.** Fix a scheme Z, and write $R = \mathcal{O}_Z$. We next study the category $\mathcal{C}_Z$ of coalgebras over R, and a certain full subcategory $\mathcal{C}'_Z$. It turns out that there is a full and faithful embedding $\mathcal{C}'_Z \to \widehat{\mathcal{X}}_Z$, and that the categorical properties of $\mathcal{C}_Z$ are in some respects superior to those of $\widehat{\mathcal{X}}_Z$. Because of this, the categories $\mathcal{C}_Z$ and $\mathcal{C}'_Z$ are often useful tools for constructing objects of $\widehat{\mathcal{X}}_Z$ with specified properties. Our use of coalgebras was inspired by their appearance in [3], although it is assumed there that R is a field, which removes many technicalities.

We will use R and Z as interchangeable subscripts, so

$$\widehat{\mathcal{X}}_R = \widehat{\mathcal{X}}_Z = \{\text{formal schemes over } Z\},$$

for example. Write $\mathcal{M}_R = \mathcal{M}_Z$ and $\mathcal{C}_R = \mathcal{C}_Z$ for the categories of modules and coalgebras over R. (All coalgebras will be assumed to be cocommutative and counital.) It is natural to think of $\mathcal{C}_Z$ as a "geometric" category, and we choose our notation to reflect this point of view. In particular, we shall see shortly that $\mathcal{C}_Z$ has finite products; we shall write them as $U \times V$, although they are actually given by the tensor product over R. We also write 1 for the terminal object, which is the coalgebra R with $\psi_R = \epsilon_R = 1_R$.

The following result is well-known when R is a field, but we outline a proof to show that nothing goes wrong for more general rings.

Proposition 4.57. The category $\mathcal{C}_Z$ has finite products, and strong colimits for all small diagrams. The forgetful functor to $\mathcal{M}_Z$ creates colimits.

Proof. Given two coalgebras U, V, we make $U \otimes V$ into a coalgebra with counit $\epsilon_U \otimes \epsilon_V : U \otimes V \to R$ and coproduct

$$U \otimes V \xrightarrow{\psi_U \otimes \psi_V} U \otimes U \otimes V \otimes V \xrightarrow{1\otimes\tau\otimes 1} U \otimes V \otimes U \otimes V.$$

This is evidently functorial in U and V. There are two projections $\pi_U = 1 \otimes \epsilon_V : U \otimes V \to U$ and $\pi_V = \epsilon_U \otimes 1 : U \otimes V \to V$, and one checks that these are coalgebra maps. One also checks that a pair of maps $f : W \to U$ and $W \to V$ yield a coalgebra map $h = (f, g) = (f \otimes g) \circ \psi_W : W \to U \otimes V$, and that this is the unique map such that $\pi_U \circ h = f$ and $\pi_V \circ h = g$. Thus, $U \otimes V$ is the categorical product of U and V. Similarly, we can make R into a coalgebra with $\psi_R = \epsilon_R = 1_R$, and this makes it a terminal object in $\mathcal{C}_Z$.

Now suppose we have a diagram of coalgebras U_i, and let $U = \varinjlim_i U_i$ denote the colimit in $\mathcal{M}_Z$. Because tensor products are right exact, we see that $U \otimes U = \varinjlim_{i,j} U_i \otimes U_j$, so there is an obvious map $U_i \otimes U_i \to U \otimes U$. By composing with the coproduct on U_i, we get a map $U_i \to U \otimes U$. These maps are compatible with the maps of the diagram, so we get a map $U = \varinjlim_i U_i \to U \otimes U$. We use this as the coproduct on U. The counit maps $U_i \to R$ also fit together to give a counit map $U \to R$, and this makes U into a coalgebra. One can check that this gives a colimit in the category $\mathcal{C}_Z$. Thus, $\mathcal{C}_Z$ has colimits and they are created in $\mathcal{M}_Z$. It is clear from the construction that $V \times \varinjlim_i U_i = \varinjlim_i (V \times U_i)$, because tensoring with V is right exact. $\square$

Let $f : R \to S = \mathcal{O}_Y$ be a map of rings, and let $T_f : \mathcal{M}_Z \to \mathcal{M}_Y$ be the functor $M \mapsto S \otimes_R M$. This clearly gives a functor $\mathcal{C}_Z \to \mathcal{C}_Y$ which preserves finite products and all colimits.

We now introduce a class of coalgebras with better than usual behaviour under duality.

Definition 4.58. Let U be a coalgebra over R, and suppose that U is free as an R-module, say $U = R\{e_i \mid i \in I\}$. For any finite set J of indices, we write $U_J = R\{e_i \mid i \in J\}$; if this is a subcoalgebra of U, we call it a *standard subcoalgebra*. We say that $\{e_i\}$ is a *good basis* if each finitely generated submodule of U is contained in a standard subcoalgebra. We write $\mathcal{C}'_Z$ for the category of those coalgebras that admit a good basis. It is easy to see that $\mathcal{C}'_Z$ is closed under finite products.

Proposition 4.59. There is a full and faithful functor $\mathrm{sch} = \mathrm{sch}_Z : \mathcal{C}'_Z \to \widehat{\mathcal{X}}_Z$, which preserves finite products and commutes with base change. Moreover, $\mathrm{sch}(U)$ is always solid and we have $\mathcal{O}_{\mathrm{sch}(U)} = U^\vee := \mathrm{Hom}_R(U, R)$.

Proof. Let U be a coalgebra in $\mathcal{C}'_Z$. For each subcoalgebra $V \le U$ such that V is a finitely generated free module over R, we define $V^\vee = \mathrm{Hom}_R(V, R)$. We can clearly make this into an R-algebra using the duals of the coproduct and counit maps, so we have a scheme $\mathrm{spec}(V^\vee)$ over Z. We define $\mathrm{sch}(U) = \varinjlim_V \mathrm{spec}(V^\vee) \in \widehat{\mathcal{X}}_Z$. If we choose a good basis $\{e_i \mid i \in I\}$for U then it is clear that the standard subcoalgebras form a cofinal family of V's, so we have $\mathrm{sch}(U) = \varinjlim_J \mathrm{spec}(U_J^\vee)$, where J runs over the finite subsets of I for which U_J is a subcoalgebra. This is clearly a directed, and thus filtered, colimit. It follows that $\mathcal{O}_{\mathrm{sch}(U)} = \varprojlim_J U_J^\vee = U^\vee$. The resulting topology on $U^\vee = \mathrm{Hom}_R(U, R)$ is just the topology of pointwise convergence, where we give R the discrete topology. We can also think of this as $\prod_I R$, and the topology is just the product topology. It is clear from this that $\mathrm{sch}(U)$ is solid.

If V is another coalgebra with good basis, then the obvious basis for $U \otimes_R V$ is also good. Moreover, if U_J and V_K are standard subcoalgebras of U and V, then $U_J \otimes_R V_J$ is a standard subcoalgebra of $U \otimes_R V$, and the subcoalgebras of this form are cofinal among all standard subcoalgebras of $U \otimes_R V$. It follows easily that $\mathrm{sch}(U \times V) = \mathrm{sch}(U \otimes_R V) = \varinjlim_{J,K} \mathrm{spec}(U_J^\vee) \times_Z \mathrm{spec}(V_K^\vee)$. As finite limits commute with filtered colimits in $\widehat{\mathcal{X}}$, this is the same as $\mathrm{sch}(U) \times_Z \mathrm{sch}(V)$.

Now consider a map $Y = \mathrm{spec}(S) \to Z$ of schemes. The claim is that the functors sch_Y and sch_Z commute with base change, in other words that $\mathrm{sch}_Y(S \otimes_R U) = Y \times_Z \mathrm{sch}_Z(U)$. As pullbacks commute with filtered colimits, the right hand side is just $\varinjlim_J \mathrm{spec}(S \otimes_R U_J)$, which is the same as the left hand side. $\square$

Definition 4.60. Let Z be an informal scheme. We write $\widehat{\mathcal{X}}'_Z$ for the image of sch_Z, which is a full subcategory of $\widehat{\mathcal{X}}_Z$. We say that a formal scheme Y is *coalgebraic* over Z if it lies in $\widehat{\mathcal{X}}'_Z$. We say that Y is *finitely coalgebraic* over Z if $\mathcal{O}_Y$ is a finitely generated free module over $\mathcal{O}_Z$, or equivalently Y is finite and very flat over Z; this easily implies that Y is coalgebraic over Z.

More generally, let Z be a formal scheme, and Y a formal scheme over Z. We say that Y is (finitely) coalgebraic over Z if for all informal schemes Z' over Z, the pullback $Z' \times_Z Y$ is (finitely) coalgebraic over Z'. We again write $\widehat{\mathcal{X}}'_Z$ for the category of coalgebraic formal schemes over Z.

Example 4.61. Let Z be a space such that $H_*(Z; \mathbb{Z})$ is a free Abelian group, concentrated in even degrees. It is not hard to check that $E_0 Z$ is a coalgebra over E^0 which admits a good basis, and that $Z_E = \mathrm{sch}_{E^0}(E_0 Z)$. Details are given in Section 8.

Remark 4.62. The functor $\mathrm{sch}_X \colon \mathcal{C}'_X \to \widehat{\mathcal{X}}'_X$ is an equivalence of categories, with inverse $Y \mapsto cY = \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X)$.

Remark 4.63. For any coalgebra U, we say that an element $u \in U$ is *group-like* if $\epsilon(u) = 1$ and $\psi(u) = u \otimes u$, or equivalently if the map $R \to U$ defined by $r \mapsto ru$ is a coalgebra map. We write $\mathrm{GL}(U) = \mathcal{C}_R(R, U)$ for the set of group-like elements. If U is a finitely generated free module over R, then it is easy to check that $\mathrm{GL}(U) = \mathrm{Alg}_R(U^\vee, R)$. From this one can deduce that

$$\widehat{\mathcal{X}}_Z(Y, \mathrm{sch}_Z(U)) = \mathrm{GL}(\mathcal{O}_Y \otimes_R U),$$

where we regard $\mathcal{O}_Y \otimes_R U$ as a coalgebra over $\mathcal{O}_Y$. This gives another useful characterisation of $\mathrm{sch}_Z(U)$.

Proposition 4.64. Let $\{U_i\}$ be a diagram in $\mathcal{C}_Z$ with colimit U, and suppose that U and U_i actually lie in $\mathcal{C}'_Z$. Then $\mathrm{sch}(U)$ is the strong colimit in $\widehat{\mathcal{X}}_Z$ of the formal schemes $\mathrm{sch}(U_i)$.

Proof. Note that $U = \varinjlim_i U_i$ as R-modules (because colimits in $\mathcal{C}_Z$ are created in $\mathcal{M}_Z$), and it follows immediately that $U^\vee = \varprojlim_i U_i^\vee$ as rings. There are apparently two possible topologies on $U^\vee$. The first is as in the definition of $\mathrm{sch}_R(U)$, where the basic neighbourhoods of zero are the submodules $\mathrm{ann}(M)$, where M runs over finitely generated submodules of U. The second is the inverse limit topology: for each index i and each finitely generated submodule N of U_i, the preimage of the annihilator of N under the evident map $U^\vee \to U_i^\vee$ is a neighbourhood of zero. This is just the same as the annihilator of the image of N in U, and neighbourhoods of this form give a basis for the inverse limit topology. Given this, it is easy to see that the two topologies in question are the same. We thus have an inverse limit of topological rings. As the category of formal schemes is just dual to the category of formal rings, we have a colimit diagram of formal schemes, so $\mathrm{sch}(U) = \varinjlim_i \mathrm{sch}(U_i)$.

We need to show that the colimit is strong, in other words that for any formal scheme T over Z we have $T \times_Z \mathrm{sch}_Z(U) = \varinjlim_i (T \times_Z \mathrm{sch}_Z(U_i))$. First suppose that $T = \mathrm{spec}(B)$ is an informal scheme. We then have $T \times_Z \mathrm{sch}_Z(U) = \mathrm{sch}_T(B \otimes_R U)$ and similarly for each U_i, and $B \otimes_R U = \varinjlim_i B \otimes_R U_i$ because tensor products are right exact. By the first part of the proof (with R replaced by B) we see that $T \times_Z \mathrm{sch}_Z(U) = \varinjlim_i (T \times_Z \mathrm{sch}_Z(U_i))$ as required.

If T is a formal scheme, we write it as a strong filtered colimit of informal schemes T_k. The colimit of the isomorphisms $T_k \times_Z \mathrm{sch}_Z(U) = \varinjlim_i (T_k \times_Z \mathrm{sch}_Z(U_i))$ is the required isomorphism $T \times_Z \mathrm{sch}_Z(U) = \varinjlim_i (T \times_Z \mathrm{sch}_Z(U_i))$. $\square$

Example 4.65. If X is coalgebraic over Y we claim that X^n_Y/Σ_n is a strong colimit for the action of Σ_n on X^n_Y. To see this, we first suppose that Y is informal and $X = \mathrm{sch}_Y(U)$ for some coalgebra U that is free over X with good basis $\{e_i \mid i \in I\}$ say. Then $X^n_Y = \mathrm{sch}_Y(U^{\otimes n})$, and the set of terms $e_{\underline{i}} = e_{i_1} \otimes \ldots \otimes e_{i_n}$ for $\underline{i} = (i_1, \ldots, i_n) \in I^n$ is a good basis for I^n. For each orbit $j \in I^n/\Sigma_n$, we choose an element $\underline{i}$ of the orbit and let f_j be the image of $e_{\underline{i}}$ in $U^{\otimes n}/\Sigma_n$. We find that the terms f_j form a good basis for $U^{\otimes n}/\Sigma_n$, so this coalgebra lies in $\mathcal{C}'_Y$. It follows from Proposition 4.64 that $X^n_Y = \mathrm{sch}_Y(U^{\otimes n}/\Sigma_n)$, and that this is a strong colimit. For a general base Y, we choose a presentation $Y = \varinjlim_i Y_i$ and write $X_i = X \times_Y Y_i$ and $Z_i = (X_i)^n_{Y_i}/\Sigma_n$. By what we have just proved, this is an object of $\widehat{\mathcal{X}}_{\{Y_i\}}$, with $\varinjlim_i Z_i = X^n_Y/\Sigma_n$. It is now easy to see that this is a strong colimit, using the ideas of Proposition 4.27.

We conclude this section with a result about gradings.

Proposition 4.66. Let Y be a coalgebraic formal scheme over an informal scheme X, and suppose that X and Y have compatible actions of $\mathbb{G}_m$. Then cY has a natural structure as a graded coalgebra over $\mathcal{O}_X$.

Proof. Write $R = \mathcal{O}_X$ and $U = cY$. Proposition 2.96 makes R into a graded ring. Next, observe that $\mathcal{O}_Y = U^\vee$ and $\mathcal{O}_{\mathbb{G}_m \times Y} = U^\vee \widehat{\otimes} \mathbb{Z}[t^{\pm 1}]$, which is the ring of doubly infinite Laurent series $\sum_{k \in \mathbb{Z}} a_k t^k$ such that $a_k \in U^\vee$ and $a_k \to 0$ as $|k| \to \infty$. Thus, the action $\alpha \colon \mathbb{G}_m \times Y \to Y$ gives a continuous homomorphism $\alpha^* \colon U^\vee \to U^\vee \widehat{\otimes} \mathbb{Z}[t^{\pm 1}]$, say $\alpha^*(a) = \sum_k a_k t^k$. The basic neighbourhoods of zero in $U^\vee$ are the kernels of the maps $U^\vee \to W^\vee$, where W is a standard subcoalgebra of U. Similarly, the basic neighbourhoods of zero in $U^\vee \widehat{\otimes} \mathbb{Z}[t^{\pm 1}]$ are the kernels of the maps to $V^\vee[t^{\pm 1}]$, where V is a standard subcoalgebra. Thus, continuity means that for every standard subcoalgebra $V \leq U$, there is a standard subcoalgebra W such that whenever $a(W) = 0$ we have $a_k(V) = 0$ for all k. In particular, it follows that the map $\pi_k \colon a \to a_k$ is continuous. Just as in the proof of Proposition 2.96, we see that $\sum_k a_k = a$ and that $\pi_j \pi_k = \delta_{jk} \pi_k$. It follows that $U^\vee$ is a kind of completed direct sum of the subgroups $\mathrm{image}(\pi_k)$. We would like to dualise this and thus split U as an honest direct sum.

First, we need to show that the maps π_i have a kind of R-linearity. Let r be an element of R, and let r_i be the part in degree i, so that $r = \sum_i r_i$ and $r_i = 0$ for almost all i. Using the compatibility of the actions, we find that $(ra)_i = \sum_j r_j a_{i-j}$ (which is really a finite sum).

Suppose that $u \in U$. Choose a standard subcoalgebra V containing u, and let W be a standard subcoalgebra such that whenever $a(W) = 0$ we have $a_i(V) = 0$ for all i.

Suppose that $a \in U^\vee$. It follows from our asymptotic condition on Laurent series that $a_i(u) = 0$ when $|i|$ is large, so we can define $\chi_k(u)(a) = \sum_i a_i(u)_{i+k} \in R$. We then have

$$\begin{aligned}
\chi_k(u)(ra) &= \sum_i ((ra)_i(u))_{i+k} \\
&= \sum_{i,j} (r_j a_{i-j}(u))_{i+k} \\
&= \sum_{i,j} r_j (a_{i-j}(u))_{i+k-j} \\
&= \sum_{m,j} r_j a_m(u)_{m+k} \\
&= r\chi_k(u)(a).
\end{aligned}$$

Thus, the map $\chi_j(u) \colon U^\vee \to R$ is R-linear. Clearly, if $a(W) = 0$ then $\chi_j(u)(a) = 0$, so $\chi_j(u)$ can be regarded as an element of $(U^\vee/\mathrm{ann}(W))^\vee = W^{\vee\vee} = W$ (because W is a finitely generated free module). More precisely, there is a unique element $u_j \in U$ such that $\chi_j(u)(a) = a(u_j)$ for all a, and in fact $u_j \in W$.

Next, we choose a finite set of elements in $U^\vee$ which project to a basis for $W^\vee$. We can then choose a number N such that $b_i(u) = 0$ whenever b lies in that set and $|i| > N$. Because $a_j(u) = 0$ for all j whenever $a(W) = 0$, we conclude that $a_i(u) = 0$ for all $a \in U^\vee$ and all i such that $|i| > N$. It follows that $u_i = 0$ when $|i| > N$. This justifies the following manipulation: $a(u) = \sum_{i,j} a_i(u)_j = \sum_j a(u_j) = a(\sum_j u_j)$. We conclude that $u = \sum_j u_j$. We define a map $\phi_i \colon U \to U$ by $\phi_i(u) = u_i$, and we define $U_i = \mathrm{image}(\phi_i)$. We leave it to the reader to check that $\phi_i\phi_j = \delta_{ij}\phi_j$, so that $U = \bigoplus_i U_i$, and that this grading is compatible with the R-module structure and the coalgebra structure. $\square$

4.9. **More mapping schemes.** Recall the functor $\mathrm{Map}_Z(X, Y)$, given in Definition 2.89. We now prove some more results which tell us when $\mathrm{Map}_Z(X, Y)$ is a scheme or a formal scheme.

First, note that for any functor W over Z, we have

$$\mathcal{F}_Z(W, \mathrm{Map}_Z(X, Y)) = \mathcal{F}_Z(W \times_Z X, Y) = \mathcal{F}_W(W \times_Z X, W \times_Z Y).$$

Indeed, if W is informal then this follows from the definitions and the Yoneda lemma, by writing W in the form $\mathrm{spec}(R)$. The general case follows from this by taking limits, because every functor is the colimit of a (not necessarily small or filtered) diagram of representable functors. It is also not hard to give a direct proof.

Conversely, suppose we have a functor M over Z and a natural isomorphism $\mathcal{F}_Z(W, M) \simeq \mathcal{F}_Z(W \times_Z X, Y)$ for all informal schemes W over Z. It is then easy to identify M with $\mathrm{Map}_Z(X, Y)$.

Lemma 4.67. Let X and Y be functors over Z, and suppose that X and Z are formal schemes. Then $\mathrm{Map}_Z(X, Y)(R)$ is a set for all R, so the functor $\mathrm{Map}_Z(X, Y)$ exists.

Proof. We have only a set of elements $z \in Z(R)$, so it suffices to check that for any such z there is only a set of maps $X_z \to Y_z$ of functors over $\mathrm{spec}(R)$. Here X_z is a formal scheme, with presentation $\{W_i\}$ say. Clearly $\mathcal{F}(W_i, Y_z) = Y_z(\mathcal{O}_{W_i})$ is a set, and $\mathcal{F}_{\mathrm{spec}(R)}(X_z, Y_z)$ is a subset of $\prod_i \mathcal{F}(W_i, Y_z)$. $\square$

Recall also from Proposition 2.94 that $\mathrm{Map}_Z(X, Y)$ is a scheme when X, Y and Z are all informal schemes, and X is finite and very flat over Z.

Definition 4.68. We say that a formal scheme Y over Z is of finite presentation if there is an equaliser diagram in $\widehat{\mathcal{X}}_Z$ of the form

$$Y \to \mathbb{A}^n \times Z \rightrightarrows \mathbb{A}^m \times Z.$$

Theorem 4.69. Let X and Y be formal schemes over Z. Then $\mathrm{Map}_Z(X, Y)$ is a formal scheme if

(a) X is coalgebraic over Z and Y is relatively informal over Z, or
(b) X is finite and very flat over Z, or
(c) X is very flat over Z and Y is of finite presentation over Z.

This will be proved at the end of the section, after some auxiliary results.

Lemma 4.70. If Z' is a functor over Z then $\mathrm{Map}_{Z'}(X \times_Z Z', Y \times_Z Z') = \mathrm{Map}_Z(X, Y) \times_Z Z'$.

Proof. If W is a scheme over Z' then

$$\begin{aligned}
\mathcal{F}_{Z'}(W, \mathrm{Map}_Z(X, Y) \times_Z Z') &= \mathcal{F}_Z(W, \mathrm{Map}_Z(X, Y)) \\
&= \mathcal{F}_Z(W \times_Z X, Y) \\
&= \mathcal{F}_{Z'}(W \times_Z X, Y \times_Z Z') \\
&= \mathcal{F}_{Z'}(W \times_{Z'} (X \times_Z Z'), Y \times_Z Z').
\end{aligned}$$

Thus, $\mathrm{Map}_Z(X, Y) \times_Z Z'$ has the required universal property. $\square$

Lemma 4.71. If X is a strong colimit of formal schemes X_i and $\mathrm{Map}_Z(X_i, Y)$ is a formal scheme and is relatively informal over Z for all i then $\mathrm{Map}_Z(X, Y)$ is a formal scheme and is equal to $\varprojlim_i \mathrm{Map}_Z(X_i, Y)$ (where the inverse limit is computed in $\widehat{\mathcal{X}}_Z$).

Note that coproducts and filtered colimits are always strong, so the lemma applies in those cases.

Proof. Because $\mathrm{Map}_Z(X_i, Y)$ is relatively informal, Proposition 4.29 allows us to form the limit $\varprojlim_i \mathrm{Map}_Z(X_i, Y)$ in $\widehat{\mathcal{X}}_Z$. If W is a formal scheme over Z then we have

$$\begin{aligned}
\widehat{\mathcal{X}}_Z(W, \varprojlim_i \mathrm{Map}_Z(X_i, Y)) &= \varprojlim_i \widehat{\mathcal{X}}_Z(W, \mathrm{Map}_Z(X_i, Y)) \\
&= \varprojlim_i \widehat{\mathcal{X}}_Z(W \times_Z X_i, Y) \\
&= \widehat{\mathcal{X}}_Z(\varinjlim_i W \times_Z X_i, Y) \\
&= \widehat{\mathcal{X}}_Z(W \times_Z X, Y).
\end{aligned}$$

This proves that $\varprojlim_i \mathrm{Map}_Z(X_i, Y) = \mathrm{Map}_Z(X, Y)$ as required. □

We leave the next lemma to the reader.

Lemma 4.72. Suppose that Y is an inverse limit of a finite diagram of formal schemes $\{Y_i\}$ over Z. Then $\mathrm{Map}_Z(X, Y) = \varprojlim_i \mathrm{Map}_Z(X, Y_i)$, where the limit is computed in $\mathcal{F}_Z$. Thus, if $\mathrm{Map}_Z(X, Y_i)$ is a formal scheme for all i, then $\mathrm{Map}_Z(X, Y)$ is a formal scheme. □

Lemma 4.73. Let $\{Z_i\}$ be a filtered system of informal schemes with colimit Z. Let X and Y be formal schemes over Z, with $X_i = X \times_Z Z_i$ and $Y_i = Y \times_Z Z_i$. If $\mathrm{Map}_{Z_i}(X_i, Y_i)$ is a formal scheme for all i then $\mathrm{Map}_Z(X, Y)$ is a formal scheme and is equal to $\varinjlim_i \mathrm{Map}_{Z_i}(X_i, Y_i)$.

Proof. Lemma 4.70 tells us that the system of formal schemes

$$M_i = \mathrm{Map}_{Z_i}(X_i, Y_i)$$

defines an object of the category $\widehat{\mathcal{X}}_{\{Z_i\}}$ of Proposition 4.27. Thus, if we define $M = \varinjlim_i M_i$ we find that $\widehat{\mathcal{X}}_Z(W, M)$ is the set of maps of diagrams $\{W \times_Z Z_i\} \to \{M_i\}$ over $\{Z_i\}$. This is the same as the set of maps of diagrams $\{W \times_Z X_i\} = \{W \times_Z Z_i \times_{Z_i} X_i\} \to \{Y_i\}$ over $\{Z_i\}$. By the adjunction in Proposition 4.27, this is the same as the set of maps $W \times_Z X = \varinjlim_i W \times_Z X_i \to Y$ over Z. Thus, M has the defining property of $\mathrm{Map}_Z(X, Y)$. □

Lemma 4.74. Let X be relatively informal over Z, and let $\{Y_i\}$ be a filtered system of formal schemes over Z with colimit Y. If $\mathrm{Map}_Z(X, Y_i)$ is a formal scheme for all i, then $\mathrm{Map}_Z(X, Y)$ is a formal scheme and is equal to $\varinjlim_i \mathrm{Map}_Z(X, Y_i)$.

Proof. Write $M = \varinjlim_i \mathrm{Map}_Z(X, Y_i)$. Let W be an informal scheme over Z. As X is relatively informal, we see that $W \times_Z X$ is informal. It follows that the functors $\widehat{\mathcal{X}}_Z(W, -)$ and $\widehat{\mathcal{X}}_Z(W \times_Z X, -)$ preserve filtered colimits. We thus have

$$\begin{aligned}
\widehat{\mathcal{X}}_Z(W, M) &= \varinjlim_i \widehat{\mathcal{X}}_Z(W, \mathrm{Map}_Z(X, Y_i)) \\
&= \varinjlim_i \widehat{\mathcal{X}}_Z(W \times_Z X, Y_i) \\
&= \widehat{\mathcal{X}}_Z(W \times_Z X, \varinjlim_i Y_i) \\
&= \widehat{\mathcal{X}}_Z(W \times_Z X, Y),
\end{aligned}$$

as required. □

Lemma 4.75. If X and Z are informal and X is very flat over Z then the functor $\mathrm{Map}_Z(X, \mathbb{A}^1 \times Z)$ is a formal scheme.

Proof. We can choose a basis for $\mathcal{O}_X$ over $\mathcal{O}_Z$ and thus write $\mathcal{O}_X$ as a filtered colimit of finitely generated free modules M_i over $\mathcal{O}_Z$. From the definitions we see that $\mathrm{Map}_Z(X, \mathbb{A}^1)(R)$ is the set of pairs (x, u), where $x \in X(R)$ (making R into an $\mathcal{O}_X$-algebra) and u is a map $R[t] \to R \otimes_{\mathcal{O}_Z} \mathcal{O}_X$ of R-algebras. This is of course equivalent to an element of $R \otimes_{\mathcal{O}_Z} \mathcal{O}_X = \varinjlim_i R \otimes_{\mathcal{O}_Z} M_i$. Thus, we see that $\mathrm{Map}_Z(X, \mathbb{A}^1 \times Z) = \varinjlim_i \mathbb{A}(M_i)$, which is a formal scheme. $\square$

Proof of Theorem 4.69. We shall prove successively that $\mathrm{Map}_Z(X, Y)$ is a formal scheme under any of the following hypotheses. Cases (3), (5) and (7) give the results claimed in the theorem.

(1) X, Y and Z are informal, and X is finite and very flat. In this case $\mathrm{Map}_Z(X, Y)$ is informal.
(2) Y is informal, and X is finite and very flat. In this case $\mathrm{Map}_Z(X, Y)$ is relatively informal.
(3) X is finite and very flat.
(4) Y and Z are informal, and X is coalgebraic. In this case, $\mathrm{Map}_Z(X, Y)$ is informal.
(5) Y is relatively informal, and X is coalgebraic. In this case, $\mathrm{Map}_Z(X, Y)$ is relatively informal.
(6) X and Z are informal, X is very flat, and Y is of finite presentation.
(7) X is very flat and Y is of finite presentation.

Proposition 2.94 gives case (1). For case (2), write $Z = \varinjlim_i Z_i$ in the usual way. Then case (1) tells us that $\mathrm{Map}_{Z_i}(X \times_Z Z_i, Y \times_Z Z_i)$ is an informal scheme. Using this and Lemma 4.73, we see that $\mathrm{Map}_Z(X, Y)$ is a formal scheme. Using case (1) and Lemma 4.70 we see that $\mathrm{Map}_Z(X, Y)$ is relatively informal. In case (3), we write Y as a filtered colimit of informal schemes Y_j. Case (2) tells us that $\mathrm{Map}_Z(X, Y_j)$ is a relatively informal scheme, so Lemma 4.74 tells us that $\mathrm{Map}_Z(X, Y)$ is a formal scheme. In case (4), it follows easily from the definitions that X can be written as the filtered colimit of a system of finite, very flat schemes X_i. It then follows from case (1) that $\mathrm{Map}_Z(X_i, Y)$ is an informal scheme. Using Lemma 4.71 we see that $\mathrm{Map}_Z(X, Y) = \varprojlim_i \mathrm{Map}_Z(X_i, Y)$. This is an inverse limit of informal schemes, and thus is an informal scheme. We deduce (5) from (4) in the same way that we deduced (2) from (1). Case (6) follows easily from Lemmas 4.75 and 4.72. Again, the argument for (1)⇒(2) also gives (6)⇒(7). $\square$

5. Formal curves

In this section, we define formal curves. We also study divisors, differentials, and meromorphic functions on such curves.

Let X be a formal scheme, and let C be a formal scheme over X. We say that C is a *formal curve* over X if it is isomorphic in $\widehat{\mathcal{X}}_X$ to $\widehat{\mathbb{A}}^1 \times X$. (In some sense, it would be better to allow formal schemes that are only isomorphic to $\widehat{\mathbb{A}}^1 \times X$ fpqc-locally on X, but this seems unnecessary for the topological applications so we omit it.) A *coordinate* on C is a map $x \colon C \to \widehat{\mathbb{A}}^1$ giving rise to an isomorphism $C \simeq \widehat{\mathbb{A}}^1 \times X$.

Example 5.1. If E is an even periodic ring spectrum then $(\mathbb{C}P^\infty)_E$ and $(\mathbb{H}P^\infty)_E$ are formal curves over S_E.

5.1. Divisors on formal curves.

Let C be a formal curve over X, and let D be a closed subscheme of X. If X is informal, we say that D is a *effective divisor of degree n on C* if D is informal, and $\mathcal{O}_D$ is a free module of rank n over $\mathcal{O}_X$. If X is a general formal scheme, we say that D is a divisor if $D \times_X X'$ is a divisor on $C \times_X X'$, for all informal schemes X' over X. If Y is a formal scheme over X, we refer to divisors on $C \times_X Y$ as divisors on C over Y.

Proposition 5.2. There is a formal scheme $\mathrm{Div}_n^+(C)$ over X such that maps $Y \to \mathrm{Div}_n^+(C)$ over X biject with effective divisors of degree n on C over Y. Moreover, a choice of coordinate on C gives rise to an isomorphism $\mathrm{Div}_n^+(C) \simeq \widehat{\mathbb{A}}^n \times X$.

Proof. This is much the same as Example 2.10. We define

$$\mathrm{Div}_n^+(C)(R) = \{(a, D) \mid a \in X(R) \text{ and } D \text{ is an effective divisor of degree } n \text{ on } C_a\}.$$

We make this a functor by pullback, just as in Example 2.10. To see that $\mathrm{Div}_n^+(C)$ is a formal scheme, choose a coordinate x on C. Given a point (a, D) as above, we find that $C_a = C \times_X \mathrm{spec}(R) = \mathrm{spf}(R[\![x]\!])$, where the topology on $R[\![x]\!]$ is defined by the ideals (x^k). We know that D is a closed subscheme of C_a, and that D is informal. It follows that $D = \operatorname{spec}(R[\![x]\!]/J)$ for some ideal J such that $x^k \in J$ for some k. Let $\lambda(x)$ be the endomorphism of $\mathcal{O}_D$ given by multiplication by x, and let $f_D(t) = \sum_{i=0}^n a_i(D)t^{n-i}$ be the characteristic polynomial of $\lambda(x)$. As $x^k \in J$, we see that $\lambda(x)^k = 0$. If R is a field, then we deduce by elementary linear algebra that $f_D(t) = t^n$. If $\mathfrak{p}$ is a prime ideal in R then by considering the divisor $\operatorname{spec}(\kappa(\mathfrak{p})) \times_{\operatorname{spec}(R)} D$, we conclude that $f_D(t) = t^n \pmod{\mathfrak{p}[t]}$. Using Proposition 2.37, we deduce that $a_i(D) \in \operatorname{Nil}(R)$ for $i > 0$. Thus, the a_i's give a map $\operatorname{Div}_n^+(C) \to \widehat{\mathbb{A}}^n \times X$. As in Example 2.10, the Cayley-Hamilton theorem tells us that $f_D(x) \in J$ and thus that $\mathcal{O}_D = R[x]/f_D(x) = R[\![x]\!]/f_D(x)$.

Conversely, suppose we have elements $b_0, \ldots, b_n$ with $b_0 = 1$ and $b_i \in \operatorname{Nil}(R)$ for $i > 0$ and we define $g(t) = \sum_i b_i t^{n-i}$ and $D = \operatorname{spf}(R[\![x]\!]/g(x))$. In $\mathcal{O}_D$ we have $x^n = -\sum_{i>0} b_i x^{n-i}$, which is nilpotent, so x is nilpotent, so $(g(x))$ is open in $R[\![x]\!]$. This means that D is informal and that $\mathcal{O}_D = R[x]/g(x)$, which is easily seen to be a free module of rank n over R. Thus, D is an effective divisor of rank n on C_a. We conclude that $\operatorname{Div}_n^+(C)$ is isomorphic to $\widehat{\mathbb{A}}^n$, and in particular is a formal scheme.

If Y is an arbitrary formal scheme over X, we can choose a presentation $Y = \varinjlim_i Y_i$, so Y_i is an informal scheme over X. The above tells us that maps $Y_i \to \operatorname{Div}_n^+(C)$ over X biject with effective divisors of degree n on C over Y_i. Thus, maps $Y \to \operatorname{Div}_n^+(C)$ over X biject with systems of divisors D_i over Y_i, such that for each map $Y_i \to Y_j$ we have $D_i = D_j \times_{Y_j} Y_i$. Using Proposition 4.27, we see that these biject with effective divisors of degree n on C over Y. □

Example 5.3. It is essentially well-known that $BU(n)_E = \operatorname{Div}_n^+(G_E)$, where $G_E = (\mathbb{C}P^\infty)_E$. A proof will be given in Section 8.

Remark 5.4. It is not hard to check that for any map $Y \to X$ of formal schemes and any formal curve C over X we have $\operatorname{Div}_n^+(C \times_X Y) = \operatorname{Div}_n^+(C) \times_X Y$ (because both sides represent the same functor $\widehat{\mathcal{X}}_Y \to \text{Sets}$).

Definition 5.5. Let D be an effective divisor on a curve C over X. We shall define an associated line bundle $J(D)$ over C. By Corollary 4.48, it is enough to do this in a sufficiently natural way when X is an informal scheme. In that case we have $\mathcal{O}_D = \mathcal{O}_C/J(D)$ for some ideal $J(D)$ in $\mathcal{O}_C$. In terms of a coordinate x, we see from the proof of Proposition 5.2 that $J(D)$ is generated by a monic polynomial $f(x)$ whose lower coefficients are nilpotent. Thus $f(x) = x^n - g(x)$ where $g(x)^k = 0$ say. If $fh = 0$ then $x^{nk}h = g^k h = 0$ so $h = 0$, so f is not a zero-divisor and $J(D)$ is free of rank one over $\mathcal{O}_C$. Thus, $J(D)$ can be regarded as a line bundle over C as required (using Remark 4.45).

Proposition 5.6. There is a natural commutative and associative addition $\sigma\colon \operatorname{Div}_j^+(C) \times_X \operatorname{Div}_k^+(C) \to \operatorname{Div}_{j+k}^+(C)$, such that $J(D + E) = J(D) \otimes J(E)$.

Proof. Let $a\colon \operatorname{spec}(R) \to X$ be an element of $X(R)$, and let D and E be effective divisors of degrees j and k on C_a over $\operatorname{spec}(R)$. We then have $D = V(J(D))$ and $E = V(J(E))$ where $J(D)$ and $J(E)$ are ideals in $\mathcal{O}_{C_a}$. We define $F = V(J(D)J(E))$. If we choose a coordinate x on C we find (as in the proof of Proposition 5.2) that $J(D) = (f_D(x))$ and $J(E) = (f_E(x))$, where f_D and f_E are monic polynomials whose lower coefficients are nilpotent. This means that $g = f_D f_E$ is a polynomial of the same type, and it follows that $F = V(g)$ is a divisor of degree $j + k$ as required. We define $\sigma(D, E) = D + E = F$. It is clear from the construction that $J(D + E) = J(D) \otimes J(E)$. □

Proposition 5.7. Let C be a formal curve over a formal scheme X. Then $\operatorname{Div}_n^+(C) = C_X^n/\Sigma_n$, and this is a strong colimit. Moreover, the quotient map $C_X^n \to C_X^n/\Sigma_n$ is faithfully flat.

Proof. First consider the case $n = 1$. Fix a ring R and a point $a \in X(R)$, and write $C_a = C \times_X \operatorname{spec}(R)$, which is a formal curve over $Y = \operatorname{spec}(R)$. A point $c \in C$ lying over a is the same as a section of the projection $C_a \to Y$. Such a section is a split monomorphism, and thus a closed inclusion; we write $[c]$ for its image, which is a closed formal subscheme of C_a. The projection $C_a \to Y$ carries $[c]$ isomorphically to Y, which shows that $[c]$ is an effective divisor of degree 1 on C over Y. Thus, this construction gives a map $C \to \operatorname{Div}_1^+(C)$. If x is a coordinate on C then it is easy to see that $x(c) \in \operatorname{Nil}(R)$ and $[c] = \operatorname{spf}(R[\![x]\!]/(x - x(c)))$. Using this, we see easily that our map is an isomorphism, giving the case $n = 1$ of the Proposition.

We now use the iterated addition map $C_X^n = \operatorname{Div}_1^+(C)_X^n \to \operatorname{Div}_n^+(C)$ to get a map $C_X^n/\Sigma_n \to \operatorname{Div}_n^+(C)$.

Next, because $C \simeq \widehat{\mathbb{A}}^1 \times X$, it is easy to see that C is coalgebraic over X and thus (by Example 4.65) that C^n_X/Σ_n is a strong colimit. Given this, we can reduce easily to the case where X is informal, say $X = \operatorname{spec}(R)$. Choose a coordinate x on C. This gives isomorphisms $\mathcal{O}_{\mathrm{Div}^+_n(C)} = R[\![a_1, \dots, a_n]\!] = S$ and $\mathcal{O}_{C^n_X} = R[\![x_1, \dots, x_n]\!] = T$ and $\mathcal{O}_{C^n_X/\Sigma_n} = T^{\Sigma_n}$. The claim is thus that the map $S \to T^{\Sigma_n}$ is an isomorphism, and that T is faithfully flat over T^{Σ_n}. The map $S \to T^{\Sigma_n}$ sends a_i to the coefficient of x^{n-i} in $\prod_j (x - x_j)$, which is (up to sign) the i'th elementary symmetric function of the variables x_j. It is thus a well-known theorem of Newton that $S = T^{\Sigma_n}$. It is also well-known that the elements of the form $\prod_{j=1}^n x_j^{d_j}$ with $0 \le d_j < j$ form a basis for T over T^{Σ_n}, so that T is indeed faithfully flat over T^{Σ_n}. $\square$

We next consider pointed curves, in other words curves C equipped with a specified "zero-section" $0\colon X \to C$ such that the composite $X \xrightarrow{0} C \to X$ is the identity. If C is such a curve and x is a coordinate on C, we say that x is *normalised* if $x(0) = 0$. If y is an unnormalised coordinate then $x = y - y(0)$ is a normalised one, so normalised coordinates always exist.

Definition 5.8. Let C be a pointed formal curve over X. Define

$$f\colon \mathrm{Div}^+_n(C) \to \mathrm{Div}^+_{n+1}(C)$$

by $f(D) = D + [0]$. For $n \in \mathbb{Z}$ with $n < 0$ we write $\mathrm{Div}^+_n(C) = \emptyset$. Define

$$\begin{aligned}
\mathrm{Div}^+(C) &= \coprod_{n \ge 0} \mathrm{Div}^+_n(C) \\
\mathrm{Div}_n(C) &= \varinjlim(\mathrm{Div}^+_n(C) \xrightarrow{f} \mathrm{Div}^+_{n+1}(C) \xrightarrow{f} \dots) \\
\mathrm{Div}(C) &= \coprod_{n \in \mathbb{Z}} \mathrm{Div}_n(C) \\
&= \varinjlim(\mathrm{Div}^+(C) \xrightarrow{f} \mathrm{Div}^+(C) \xrightarrow{f} \dots).
\end{aligned}$$

It is not hard to see that f^k induces an isomorphism $\mathrm{Div}_n(C) \simeq \mathrm{Div}_{n+k}(C)$, so $\mathrm{Div}(C)$ can be identified with $\coprod_n \mathrm{Div}_0(C) = \underline{\mathbb{Z}} \times \mathrm{Div}_0(C)$.

A choice of normalised coordinate on C gives an isomorphism $\mathrm{Div}^+_n(C) \simeq X \times \widehat{\mathbb{A}}^n$. Under this identification, f becomes the map

$$(x, a_1, \dots, a_n) \mapsto (x, a_1, \dots, a_n, 0).$$

We thus have an isomorphism $\mathrm{Div}_0(C) = \widehat{\mathbb{A}}^{(\infty)}$ (using the notation of Example 4.4) and thus $\mathrm{Div} = \underline{\mathbb{Z}} \times \widehat{\mathbb{A}}^{(\infty)}$.

Definition 5.9. Given a divisor D on a pointed curve C over X, we define the *Thom sheaf* of D to be the line bundle $L(D) = 0^* J(D)$ over X. It is clear that $L(D + E) = L(D) \otimes L(E)$. Note that a coordinate on C gives a generator $f_D(x)$ for $J(D)$ and thus a generator u_D for $L(D)$, which we call the *Thom class*. This is natural for maps of X, and satisfies $u_{D+E} = u_D \otimes u_E$.

Definition 5.10. If C is a pointed formal curve over X, we define a functor $\mathrm{Coord}(C) \in \mathcal{F}_X$ by

$$\mathrm{Coord}(C)(R) = \{(a, x) \mid a \in X(R) \text{ and } x \text{ is a normalised coordinate on } C_a \ \}.$$

Proposition 5.11. The functor $\mathrm{Coord}(C)$ is a formal scheme over X, and is unnaturally isomorphic to $\mathbb{G}_m \times \mathbb{A}^\infty \times X$.

Proof. Choose a normalised coordinate x on C, and suppose that $a \in X(R)$. Then any normalised function $y\colon C_a \to \widehat{\mathbb{A}}^1$ has the form

$$y(c) = f(x(c)) = \sum_{k > 0} u_k x(c)^k$$

for a uniquely determined sequence of coefficients u_k. Moreover, y is a coordinate if and only if $f\colon \widehat{\mathbb{A}}^1 \times \operatorname{spec}(R) \to \widehat{\mathbb{A}}^1 \times \operatorname{spec}(R)$ is an isomorphism, if and only if there is a power series g with $g(f(t)) = t = f(g(t))$.

It is well-known that this happens if and only if u_1 is invertible. Thus, the set of coordinates on C_a bijects naturally with $(\mathbb{G}_m \times \mathbb{A}^\infty)(R)$, and $\mathrm{Coord}(C) \simeq \mathbb{G}_m \times \mathbb{A}^\infty \times X$ is a formal scheme, as required. $\square$

Remark 5.12. We will see later that when E is an even periodic ring spectrum and $G_E = (\mathbb{C}P^\infty)_E$ we have $\mathrm{Coord}(G_E) = \mathrm{spec}(E_0 MP)$.

5.2. **Weierstrass preparation.**

Definition 5.13. A *Weierstrass series* over a ring R is a formal power series $g(x) = \sum_k a_k x^k \in R[\![x]\!]$ such that there exists an integer n such that a_k is nilpotent for $k < n$, and a_n is a unit. The integer n is called the *Weierstrass degree* of $g(x)$. (It is clearly well-defined unless $R = 0$). A *Weierstrass polynomial* over a ring R is a monic polynomial $h(x) = \sum_{k=0}^n b_k x^k$ such that b_k is nilpotent for $k < n$.

The following result is a version of the Weierstrass Preparation Theorem; see [6, Theorem 3] (for example) for a more classical version.

Lemma 5.14. Let R be a ring, and let $g(x)$ be a Weierstrass series over R, of Weierstrass degree n. Then there is a unique ring map $\alpha\colon R[\![y]\!] \to R[\![x]\!]$ sending y to $g(x)$, and this makes $R[\![x]\!]$ into a free module over $R[\![y]\!]$ with basis $\{1, x, \dots, x^{n-1}\}$.

Proof. We can easily reduce to the case where $a_n = 1$. It is also easy to check that there is a unique map α sending y to $g(x)$, and that it sends any series $\sum_j b_j y^j$ to the sum $\sum_j b_j g(x)^j$, which is x-adically convergent.

For any $j \geq 0$ and $0 \leq k < n$ we define $z_{jk} = g(x)^j x^k$. Given any $m \geq 0$ we can write $m = nj + k$ for some $j \geq 0$ and $0 \leq k < n$, and we put $w_m = z_{jk}$. For any R-module M, we define a map

$$\beta_M\colon \prod_m M \to M[\![x]\!]$$

by $\beta_M(b) = \sum_m b_m w_m$. It is easy to check that this sum is again x-adically convergent. The claim in the lemma is equivalent to the statement that β_R is an isomorphism.

Write $I = (a_0, \dots, a_{n-1})$. This is finitely generated, so the same is true of I^r for all r, and it follows that $I^r \prod_m M = \prod_m I^r M$ and so on. We also see that $w_m = x^m \pmod{I, x^{m+1}}$.

Now consider a module M with $IM = 0$, so that $bw_m = bx^m \pmod{x^{m+1}}$ for $b \in M$. Given any series $c(x) = \sum_m c_m x^m \in M[\![x]\!]$, we see by induction on m that there is a unique sequence (b_j) such that $\sum_{j<m} b_m w_m = c(x) \pmod{x^m}$ for all m. It follows that β_M is an isomorphism whenever $IM = 0$. Next, whenever we have a short exact sequence $L \rightarrowtail M \twoheadrightarrow N$ we have short exact sequences $\prod_m L \rightarrowtail \prod_m M \twoheadrightarrow \prod_m N$ and $L[\![x]\!] \rightarrowtail M[\![x]\!] \twoheadrightarrow N[\![x]\!]$, and we can use the five-lemma to see that β_M is iso if β_L and β_N are. Using this we see by induction that β_{R/I^r} is iso for all r. On the other hand, when R is large we have $I^r = 0$ and so β_R is an isomorphism. $\square$

Corollary 5.15. In the situation of the lemma, the quotient ring $R[\![x]\!]/g(x)$ is a free module of rank n over R, with basis $\{1, \dots, x^{n-1}\}$. $\square$

Corollary 5.16. If $g(x)$ is a Weierstrass series over a ring R then there is a unique factorisation $g(x) = h(x)u(x)$, where $h(x)$ is a Weierstrass polynomial, and $u(x)$ is invertible.

Proof. By the previous corollary, we have $-x^n = \sum_{j=0}^{n-1} b_j x^j \pmod{g(x)}$ for some unique sequence $b_0, \dots, b_{n-1} \in R$. Put $h(x) = x^n + \sum_j b_j x^j$, so h is a monic polynomial of degree n with $h(x) = 0 \pmod{g(x)}$, say $h(x) = g(x)v(x)$. Now write $g(x)$ in the form $\sum_k a_k x^k$ and put $I = (a_0, \dots, a_{n-1})$, so I is a nilpotent ideal. Modulo I we find that $g(x)$ is a unit multiple of x^n, so $x^n = 0 \pmod{I, g(x)}$. The uniqueness argument applied mod I now tells us that $h(x) = x^n \pmod{I}$, so $h(x)$ is a Weierstrass polynomial. It is also clear that $v(x)$ becomes a unit mod $I[\![x]\!]$, but $I[\![x]\!]$ is nilpotent so $v(x)$ is a unit. We can thus take $u(x) = 1/v(x)$ to get the required factorisation. $\square$

We now give a more geometric restatement of the above results.

Definition 5.17. Let $C \xrightarrow{q} X$ and $D \xrightarrow{r} X$ be formal curves over a formal scheme X, and let $f\colon C \to D$ be a map over X. We then have a curve $r^*C = C \times_X D$ over D, with projection map $s\colon (c, d) \mapsto d$. We also have a map $f'\colon C \to r^*C$ of formal schemes over D, given by $f'(c) = (c, f(c))$. We say that f is an *isogeny* if the map f' makes C into a divisor on r^*C over D. This implies in particular that f is finite and very flat.

Lemma 5.18. Let X be an informal scheme, and let $f\colon C \to D$ be a map of formal curves over X. Let x and y be coordinates on C and D respectively, and suppose that $f^*y = g(x)$ for some Weierstrass series $g(x)$. Then f is an isogeny.

Proof. Write $R = \mathcal{O}_X$, and let n be the Weierstrass degree of $g(x)$. We then have $C = \operatorname{spf}(R[\![x]\!])$ and $D = \operatorname{spf}(R[\![y]\!])$ and $r^*C = \operatorname{spf}(R[\![x,y]\!])$. In this last case we think of x as the coordinate on r^*C and y as a parameter on the base. The map f' corresponds to the map $\alpha\colon R[\![x,y]\!] \to R[\![x]\!]$ that sends x to x and y to $g(x)$. We thus need to show that α is surjective (making f' a closed inclusion) and that it makes $A[\![x]\!]$ into a free module of rank n over $A[\![y]\!]$. The surjectivity is clear, and the freeness follows from Lemma 5.14. □

Example 5.19. One can check that the evident map $\mathbb{C}P^\infty \to \mathbb{H}P^\infty$ gives an isogeny $(\mathbb{C}P^\infty)_E \to (\mathbb{H}P^\infty)_E$ of formal curves.

Definition 5.20. Let X be an informal scheme, and C a formal curve over X. We then let $\mathcal{M}_{C/X}$ be the ring obtained from $\mathcal{O}_C$ by inverting all coordinates on C. We refer to this as the ring of *meromorphic functions* on C.

Lemma 5.21. Let X be an informal scheme, and C a formal curve over X, and x a coordinate on C. Then $\mathcal{M}_{C/X} = \mathcal{O}_C[1/x]$, which is the ring of series $\sum_{k\in\mathbb{Z}} a_k x^k$ such that $a_k \in \mathcal{O}_X$ and $a_k = 0$ for $k \ll 0$.

Proof. Let y be another coordinate on C; it will suffice to check that y becomes invertible in $\mathcal{O}_C[1/x]$. As x and y are both coordinates, we find that $y = \sum_{k\geq 0} a_k x^k$ for some series such that a_0 is nilpotent and a_1 is a unit. In other words, we have $y = b + xc(x)$, where b is nilpotent and $c(0)$ is invertible in $\mathcal{O}_X$, so $c(x)$ is invertible in $\mathcal{O}_C$. It is thus clear that $y - b$ has inverse $x^{-1}c(x)^{-1}$ in $\mathcal{O}_C[1/x]$. The sum of a unit and a nilpotent element is always invertible, so y is a unit as required. □

Remark 5.22. The elements of $\mathcal{M}_{C/X}$ should be thought of as Laurent expansions of functions whose poles are all very close to the origin, the expansion being valid outside a small disc containing all the poles.

Lemma 5.23. Let x be a coordinate on C, and let $f(x) = \sum_{k\in\mathbb{Z}} a_k x^k$ be an element of $\mathcal{M}_{C/X}$. Then f is invertible in $\mathcal{M}_{C/X}$ if and only if X can be written as a coproduct $X = \coprod_{k\in\mathbb{Z}} X_k$, where $X_k = \emptyset$ for almost all k, such that a_j is nilpotent on X_k for $j < k$, and a_k is invertible on X_k.

Proof. First suppose that $f(x)$ is invertible, say $f(x)g(x) = 1$ with $g(x) = \sum_{j\in\mathbb{Z}} b_j x^j$. Write $I_j = (a_j b_{-j})$ and $J_j = \sum_{k\neq j} I_j$. Because $f(x)g(x) = 1$ it is clear that $\sum_j I_j = \mathcal{O}_X$ and thus $J_i + J_j = \mathcal{O}_X$ when $i \neq j$. There exists K such that $a_{-j} = b_{-j} = 0$ when $j > K$. It follows that $I_j = 0$ and $J_j = \mathcal{O}_X$ when $|j| > K$. Next, let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_X$. As $\mathcal{O}_X/\mathfrak{p}$ is an integral domain, it is clear that modulo $\mathfrak{p}$ we must have $f(x) = a_k x^k + \ldots$ and $g(x) = b_{-k}x^{-k} + \ldots$ for some k. This implies that $a_i b_j \in \mathfrak{p}$ whenever $i + j < 0$. As the intersection of all prime ideals is the set of nilpotents, the elements $a_i b_j$ must be nilpotent when $i + j < 0$. If $i \neq j$ then either $i - j$ or $j - i$ is negative, so $a_i b_{-i} a_j b_{-j}$ is nilpotent. It follows that $I_i I_j$ is nilpotent when $i \neq j$, and thus that $\bigcap_j J_j$ is nilpotent. It follows from the results of Section 2.5 that there are unique ideals J'_j such that $J_j \leq J'_j \leq \sqrt{J_j}$ and $\mathcal{O}_X = \prod_j \mathcal{O}_X/J'_j$. We take $X_j = \operatorname{spec}(\mathcal{O}_X/J'_j)$; one can check that this has the claimed properties.

Conversely, suppose that X has a decomposition of the type discussed. We reduce easily to the case where $X = X_k$ for some k. After replacing f by $x^{-k}f$, we may assume that $k = 0$. This means that $f(x) = \sum_{j\in\mathbb{Z}} a_j x^j$, where a_0 is invertible and a_j is nilpotent for $j < 0$ and $a_j = 0$ for $j \ll 0$. The invertibility or otherwise of f is unaffected if we subtract off a nilpotent term, so we may assume that $a_j = 0$ for $j < 0$. The resulting series is invertible in $\mathcal{O}_C$ and thus certainly in $\mathcal{M}_{C/X}$. □

Definition 5.24. Let x be a coordinate on C, and let f be an invertible element of $\mathcal{M}_{C/X}$, so we have a decomposition $X = \coprod_k X_k$ as above. If $X = X_k$ then we say that f has constant degree k. More generally, we let $\deg(f)$ be the map from X to the constant scheme $\underline{\mathbb{Z}}$ that takes the value k on X_k. One can check that these definitions are independent of the choice of coordinate.

Lemma 5.25. Let x be a coordinate on C, and let f be an invertible element of $\mathcal{M}_{C/X}$, with constant degree k. Then there is a unique factorisation $f(x) = x^k u(x) g(x)$, where $u(x) \in \mathcal{O}_C^\times$, and $g(x) = \sum_{j\geq 0} b_j x^{-j}$ where $b_0 = 1$ and b_j is nilpotent for $j > 0$ and $b_j = 0$ for $j \gg 0$.

Proof. Clearly we have $h(x) = x^N f(x) \in \mathcal{O}_C$ for some $N > 0$. We see from Lemma 5.23 that $h(x)$ is a Weierstrass power series of Weierstrass degree $N + k$. It follows from Corollary 5.16 that $h(x)$ has a unique factorisation of the form $h(x) = k(x)u(x)$, where $k(x)$ is a Weierstrass polynomial of degree $N + k$, and $u(x) \in \mathcal{O}_C^\times$. We write $g(x) = x^{-N-k}h(x)$; this clearly gives a factorisation of the required type, and one can check that it is unique. □

Proposition 5.26. Let C be a formal curve over a formal scheme X. For any ring R, we define

$$\mathrm{Mer}(C, \mathbb{G}_m)(R) = \{(u, f) \mid u \in X(R)\ ,\ f \in \mathcal{M}^\times_{C_u/\mathrm{spec}(R)}\}.$$

Then $\mathrm{Mer}(C, \mathbb{G}_m)$ is a formal scheme over X, and there is a short exact sequence of formal groups

$$\mathrm{Map}(C, \mathbb{G}_m) \rightarrowtail \mathrm{Mer}(C, \mathbb{G}_m) \twoheadrightarrow \mathrm{Div}(C),$$

which admits a non-canonical splitting.

Proof. As $\mathrm{Map}(C, \mathbb{G}_m)(R) = \{(u, f) \mid u \in X(R)\ ,\ f \in \mathcal{O}^\times_{C_u}\}$, there is an obvious inclusion $\mathrm{Map}(C, \mathbb{G}_m) \to \mathrm{Mer}(C, \mathbb{G}_m)$ of group-valued functors. Next, let $Y(R)$ be the set of series $g(x) = \sum_{j\geq 0} b_j x^{-j}$ such that $b_0 = 1$ and b_j is nilpotent for $j > 0$ and $b_j = 0$ for $j \gg 0$. Then $Y = \varinjlim_k \prod_{0<j<k} \widehat{\mathbb{A}}^1$ is a formal scheme, and Lemma 5.25 gives an isomorphism $\mathrm{Mer}(C, \mathbb{G}_m) \simeq \mathrm{Map}(C, \mathbb{G}_m) \times \underline{\mathbb{Z}} \times Y$. This shows that $\mathrm{Mer}(C, \mathbb{G}_m)$ is a formal scheme. We next define a map $\mathrm{div}\colon \mathrm{Mer}(C, \mathbb{G}_m) \to \mathrm{Div}(C)$. Suppose that $f \in \mathcal{O}_{C_u}$ is such that $\mathcal{O}_{C_u}/f$ is a free module of rank n over R. Then $D = \mathrm{spf}(\mathcal{O}_{C_u}/f) \in \mathrm{Div}_n(G)(R)$and we define $\mathrm{div}(f) = D$. Given another such function $g \in \mathcal{O}_{C_u}$, we define $\mathrm{div}(f/g) = \mathrm{div}(f) - \mathrm{div}(g)$. This is well-defined, because if $f/g = f'/g'$ then $fg' = f'g$ (because series of this form are never zero-divisors) and thus $\mathrm{div}(f) + \mathrm{div}(g') = \mathrm{div}(f') + \mathrm{div}(g)$ and so $\mathrm{div}(f) - \mathrm{div}(g) = \mathrm{div}(f') - \mathrm{div}(g')$. It is easy to see that we get a well-defined homomorphism $\mathrm{div}\colon \mathrm{Mer}(C, \mathbb{G}_m) \to \mathrm{Div}(C)$, which is zero on $\mathrm{Map}(C, \mathbb{G}_m)$. Conversely, suppose that $\mathrm{div}(f/g) = 0$, so that $\mathrm{div}(f) = \mathrm{div}(g)$. Then f and g are non-zero-divisors and they generate the same ideal in $\mathcal{O}_{C_u}$, so they are unit multiples of each other and thus $f/g \in \mathrm{Map}(C, \mathbb{G}_m)(R)$. Thus $\ker(\mathrm{div}) = \mathrm{Map}(C, \mathbb{G}_m)$.

Now let $j\colon C \to \mathrm{Div}(C)$ be the evident inclusion. Given a point $a \in C(R)$, we also define $\sigma(a) = x - x(a) = x(1 - x(a)/x) \in \mathrm{Mer}(C)(R)$. This gives a map $\sigma\colon C \to \mathrm{Mer}(C, \mathbb{G}_m)$, and it is clear that $\mathrm{div} \circ \sigma = j$. As $\mathrm{Div}(C)$ is the free Abelian formal group generated by C, we see that there is a unique homomorphism $\tau\colon \mathrm{Div}(C) \to \mathrm{Mer}(C, \mathbb{G}_m)$ with $\tau \circ j = \sigma$. We thus have $\mathrm{div} \circ \tau \circ j = j$ and thus $\mathrm{div} \circ \tau = 1$. It follows that the sequence $\mathrm{Map}(C, \mathbb{G}_m) \rightarrowtail \mathrm{Mer}(C, \mathbb{G}_m) \twoheadrightarrow \mathrm{Div}(C)$ is a split exact sequence. The splitting depends on a choice of coordinate, but the other maps are canonical. □

5.3. **Formal differentials.** We next generalise Definition 2.65 to a certain (rather small) class of formal schemes.

Definition 5.27. We say that a formal scheme W over X is *formally smooth* of dimension n over X if it is isomorphic in $\widehat{\mathcal{X}}_X$ to $\widehat{\mathbb{A}}^n \times X$. In particular, W is formally smooth of dimension one if and only if it is a formal curve.

Definition 5.28. Let W be formally smooth of dimension n over X; we shall define a vector bundle $\Omega^1_{W/X}$ of rank n over W. By Corollary 4.48, it suffices to do this in a sufficiently natural way whenever X is an informal scheme. In that case, we let J be the kernel of the multiplication map $\mathcal{O}_{W\times_X W} = \mathcal{O}_W \widehat{\otimes}_{\mathcal{O}_X} \mathcal{O}_W \to \mathcal{O}_W$, so that $V(J)$ is the diagonal subscheme in $W \times_X W$. We then write $\Omega^1_{W/X} = J/J^2$, which is a module over $\mathcal{O}_{W\times_X W}/J = \mathcal{O}_W$. If $f \in \mathcal{O}_W$ then we write $d(f) = f \otimes 1 - 1 \otimes f + J^2 \in \Omega^1_{W/X}$, and note that $d(fg) = fd(g) + gd(f)$ as usual. As W is formally smooth, we can choose $x_1, \dots, x_n \in \mathcal{O}_W$ giving an isomorphism $W \simeq \widehat{\mathbb{A}}^n \times X$ and thus $\mathcal{O}_W \simeq \mathcal{O}_X[\![x_1, \dots, x_n]\!]$. One checks that $\Omega^1_{W/X}$ is freely generated over $\mathcal{O}_W$ by $\{d(x_1), \dots, d(x_n)\}$. Thus, $\Omega^1_{W/X}$ can be regarded as a vector bundle of rank n over W, as required. We write $\Omega^k_{W/X}$ for the k'th exterior power of $\Omega^1_{W/X}$.

Any map $f\colon V \to W$ of formally smooth schemes over X gives rise to a map $f^*\colon f^*\Omega^1_{W/X} \to \Omega^1_{V/X}$ of vector bundles over V. If we have coordinates x_i on W and y_j on V then $x_i \circ f = u_i(y_1, \dots, y_d)$ for certain power series u_i over $\mathcal{O}_X$, and we have $f^*(dx_i) = \sum_j (\partial u_i/\partial y_j)dy_j$. Thus, f^* is a coordinate-free way of encoding the partial derivatives of the series u_i.

If W is formally smooth over X and $g\colon Y \to X$ then $g^*W = Y \times_X W$ is easily seen to be formally smooth over Y, with $\Omega^1_{g^*W/Y} = h^*\Omega^1_{W/X}$, where $h\colon g^*W \to W$ is the evident projection map.

Definition 5.29. If R is an $\mathbb{F}_p$-algebra, then we have a ring map ϕ_R from R to itself defined by $\phi_R(a) = a^p$. We call this the *algebraic Frobenius map*. Now let X be a functor over $\operatorname{spec}(\mathbb{F}_p)$. If R is an $\mathbb{F}_p$-algebra, we define $(F_X)_R = X(\phi_R)\colon X(R) \to X(R)$. If R is not an $\mathbb{F}_p$-algebra then $\operatorname{spec}(\mathbb{F}_p)(R) = \emptyset$ and thus $X(R) = \emptyset$ and we define $(F_X)_R = 1\colon X(R) \to X(R)$. This gives a map $F_X\colon X \to X$, which we call the *geometric Frobenius map*.

Remark 5.30. If $h\colon X \to Y$ is a map of functors over $\operatorname{spec}(\mathbb{F}_p)$ then one can check that $F_Y \circ h = h \circ F_X$. If X is a scheme then F_X is characterised by the fact that $g(F_X(a)) = g(a)^p$ for all rings R, points $a \in X(R)$, and functions $g \in \mathcal{O}_X$. If $X = \operatorname{spec}(A)$ then $F_X = \operatorname{spec}(\phi_A)$.

Definition 5.31. Let X be a functor over $\operatorname{spec}(\mathbb{F}_p)$, and let W be functor over X, with given map $q\colon W \to X$. We then have a functor F_X^*W over X defined by

$$(F_X^*W)(R) = \{(a,b) \in W(R) \times X(R) \mid q(a) = F_X(b)\}.$$

We define a map $F_{W/X}\colon W \to F_X^*W$ by $F_{W/X}(a) = (F_W(a), q(a))$. Note that if W is formally smooth over X then the same is true of F_X^*W. Moreover, if we have coordinates x_i on W and we use the obvious resulting coordinates y_i on F_X^*W then we have $y_i(F_{W/X}(a)) = x_i(a)^p$.

Proposition 5.32. Let $f\colon V \to W$ be a map of formally smooth schemes over X, and suppose that $f^* = 0\colon \Omega^1_{W/X} \to \Omega^1_{V/X}$.

(a) If X lies over $\operatorname{spec}(\mathbb{Q})$ then there is a unique map $g\colon X \to W$ of schemes over X such that f is the composite $V \to X \xrightarrow{g} W$. In other words, f is constant on the fibres of V.
(b) If X lies over $\operatorname{spec}(\mathbb{F}_p)$ for some prime p, then there is a unique map $f'\colon F_X^*W \to V$ such that $f = f' \circ F_{W/X}$.

Proof. Choose coordinates x_i on W and y_j on V, so $x_i \circ f = u_i(y_1, \dots, y_d)$ for certain power series u_i over $\mathcal{O}_X$. We have $0 = f^*(dx_i) = \sum_j (\partial u_i/\partial y_j) dy_j$, so $\partial u_i/\partial y_j = 0$ for all i and j. In the rational case we conclude that the series u_i are constant, and in the mod p case we conclude that $u_i(y_1, \dots, y_d) = v_i(y_1^p, \dots, y_d^p)$ for some unique series v_i. The conclusion follows easily. $\square$

5.4. **Residues.** We now describe an algebraic theory of residues, which is essentially the same as that discussed in [10] and presumably identical to the unpublished definition by Cartier mentioned in [19].

Definition 5.33. If $f(x) = \sum_{k\in\mathbb{Z}} a_k x^k \in R[\![x]\!][1/x]$, we define $\rho(f) = a_{-1}$.

Remark 5.34. Recall from Remark 5.22 that the elements of $R[\![x]\!][1/x]$ should be compared with meromorphic functions on a neighbourhood of zero in $\mathbb{C}$ of moderate size, whose poles are concentrated very near the origin. The expansion in terms of x should be thought of as a Laurent expansion that is valid outside a tiny disc containing all the poles. Thus, the coefficient of $1/x$ is the sum of the residues at all the poles, and not just the pole at the origin. To justify this, note that if a is nilpotent (say $a^N = 0$) we have $1/(x-a) = \sum_{k=0}^{N-1} a^k/x^{k+1}$ so $\rho(1/(x-a)) = 1$.

Proposition 5.35. For any $f \in R[\![x]\!][1/x]$ we have $\rho(f') = 0$. If f is invertible we have $\rho(f'/f) = \deg(f)$, where $\deg(f)$ is as in Definition 5.24. Moreover, we have $\rho(f^n.f') = 0$ for $n \neq -1$.

Proof. It is immediate from the definitions that $\rho(f') = 0$. Now let f be invertible; we may assume that it has constant degree d say. Lemma 5.25 gives a factorisation $f(x) = x^d u(x) g(x)$, where $u(x) \in R[\![x]\!]^\times$, and $g(x) = \sum_{j\geq 0} b_j x^{-j}$ where $b_0 = 1$ and b_j is nilpotent for $j > 0$ and $b_j = 0$ for $j \gg 0$. We then have $f'/f = d/x + u'/u + g'/g$. It is clear that $u'/u \in R[\![x]\!]$ so $\rho(u'/u) = 0$. Similarly, we find that g' only involves powers x^k with $k < -1$. Moreover, if $h(x) = 1 - g(x)$ then h is a polynomial in $1/x$ and is nilpotent, and $1/g = \sum_{k=0}^N h^k$ for some N so $1/g$ is a polynomial in $1/x$. It follows that g'/g only involves powers x^k with $k < -1$, so $\rho(g'/g) = 0$. Thus $\rho(f'/f) = d$ as claimed.

Finally, suppose that $n \neq -1$. Note that $(n+1)\rho(f^n.f') = \rho((f^{n+1})') = 0$. If R is torsion-free we conclude that $\rho(f^n.f') = 0$. If R is not torsion-free, we recall that $f(x)$ has the form $\sum_{i=m}^\infty a_i x^i$ for some m, where a_i is nilpotent for $i < d$ and a_d is invertible. Thus there is some $N > 0$ such that $a_i^N = 0$ for all $i < d$.

Define $R' = \mathbb{Z}[b_i \mid i \geq m][1/b_d]/(b_i^N \mid m \leq i < d)$ and $g(x) = \sum_i b_i x^i \in R'[\![x]\!][1/x]^\times$. It is clear that R' is torsion-free and thus that $\rho(g^n.g') = 0$. There is an evident map $R' \to R$ carying g to f, so we deduce that $\rho(f^n.f') = 0$ as claimed. □

Corollary 5.36. If $g(x) \in R[\![x]\!]$ is a Weierstrass series of degree $d > 0$ and $f(x) \in R[\![x]\!][1/x]$ then $\rho(f(g(x))g'(x)) = d\rho(f(x))$.

Proof. Suppose that $f(x) = \sum_{k \geq m} a_k x^k$. We first observe that the claim makes sense: as g is a Weierstrass series of degree $d > 0$ we know that $g(0)$ is nilpotent, so $g^N \in R[\![x]\!]x$ for some N, so $g^{Nk} \in R[\![x]\!]x^k$ for $k \geq 0$. Moreover, Lemma 5.23 implies that g is invertible in $R[\![x]\!][1/x]$. Thus, the terms in the sum $f(g(x)) = \sum_{k \geq m} a_k g(x)^k$ are all defined, and the sum is convergent. We thus have

$$\rho(f(g(x))) = \sum_k a_k \rho(g^k.g') = d\, a_{-1} = d\rho(f)$$

as required. □

Definition 5.37. Let C be a formal curve over an affine scheme X. We write $\mathcal{M}\Omega^1_{C/X}$ for $\mathcal{M}_{C/X} \otimes_{\mathcal{O}_C} \Omega^1_{C/X}$, which is a free module of rank one over $\mathcal{M}_{C/X}$. It is easy to check that there is a unique map $d\colon \mathcal{M}_{C/X} \to \mathcal{M}\Omega^1_{C/X}$ extending the usual map $d\colon \mathcal{O}_C \to \Omega^1_{C/X}$ and satisfying $d(fg) = f\,d(g) + g\,d(f)$.

Corollary 5.38. Let C be a formal curve over an affine scheme X. Then there is a natural residue map $\mathrm{res} = \mathrm{res}_{C/X} \colon \mathcal{M}\Omega^1_{C/X} \to \mathcal{O}_X$ such that

(a) $\mathrm{res}(df) = 0$ for all $f \in \mathcal{M}_{C/X}$.
(b) $\mathrm{res}((df)/f) = \deg(f)$ for all $f \in \mathcal{M}^\times_{C/X}$.
(c) If $q\colon C \to C'$ is an isogeny then $\mathrm{res}(q^*\alpha) = \deg(q)\mathrm{res}(\alpha)$ for all α.

Proof. Choose a coordinate x on C, so that any $\alpha \in \mathcal{M}_{C/X} \otimes_{\mathcal{O}_C} \Omega^1_{C/X}$ has a unique expression $\alpha = f(x)dx$ for some $f \in \mathcal{O}_X[\![t]\!][1/t]$. Define $\mathrm{res}(\alpha) = \rho(f)$. If y is a different coordinate then $x = g(y)$ for some Weirstrass series g of degree 1 and $dx = g'(y)dy$ so $\alpha = f(g(y))g'(y)dy$ and we know that $\rho(f(g(y))g'(y)) = \rho(f)$ so our definition is independent of the choice of the coordinate. The rest of the corollary is just a translation of the properties of ρ. □

See Remark 8.34 for a topological application of this.

6. Formal groups

A *formal group* over a formal scheme X is just a group object in the category $\widehat{\mathcal{X}}_X$. In this section, we will study formal groups in general. In the next, we will specialise to the case of commutative formal groups G over X with the property that the underlying scheme is a formal curve; we shall call these *ordinary formal groups*. For technical reasons, it is convenient to compare our formal groups with group objects in suitable categories of coalgebras. To combine these cases, we start with a discussion of Abelian group objects in an arbitrary category with finite products. We then discuss the existence of free Abelian formal groups, or of schemes of homomorphisms between formal groups. As a special case, we discuss the Cartier duality functor $G \mapsto \mathrm{Hom}(G, \mathbb{G}_m)$. Finally, we define torsors over a commutative formal group, and show that they form a strict Picard category.

6.1. Group objects in general categories.

Let $\mathcal{D}$ be a category with finite products (including an empty product, in other words a terminal object). There is an evident notion of an Abelian group object in $\mathcal{D}$; we write $\mathrm{Ab}\,\mathcal{D}$ for the category of such objects. We also consider the category $\mathrm{Mon}\,\mathcal{D}$ of Abelian monoids in $\mathcal{D}$. A *basepoint* for an object U of $\mathcal{D}$ is a map from the terminal object to U. We write $\mathrm{Based}\,\mathcal{D}$ for the category of objects of $\mathcal{D}$ equipped with a specified basepoint. There are evident forgetful functors

$$\mathrm{Ab}\,\mathcal{D} \to \mathrm{Mon}\,\mathcal{D} \to \mathrm{Based}\,\mathcal{D} \to \mathcal{D}.$$

If $U \in \mathcal{D}$ and $G \in \mathrm{Ab}\,\mathcal{D}$ then the set $\mathcal{D}(U, G)$ has a natural Abelian group structure. In fact, to give such a group structure is equivalent to giving maps $1 \xrightarrow{0} G \xleftarrow{\sigma} G \times G$ making it an Abelian group object, as one sees easily from Yoneda's lemma.

Let $\{G_i\}$ be a diagram in $\mathrm{Ab}\,\mathcal{D}$, and suppose that the underlying diagram in $\mathcal{D}$ has a limit G. Then $\mathcal{D}(U,G) = \varprojlim_i \mathcal{D}(U,G_i)$ has a natural Abelian group structure. It follows that there is a unique way to make G into an Abelian group object such that the maps $G \to G_i$ become homomorphisms, and with this structure G is also the limit in $\mathrm{Ab}\,\mathcal{D}$. In other words, the forgetful functor $\mathrm{Ab}\,\mathcal{D} \to \mathcal{D}$ creates limits. Similarly, we see that all the functors $\mathrm{Ab}\,\mathcal{D} \to \mathrm{Mon}\,\mathcal{D} \to \mathrm{Based}\,\mathcal{D} \to \mathcal{D}$ and their composites create limits.

Suppose that G, H and K are Abelian group objects in $\mathcal{D}$ and that $f\colon G \to K$ and $g\colon H \to K$ are homomorphisms. One checks that the composite $G \times H \xrightarrow{f\times g} K \times K \xrightarrow{\sigma} K$ is also a homomorphism, and that it is the unique homomorphism whose composites with the inclusions $G \to G \times H$ and $H \to G \times H$ are f and g. This means that $G \times H$ is the coproduct of G and H in $\mathrm{Ab}\,\mathcal{D}$, as well as being their product.

We next investigate another type of colimit in $\mathrm{Ab}\,\mathcal{D}$.

Definition 6.1. A *reflexive fork* in any category $\mathcal{D}$ is a pair of objects U, V, together with maps $d_0, d_1 : U \to V$ and $s\colon V \to U$ such that $d_0 s = 1 = d_1 s$. The *coequaliser* of such a fork means the coequaliser of the maps d_0 and d_1.

Proposition 6.2. Let $\mathcal{D}$ be a category with finite products. Let

$$V \xrightarrow{s} U \underset{d_1}{\overset{d_0}{\rightrightarrows}} V$$

be a reflexive fork in $\mathrm{Mon}\,\mathcal{D}$, and let $U \underset{d_1}{\overset{d_0}{\rightrightarrows}} V \xrightarrow{e} W$ be a strong coequaliser in $\mathcal{D}$. Then there is a monoid structure on W such that e is a homomorphism, and this makes the above diagram into a coequaliser in $\mathrm{Mon}\,\mathcal{D}$.

Proof. Let $\sigma_U : U \times U \to U$ and $\sigma_V : V \times V \to V$ be the addition maps. We have a commutative diagram as follows:

$$\begin{tikzcd}
V \times U \arrow[r, "s\times 1"] \arrow[d, shift right, "1\times d_0"'] \arrow[d, shift left, "1\times d_1"] & U \times U \arrow[r, "\sigma_U"] \arrow[d, shift right, "d_0\times d_0"'] \arrow[d, shift left, "d_1\times d_1"] & U \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
V \times V \arrow[r, equal] & V \times V \arrow[r, "\sigma_V"'] & V
\end{tikzcd}$$

The right hand square commutes because d_0 and d_1 are homomorphisms, and the left hand one because $d_0 s = d_1 s = 1$. Using this, we see that $e\sigma_V(1 \times d_0) = e\sigma_V(1 \times d_1)$, and a similar proof shows that $e\sigma_V(d_0 \times 1) = e\sigma_V(d_1 \times 1)$. In terms of elements, this just says that $e(d_0(u) + v) = e(d_1(u) + v)$. As our coequaliser diagram was assumed to be strong, we see that the diagram

$$V \times U \underset{1\times d_1}{\overset{1\times d_0}{\rightrightarrows}} V \times V \xrightarrow{1\times e} V \times W$$

is a coequaliser. This implies that there is a unique map $\tau\colon V \times W \to W$ with $\tau(1 \times e) = e\sigma_V : V \times V \to W$. Now consider the diagram

$$\begin{tikzcd}
U \times V \arrow[r, shift left, "d_0\times 1"] \arrow[r, shift right, "d_1\times 1"'] \arrow[d, "1\times e"'] & V \times V \arrow[r, "\sigma_V"] \arrow[d, "1\times e"] & V \arrow[d, "e"] \\
U \times W \arrow[r, shift left, "d_0\times 1"] \arrow[r, shift right, "d_1\times 1"'] & V \times W \arrow[r, "\tau"'] & W.
\end{tikzcd}$$

We have already seen that $e\sigma_V(d_0 \times 1) = e\sigma_V(d_1 \times 1)$, and it follows that $\tau(d_0 \times 1)(1 \times e) = \tau(d_1 \times 1)(1 \times e)$. As the relevant coequaliser is preserved by the functor $U \times (-)$, we see that $1_U \times e$ is an epimorphism, so we can conclude that $\tau(d_0 \times 1) = \tau(d_1 \times 1)$. As the functor $(-) \times W$ preserves our coequaliser, this gives us a unique map $\sigma_W : W \times W \to W$ such that $\sigma_W(e \times 1) = \tau\colon V \times W \to W$. One checks that this makes W into an Abelian group object, and that e is a homomorphism. One can also check that this makes W into a coequaliser in $\mathrm{Ab}\,\mathcal{D}$. $\square$

Remark 6.3. The same result holds, with essentially the same proof, with $\mathrm{Mon}\,\mathcal{D}$ replaced by $\mathrm{Ab}\,\mathcal{D}$ or $\mathrm{Based}\,\mathcal{D}$. The same methods also show that a reflexive fork in the category of R-algebras (for any ring R) has the same coequaliser when computed in the category of R-algebras, or of R-modules, or of sets.

We next try to construct free Abelian groups or monoids on objects of $\mathcal{D}$ or Based $\mathcal{D}$. If $U \in \mathcal{D}$ and $V \in \text{Based}\,\mathcal{D}$, we "define" objects $M^+(U), N^+(V) \in \text{Mon}\,\mathcal{D}$ and $M(U), N(V) \in \text{Ab}(\mathcal{D})$ by the equations

$$\begin{aligned}
\text{Mon}\,\mathcal{D}(M^+(U), M) &= \mathcal{D}(U, M)\\
\text{Mon}\,\mathcal{D}(N^+(V), M) &= \text{Based}\,\mathcal{D}(V, M)\\
\text{Ab}\,\mathcal{D}(M(U), G) &= \mathcal{D}(U, G)\\
\text{Ab}\,\mathcal{D}(N(V), G) &= \text{Based}\,\mathcal{D}(V, G).
\end{aligned}$$

More precisely, if there is an object $H \in \text{Ab}\,\mathcal{D}$ with a natural isomorphism

$$\text{Ab}\,\mathcal{D}(H, G) = \text{Based}\,\mathcal{D}(V, G)$$

for all $G \in \text{Ab}\,\mathcal{D}$, then H is unique up to canonical isomorphism, and we write $N(V)$ for H. Similar remarks apply to the other three cases. Given a monoid object M, we also "define" its group completion $G(M) \in \text{Ab}\,\mathcal{D}$ by the equation $\text{Ab}\,\mathcal{D}(G(M), H) = \text{Mon}\,\mathcal{D}(M, H)$.

There are fairly obvious ways to try to construct free group and monoid objects, using a mixture of products and colimits. However, there are two technical points to address. Firstly, it turns out that we need our colimits to be strong colimits in the sense of Definition 2.18. Secondly, in some places we can arrange to use reflexive coequalisers, which is technically convenient.

Proposition 6.4. Let U be an object of $\mathcal{D}$. For each $k \geq 0$, the symmetric group Σ_k acts on U^k. Suppose that the quotient U^k/Σ_k exists as a strong colimit and also that $L = \coprod_{k\geq 0} U^k/\Sigma_k$ exists as a strong coproduct. Then $L = M^+(U)$.

Proof. Let $\mathcal{I}$ be the category with object set $\mathbb{N}$, and with morphisms

$$\mathcal{I}(j,k) = \begin{cases} \emptyset & \text{if } j \neq k \\ \Sigma_k & \text{if } j = k. \end{cases}$$

It is easy to see that there is a functor $k \mapsto U^k$ from $\mathcal{I}$ to $\mathcal{D}$, and that L is a strong colimit of this functor. It follows that $L \times U^m$ is the colimit of the functor $k \mapsto U^k \times U^m$, and thus (using the "Fubini theorem" for colimits) that

$$L \times L = \varinjlim_{(k,m)\in\mathcal{I}\times\mathcal{I}} U^k \times U^m.$$

Similarly, $L \times L \times L$ is the colimit of the functor $(k, m, n) \mapsto U^k \times U^m \times U^n$ from $\mathcal{I} \times \mathcal{I} \times \mathcal{I}$ to $\mathcal{D}$.

Let $j_k : U^k \to L$ be the evident map. We then have maps $U^k \times U^m \simeq U^{k+m} \xrightarrow{j_{k+m}} L$, and these fit together to give a map $\sigma : L \times L \to L$. We also have a zero map $0 = j_0 : 1 = U^0 \to L$. We claim that this makes L into a commutative monoid object in $\mathcal{D}$. To check associativity, for example, we need to show that $\sigma \circ (\sigma \times 1) = \sigma \circ (1 \times \sigma) : L^3 \to L$. By the above colimit description of L^3, it is enough to check this after composing with the map $j_k \times j_m \times j_n : U^{k+m+n} \to L^3$, and it is easy to check that both the resulting composites are just j_{k+m+n}. We leave the rest to the reader.

Now suppose we have a monoid $M \in \text{Mon}\,\mathcal{D}$ and a map $f : U \to M$ in $\mathcal{D}$. We then have maps $f_k = (U^k \xrightarrow{f^k} M^k \xrightarrow{\sigma} M)$, which are easily seen to be invariant under the action of Σ_k, so we get an induced map $f' : L \to M$ in $\mathcal{D}$. We claim that this is a homomorphism. It is clear that $f' \circ 0 = 0$, so we need only check that $f \circ \sigma = \sigma \circ (f \times f) : L^2 \to M$. Again, we need only check this after composing with the map $j_k \times j_m : U^{k+m} \to L^2$, and it then becomes easy. We also claim that f' is the unique homomorphism $g : L \to M$ such that $g \circ j_1 = f$. Indeed, we have $j_k = (U^k \xrightarrow{j_1^k} L^k \xrightarrow{\sigma} L)$, so for any such g we have $g \circ j_k = (U^k \xrightarrow{f^k} M^k \xrightarrow{\sigma} M) = f' \circ j_k$. By our colimit description of L, we see that $g = f'$ as claimed.

This shows that monoid maps $g : L \to M$ biject naturally with maps $f : U \to M$, by the correspondence $g \mapsto g \circ j_1$. This means that $L = M^+(U)$ as claimed. $\square$

Proposition 6.5. Let V be an object of Based $\mathcal{D}$, and suppose that $V_k = V^k/\Sigma_k$ exists as a strong colimit for all $k \geq 0$. The basepoint of U then induces maps $V_k \to V_{k+1}$. Suppose also that the sequence of V_k's has a strong colimit L. Then $L = N^+(V)$.

Proof. This is essentially the same as the proof of Proposition 6.4, and is left to the reader. $\square$

We next try to construct group completions of monoid objects. We digress briefly to introduce some convenient language. Let M be a monoid object, so that $\mathcal{D}(U, M)$ is naturally a monoid for all U. We thus have a map $f_U : \mathcal{D}(U, M^3) = \mathcal{D}(U, M)^3 \to \mathcal{D}(U, M^2)$ defined by $f(a, b, c) = (c + 2a, 3b + c)$ (for example). This is natural in U, so Yoneda's lemma gives us a map $f : M^3 \to M^2$. From now on, we will allow ourselves to abbreviate this definition by saying "let $f : M^3 \to M^2$ be the map $(a, b, c) \mapsto (c + 2a, 3b + c)$". This is essentially the same as thinking of $\mathcal{D}$ as a subcategory of $[\mathcal{D}^{\mathrm{op}}, \mathrm{Sets}]$, by the Yoneda embedding.

Given a monoid object M, we define maps $d_0, d_1 : M^3 \to M^2$ and $s : M^2 \to M$ by

$$\begin{aligned} d_0(a, b, x) &= (a, b) \\ d_1(a, b, x) &= (a + x, b + x) \\ s(a, b) &= (a, b, 0). \end{aligned}$$

This is clearly a reflexive fork in $\mathrm{Mon}\,\mathcal{D}$.

Proposition 6.6. If the above fork has a strong coequaliser $q : M^2 \to H$ in $\mathcal{D}$, then H has a unique group structure making q into a homomorphism of monoids, and with that group structure we have $H = G(M)$.

Proof. Firstly, Proposition 6.2 tells us that there is a unique monoid structure on H making q into a monoid map, and that this makes H into the coequaliser in $\mathrm{Mon}\,\mathcal{D}$. We define a monoid map $\nu' : M^2 \to H$ by $\nu'(a, b) = q(b, a)$. Clearly $\nu' d_0(a, b, x) = q d_0(b, a, x)$ and $\nu' d_1(a, b, x) = q d_1(b, a, x)$ but $q d_0 = q d_1$ so $\nu' d_0 = \nu' d_1$, so there is a unique map $\nu : H \to H$ with $\nu' = \nu q$. We then have

$$\begin{aligned} q(a, b) + \nu q(a, b) &= q(a, b) + q(b, a) \\ &= q(a + b, a + b) \\ &= q d_1(0, 0, a + b) \\ &= q d_0(0, 0, a + b) \\ &= q(0, 0) = 0. \end{aligned}$$

This shows that $(1 + \nu)q = 0$, but q is an epimorphism so $1 + \nu = 0$. This means that ν is a negation map for H, making it into a group object. We let $j : M \to H$ be the map $a \mapsto q(a, 0)$, which is clearly a homomorphism of monoids. Clearly $q(a, b) = q(a, 0) + q(0, b) = j(a) + \nu j(b) = j(a) - j(b)$.

Now let K be another Abelian group object, and let $f : M \to K$ be a homomorphism of monoids. We define $f' : M^2 \to K$ by $f'(a, b) = f(a) - f(b)$. It is clear that $f' d_0 = f' d_1$, so we get a unique monoid map $f'' : H \to K$ with $f'' q = f'$. In particular, we have $f'' j(a) = f'' q(a, 0) = f'(a, 0) = f(a)$, so that $f'' j = f$. If $g : H \to K$ is another homomorphism with $gj = f$ then $gq(a, b) = g(j(a) - j(b)) = f(a) - f(b) = f'' q(a, b)$, and q is an epimorphism so $g = f''$.

This shows that group maps $H \to K$ biject with monoid maps $M \to K$ by the correspondence $g \mapsto gj$, which means that $H = G(M)$ as claimed. $\square$

6.2. **Free formal groups.** We next discuss the existence of free Abelian formal groups.

Proposition 6.7. Let Y be a formal scheme over a formal scheme X. Write X as a filtered colimit of informal schemes X_i, and put $Y_i = Y \times_X X_i$. If $M^+(Y_i)$ exists in $\mathrm{Mon}\,\widehat{\mathcal{X}}_{X_i}$ for all i, then $M^+(Y)$ exists and is equal to $\varinjlim_i M^+(Y_i)$. Similar remarks apply to $M(Y)$ and (if Y has a given section $0 : X \to Y$) to $N^+(Y)$ and $N(Y)$.

Proof. We use the notation of Definition 4.26 and Proposition 4.27. It is clear that $\{M^+(Y_i)\}$ is the free Abelian monoid object on $\{Y_i\}$ in the category $\mathcal{D}_{\{X_i\}}$. As the functor $F : \mathcal{D}_{\{X_i\}} \to \widehat{\mathcal{X}}_X$ preserves finite limits, we see that $L = \varinjlim_i M^+(Y_i) = F\{M^+(Y_i)\}$ is an Abelian monoid object in $\widehat{\mathcal{X}}_X$. Using the fact that F preserves finite products and is left adjoint to G, we see that

$$\widehat{\mathcal{X}}_X(L^m, Z) = \mathcal{D}_{\{X_i\}}(\{M^+(Y_i)^m_{X_i}\}, \{Z \times_X X_i\})$$

for all $Z \in \widehat{\mathcal{X}}_Z$. Using this, one can check that

$$\begin{aligned}\operatorname{Mon}\widehat{\mathcal{X}}(L,M) &= \operatorname{Mon}\mathcal{D}_{\{X_i\}}(\{M^+(Y_i)\},\{M\times_X X_i\}) \\ &= \mathcal{D}_{\{X_i\}}(\{Y_i\},\{M\times_X X_i\}) = \widehat{\mathcal{X}}_X(Y,M),\end{aligned}$$

as required. We leave the case of $M(Y)$ and so on to the reader. □

Proposition 6.8. If Y is a coalgebraic formal scheme over X, then the free Abelian monoid scheme $M^+(Y)$ exists. If Y also has a specified section $0\colon X\to Y$ (making it an object of $\mathrm{Based}\,\widehat{\mathcal{X}}_X$) then $N^+(Y)$ exists.

Proof. By the previous proposition, we may assume that X is informal, and that $Y = \operatorname{sch}_X(U)$ for some coalgebra U over $R = \mathcal{O}_X$ with a good basis $\{e_i \mid i \in I\}$. We know from Example 4.65 that Y_X^k/Σ_k is a strong colimit for the action of Σ_k on Y_X^k. Moreover, $\coprod_k Y^k/\Sigma_k$ exists as a strong coproduct by Corollary 4.42. We conclude from Proposition 6.4 that $M^+(Y) = \coprod_k Y^k/\Sigma_k$. In the based case, we observe that the diagram $\{Y^k/\Sigma_k\}$ is just indexed by $\mathbb{N}$ and thus is filtered, and filtered colimits exists and are strong in $\widehat{\mathcal{X}}_X$ by Proposition 4.12. Given this, Proposition 6.5 completes the proof. □

Remark 6.9. If X is informal we see that the coalgebra $cM^+(Y)$ is just the symmetric algebra generated by cY over $\mathcal{O}_X$. In the based case, if $e_0 \in cY$ is the basepoint then $cN^+(Y) = cM^+(Y)/(e_0 - 1)$.

We next show that in certain cases of interest, the free Abelian monoid $N^+(Y)$ constructed above is actually a group.

Definition 6.10. A *good filtration* of a coalgebra U over a ring R is a sequence of submodules F_sU for $s \geq 0$ such that

(a) $\epsilon\colon F_0U \to A$ is an isomorphism.
(b) For $s > 0$ the quotient $G_sU = F_sU/F_{s-1}U$ is a finitely generated free module over R.
(c) $\bigcup_s F_sU = U$
(d) $\psi(F_sU) \subseteq \sum_{s=t+u} F_sU \otimes F_tU$.

We write $\mathcal{C}'' = \mathcal{C}''_R = \mathcal{C}''_Z$ for the category of coalgebras that admit a good filtration. Given a good filtration, we write η for the composite $A \xrightarrow{\epsilon^{-1}} F_0U \rightarrowtail U$. One can check that this is a coalgebra map, so it makes U into a based coalgebra. A *good basepoint* for U is a basepoint which arises in this way. We say that a *very good basis* for U is a basis $\{e_0, e_1, \ldots\}$ for U over R such that

(i) $e_0 = \eta(1)$
(ii) $\epsilon(e_i) = 0$ for $i > 0$
(iii) There exist integers N_s such that $\{e_i \mid i < N_s\}$ is a basis for F_sU.

One can check that very good bases exist, and that a very good basis is a good basis.

Proposition 6.11. If U and V lie in $\mathcal{C}''_Z$ then so do $U\times V$ and U^k/Σ_k. If we choose a good basepoint for U then we can define $N^+(U)$, and it again lies in $\mathcal{C}''_Z$.

Proof. Choose good filtrations on U and V. Define a filtration on $U \times V = U\otimes V$ by setting $F_s(U\otimes V) = \sum_{s=t+u} F_tU \otimes F_uV$. It is not hard to check that this is good. Essentially the same procedure gives a filtration of $U^{\otimes m}$. This is invariant under the action of the symmetric group Σ_m, so we get an induced filtration of the group of coinvariants $U^{\otimes m}_{\Sigma_m}$. Our filtrations on these groups are compatible as m varies, so we get an induced filtration of $N(U) = \varinjlim_m U^{\otimes m}_{\Sigma_m}$. Using a very good basis for U and the associated monomial basis for $N(U)$, we can check that the filtration of $N(U)$ is good. □

Proposition 6.12. Let U be an Abelian monoid object in $\mathcal{C}_Z$, with addition map $\sigma\colon U\times U = U\otimes U \to U$. If U admits a good filtration such that the basepoint is good and $\sigma(F_sU\otimes F_tU) \subseteq F_{s+t}U$ for all $s,t \geq 0$, then U is actually an Abelian group object.

Proof. First note that we can use σ to make U into a ring. We need to construct a negation map (otherwise known as an antipode) $\chi\colon U \to U$, which must be a coalgebra map satisfying $\sigma(1\otimes\chi)\psi = \eta\epsilon$. In terms of elements, if $\psi(a) = 1\otimes a + \sum a'\otimes a''$ then the requirement is that $\chi(a) = \eta\epsilon(a) - \sum a'\chi(a'')$. The idea is to use this formula to define χ on F_sU by recursion on s.

Write $\overline{\psi} = \psi - \eta \otimes 1 \colon U \to U \otimes U$. Note that $\overline{\psi}(F_s U) \subseteq \sum_{t=0}^{s} F_{s-t}U \otimes F_t U$, and that $(\epsilon \otimes 1)\overline{\psi} = 0$. Choose a very good basis $\{e_i\}$ for U, and write $\overline{\psi}(e_i) = \sum_{j,k} a_{ijk} e_j \otimes e_k$. Suppose that $N_{s-1} \leq i < N_s$, so that $e_i \in F_s U \setminus F_{s-1}U$. If $j > 0$ and $k \geq N_{s-1}$ then $e_j \otimes e_k \notin F_s(U \otimes U)$ so $a_{ijk} = 0$. On the other hand, the equation $(\epsilon \otimes 1)\overline{\psi}(e_i) = 0$ gives $\sum_m a_{i0m} e_m = 0$ for all m, so $a_{i0k} = 0$, so $a_{ijk} = 0$ for all j. This applies for all $k \geq N_{s-1}$, and thus in particular for $k \geq i$.

We now define $\chi(e_i)$ recursively by $\chi(e_0) = e_0$ and

$$\chi(e_i) = -\sum_{0 \leq k < i} a_{ijk} e_j \chi(e_k)$$

for $i > 0$. By the previous paragraph, we actually have $\chi(e_i) = -\sum_{k \geq 0} a_{ijk} e_j \chi(e_k)$, and it follows that $\sigma(1 \otimes \chi)\psi = \eta\epsilon$ as required. We still have to check that χ is a coalgebra map. For the counit, it is clear that $\epsilon\chi(e_0) = \epsilon(e_0)$. If we assume inductively that $\epsilon(\chi(e_k)) = \epsilon(e_k) = 0$ for $0 < k < i$ then we find that

$$\epsilon\chi(e_i) = -\sum_{0 \leq k < i} a_{ijk} \epsilon(e_j)\epsilon\chi(e_k) = a_{i00} = (\epsilon \otimes \epsilon)\psi(e_i) = \epsilon(e_i) = 0.$$

A similar, but slightly more complicated, induction shows that $\psi\chi = (\chi \otimes \chi)\psi$, so χ is a coalgebra map as required. □

Proposition 6.13. Let C be a pointed formal curve over a formal scheme X. Then there are natural isomorphisms

$$\begin{aligned} M^+(C) &= \mathrm{Div}^+(C) \\ N^+(C) &= N(C) = \mathrm{Div}_0(C) \\ M(C) &= \mathrm{Div}(C). \end{aligned}$$

Proof. This follows easily from the constructions in Section 6.1 and the results above. □

6.3. **Schemes of homomorphisms.**

Definition 6.14. Given formal groups G and H over X and a ring R, we let $\mathrm{Hom}_X(G,H)(R)$ be the set of pairs (x, u), where $x \in X(R)$ and $u \colon G_x \to H_x$ is a homomorphism of formal groups over $\mathrm{spec}(R)$. This is a subfunctor of $\mathrm{Map}_X(G,H)$, so we have defined an object $\mathrm{Hom}_X(G,H) \in \mathcal{F}$. It is not hard to define an equaliser diagram

$$\mathrm{Hom}_X(G,H) \to \mathrm{Map}_X(G,H) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathrm{Map}_X(G \times_X G, H).$$

In more detail, note that a point of $\mathrm{Map}_X(G,H)$ is a map $x \colon \mathrm{spec}(R) \to X$ together with a map $f \colon G_x \to H_x$ of schemes over $\mathrm{spec}(R)$. Given such a pair (x, f), we define $g, h \colon G_x \times_{\mathrm{spec}(R)} G_x \to H_x$ by $g(a,b) = f(a+b)$ and $h(a,b) = f(a) + f(b)$, and then we define d_i by $d_0(f) = g$ and $d_1(f) = h$.

Proposition 6.15. Let G and H be formal groups over X. If G is finite and very flat over X, or if G is coalgebraic and H is relatively informal, or if G is very flat and H is of finite presentation, then $\mathrm{Hom}_X(G,H)$ is a formal scheme and there is a natural isomorphism

$$\widehat{\mathcal{X}}_X(Y, \mathrm{Hom}_X(G,H)) = \mathrm{Ab}\,\widehat{\mathcal{X}}_Y(G \times_X Y, H \times_X Y)$$

for all $Y \in \widehat{\mathcal{X}}_X$.

Proof. Theorem 4.69 tells us that $\mathrm{Map}_X(G,H)$ and $\mathrm{Map}_X(G \times_X G, H)$ are formal schemes, and $\widehat{\mathcal{X}}_X$ is closed under finite limits in $\mathcal{F}$, so $\mathrm{Hom}_X(G,H)$ is a formal scheme. The natural isomorphism comes from the Yoneda lemma when Y is informal, and follows in general by passage to colimits. □

Example 6.16. Let $\widehat{\mathbb{G}}_a$ be the additive formal group (over the terminal scheme $1 = \mathrm{spec}(\mathbb{Z})$) defined by $\widehat{\mathbb{G}}_a(R) = \mathrm{Nil}(R)$, with the usual addition. Thus, the underlying scheme of $\widehat{\mathbb{G}}_a$ is just $\widehat{\mathbb{A}}^1$. This is coalgebraic over 1, so we see that $\mathrm{End}(\widehat{\mathbb{G}}_a) = \mathrm{Hom}_1(\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a)$ exists. One checks that any map $\widehat{\mathbb{A}}^1 \times Y \to \widehat{\mathbb{A}}^1 \times Y$ over Y is given by a unique power series $f(x) \in \mathcal{O}_Y[\![x]\!]$ such that $f(0)$ is nilpotent. It follows easily that $\mathrm{End}(\widehat{\mathbb{G}}_a)(R)$ is the set of power series $f \in R[\![x]\!]$ such that $f(x+y) = f(x) + f(y) \in R[\![x,y]\!]$. If R is an algebra over $\mathbb{F}_p$, then a well-known lemma says that $f(x+y) = f(x) + f(y)$ if and only if f can be written in the form $f(x) = \sum_k a_k x^{p^k}$, for uniquely determined coefficients $a_k \in R$. One can deduce that $\mathrm{spec}(\mathbb{F}_p) \times \mathrm{End}(\widehat{\mathbb{G}}_a) = \mathrm{spec}(\mathbb{F}_p[a_k \mid k \geq 0])$.

Example 6.17. A similar analysis shows that $\mathrm{End}(\mathbb{G}_m)(R)$ is the set of Laurent polynomials $f \in R[u^{\pm 1}]$ such that $f(u)f(v) = f(uv)$ and $f(1) = 1$. If $f(u) = \sum_k e_k u^k$, we find that the elements $e_k \in R$ are orthogonal idempotents with $\sum_k e_k = 1$. It follows that $\mathrm{End}(\mathbb{G}_m)$ is the constant formal scheme $\underline{\mathbb{Z}}$, with the n'th piece in the coproduct corresponding to the endomorphism $u \mapsto u^n$.

Example 6.18. We can also form the scheme $\mathrm{Exp} = \mathrm{Hom}(\widehat{\mathbb{G}}_a, \mathbb{G}_m)$. In this case, $\mathrm{Exp}(R)$ is the set of power series $f(x) = \sum_k a^{[k]} x^k$ such that $f(0) = 1$ and $f(x+y) = f(x)f(y)$, or equivalently $a^{[0]} = 1$ and $a^{[i]}a^{[j]} = \begin{pmatrix} i+j \\ i \end{pmatrix} a^{[i+j]}$. In other words, a point of $\mathrm{Exp}(R)$ is an element $a = a^{[1]}$ of R together with a specified system of divided powers for a. Clearly, if R is a $\mathbb{Q}$-algebra then there is a unique possible system of divided powers, viz. $a^{[k]} = a^k/k!$, so $\mathrm{spec}(\mathbb{Q}) \times \mathrm{Exp} \simeq \mathrm{spec}(\mathbb{Q}) \times \mathbb{A}^1$.

Now let R be an $\mathbb{F}_p$-algebra. Given an element $a \in R$ with $a^p = 0$, we define $T(a)(x) = \sum_{j=0}^{p-1} a^j x^j/j!$; it is not hard to see that $T(a) \in \mathrm{Exp}(R)$. Given a sequence of such elements $\underline{a} = (a_0, a_1, \ldots)$, we define $T(\underline{a})(x) = \prod_i T(a_i)(x^{p^i})$; it is not hard to check that the product is convergent in the x-adic topology on $R[\![x]\!]$, and that $T(\underline{a}) \in \mathrm{Exp}(R)$. Thus T defines a map $\mathrm{spec}(\mathbb{F}_p) \times D_p^{\mathbb{N}} \to \mathrm{spec}(\mathbb{F}_p) \times \mathrm{Exp}$. It can be shown that this is an isomorphism.

More generally, we have $\mathrm{Exp} = \mathrm{spec}(D_{\mathbb{Z}}[a])$, where $D_{\mathbb{Z}}[a]$ is the divided-power algebra on one generator a over $\mathbb{Z}$. The previous paragraph is equivalent to the fact that $D_{\mathbb{F}_p}[a] = D_{\mathbb{Z}}[a]/p = \mathbb{F}_p[a_k \mid k \geq 0]/(a_k^p)$, where $a_k = a^{[p^k]}$.

6.4. **Cartier duality.** Let G be a coalgebraic commutative formal group over a formal scheme X. By Proposition 6.15, we can define the group scheme $DG = \mathrm{Hom}_X(G, \mathbb{G}_m \times X)$. We call this the *Cartier dual* of G. Note also that the product structure on G makes cG into commutative group in the category of coalgebras, in other words a Hopf algebra, and in particular an algebra over $\mathcal{O}_X$. We can thus define $H = \mathrm{spec}(cG)$, which is an informal scheme over X. The coproduct on cG gives a product on H, making it into a group scheme over X. Moreover, we know that cG is a free module over $\mathcal{O}_X$, so that H is very flat over $\mathcal{O}_X$. Thus, by Proposition 6.15, we can define a formal group scheme $DH = \mathrm{Hom}_X(H, \mathbb{G}_m \times X)$. We again call this the Cartier dual of H. These definitions appear in various levels of generality in many places in the literature; the treatment in [3] is similar in spirit to ours, although restricted to the case where $\mathcal{O}_X$ is a field.

Proposition 6.19. If G and H are as above, then $DG = H$ and $DH = G$.

Proof. First suppose that $X = \mathrm{spec}(R)$ is informal. We shall analyse the set $\widehat{\mathcal{X}}_X(X, DG)$ of sections of the map $DG \to X$. From the definitions, we see that a section of the map $DG \to X$ is the same as a map $G \to \mathbb{G}_m \times X$ of formal groups over X, or equivalently a map of Hopf algebras $\mathcal{O}_{\mathbb{G}_m \times X} \to \mathcal{O}_G$. As $\mathcal{O}_{\mathbb{G}_m \times X} = R[u^{\pm 1}]$ with $\epsilon(u) = 1$ and $\psi(u) = u \otimes u$, such a map is equivalent to an element $v \in \mathcal{O}_G^\times$ with $\epsilon(v) = 1$ and $\psi(v) = v \otimes v$. In fact, if v is any element with $\epsilon(v) = 1$ and $\psi(v) = v \otimes v$ then the Hopf algebra axioms imply that $v\chi(v) = 1$ so we do not need to require separately that v be invertible. As G is coalgebraic we have $\mathcal{O}_G = \mathrm{Hom}_R(cG, R)$, so we can regard v as a map $cG \to R$ of R-modules. The conditions $\epsilon(v) = 1$ and $\psi(v) = v \otimes v$ then become $v(1) = 1$ and $v(ab) = v(a)v(b)$, so the set of such v's is just $\mathrm{Alg}_R(cG, R) = \widehat{\mathcal{X}}_X(X, H)$.

Now let X be arbitrary. The above (together with the commutation of various constructions with pullbacks, which we leave to the reader) shows that for any informal scheme W over X we have $\widehat{\mathcal{X}}_X(W, DG) = \widehat{\mathcal{X}}_W(W, D(G \times_X W)) = \widehat{\mathcal{X}}_W(W, H \times_X W) = \widehat{\mathcal{X}}_X(W, H)$. It follows that $DG = H$ as claimed.

We now show that $DH = G$. Just as previously, we may assume that $X = \mathrm{spec}(R)$ is informal, and it is enough to show that DH and G have the same sections. Again, the sections of DH are just the elements $v \in \mathcal{O}_H = cG$ with $\epsilon(v) = 1$ and $\psi(v) = v \otimes v$. In this case, we identify cG with the continuous dual of $\mathcal{O}_G$, so v is a continuous map $\mathcal{O}_G \to R$ of R-algebras, and thus a section of $\mathrm{spf}(\mathcal{O}_G) = G$ as required. $\square$

6.5. **Torsors.** Let G be a formal group over a formal scheme X. Let T be a formal scheme over X with an action of G. More explicitly, we have an action map $\alpha \colon G \times_X T \to T$, so whenever g and t are points of G and T with the same image in X, we can define $g.t = \alpha(g,t)$. This is required to satisfy $1.t = t$ and $g.(h.t) = (gh).t$ (whenever g, h and t all have the same image in X). We write $G\widehat{\mathfrak{X}}_X$ for the category of such T. Note that G itself can be regarded as an object of $G\widehat{\mathfrak{X}}_X$.

If Y is a scheme with a specified map $p\colon Y \to X$ we shall allow ourselves to write $G\widehat{\mathfrak{X}}_Y$ instead of $(p^*G)\widehat{\mathfrak{X}}_Y$. It is easy to see that p^* gives a functor $G\widehat{\mathfrak{X}}_X \to G\widehat{\mathfrak{X}}_Y$.

Definition 6.20. Let G be a formal group over a formal scheme X, and let T be a formal scheme over X with an action of G. We say that T is a *G-torsor over X* if there exists a faithfully flat map $p\colon Y \to X$ such that $p^*T \simeq p^*G$ in $G\widehat{\mathfrak{X}}_Y$. We write $G\mathfrak{T}_X$ for the category of G-torsors over X.

Example 6.21. Let M be a vector bundle over X of rank d, and let $\mathrm{Bases}(M)$ be as in Example 2.85. Let GL_d be the group scheme of invertible $d \times d$ matrices. Then $\mathrm{GL}_d \times X$ acts on $\mathrm{Bases}(M)$, and if M is free then $\mathrm{Bases}(M) \simeq \mathrm{GL}_d \times X$. As we can always pull back along a faithfully flat map $p\colon Y \to X$ to make M free, and $\mathrm{Bases}(p^*M) = p^*\,\mathrm{Bases}(M)$, we find that $\mathrm{Bases}(M)$ is a torsor for $\mathrm{GL}_d \times X$.

Example 6.22. Let C be a pointed formal curve over X, let $\mathrm{Coord}(C)$ be as in Definition 5.10, and let IPS be as in Example 2.9. Then $\mathrm{Coord}(C)$ is a torsor for group scheme $\mathrm{IPS} \times X$. In fact, this torsor is trivialisable (i.e. isomorphic to $\mathrm{IPS} \times X$ even without pulling back) but not canonically so.

Proposition 6.23. Every morphism in $G\mathfrak{T}_X$ is an isomorphism, so $G\mathfrak{T}_X$ is a groupoid.

Proof. First, let $u\colon G \to G$ be a map of G-torsors. As u is G-equivariant we have $u(g) = g.u(1)$, so $h \mapsto h.u(1)^{-1}$ is an inverse for u. Now let $u\colon S \to T$ be an arbitrary map of G-torsors. Then there is a faithfully flat map $p\colon Y \to X$ such that $p^*S \simeq p^*T \simeq p^*G$, so the first case tells us that p^*u is an isomorphism. As p is faithfully flat, we see that p^* reflects isomorphisms, so u is an isomorphism. ☐

Proposition 6.24. Every homomorphism $\phi\colon G \to H$ of formal groups over X gives rise to functors

$$\begin{aligned} \phi^\bullet &\colon H\widehat{\mathfrak{X}}_X \to G\widehat{\mathfrak{X}}_X \\ \phi_\bullet &\colon G\mathfrak{T}_X \to H\mathfrak{T}_X, \end{aligned}$$

such that

$$H\widehat{\mathfrak{X}}_X(\phi_\bullet T, U) = G\widehat{\mathfrak{X}}_X(T, \phi^\bullet U)$$

for all $U \in H\widehat{\mathfrak{X}}_X$.

Proof. The functor $\phi^\bullet$ is just $\phi^\bullet U = U$, with G-action $g.u := \phi(g).u$. Let $\phi_\bullet T$ be the coequaliser of the maps $(h,g,t) \mapsto (h\phi(g),t)$ and $(h,g,t) \mapsto (h,g.t)$ from $H \times_X G \times_X T$ to $H \times_X T$. Note that these maps have a common splitting $(h,t) \mapsto (h,1,t)$, so we have a reflexive fork. In the case $T = G$, the coequaliser is just the map $H \times_X G \to H$ given by $(h,g) \mapsto h\phi(g)$. In fact, this coequaliser is split by the maps $h \mapsto (h,1)$ and $(h,g) \mapsto (h,g,1)$, so it is a strong coequaliser.

Now consider a general G-torsor T. We claim that the coequaliser that defines $\phi_\bullet T$ is strong. By proposition 2.69, we can check this after pulling back along a faithfully flat map $p\colon Y \to X$. We can choose p so that $p^*T \simeq p^*G$, and then the claim follows from the previous paragraph.

We can let H act on the left on $H \times_X G \times_X T$ and $H \times_X G$, and then the maps whose coequaliser defines $\phi_\bullet T$ are both H-equivariant. The reader can easily check that if a fork in $H\widehat{\mathfrak{X}}_X$ has a strong coequaliser in $\widehat{\mathfrak{X}}_X$ then the coequaliser has a unique H-action making it the coequaliser in $H\widehat{\mathfrak{X}}_X$. This implies that $\phi_\bullet T$ is the coequaliser of our fork in $H\widehat{\mathfrak{X}}_X$, and one can deduce that

$$H\widehat{\mathfrak{X}}_X(\phi_\bullet T, U) = G\widehat{\mathfrak{X}}_X(T, \phi^\bullet U)$$

for all $U \in H\widehat{\mathfrak{X}}_X$.

All that is left is to check that $\phi_\bullet T$ is a torsor. For this, we just choose a faithfully flat map p such that $p^*T \simeq p^*G$, and observe that $p^*\phi_\bullet T = \phi_\bullet p^*T \simeq p^*H$. ☐

Proposition 6.25. If G is an Abelian formal group over X, then there is a functor $\otimes\colon G\mathfrak{T}_X \times G\mathfrak{T}_X \to G\mathfrak{T}_X$ which makes $G\mathfrak{T}_X$ into a symmetric monoidal category with unit G. Moreover, the twist map of $T \otimes T$ is always the identity, and every object is invertible under $\otimes$, so that $G\mathfrak{T}_X$ is a strict Picard category.

Proof. If S and T are G-torsors over X, then it is easy to see that $S \times_X T$ has a natural structure as a $G \times_X G$-torsor. As G is Abelian, the multiplication map $\mu\colon G \times_X G \to G$ is a homomorphism, so we can define $S \otimes T = \mu_\bullet(S \times_X T)$. We leave it to the reader to check that this gives a symmetric monoidal structure with unit G. If we let $\chi\colon G \to G$ denote the map $g \mapsto g^{-1}$ then χ is also a homomorphism, so we can define $T^{-1} = \chi_\bullet T$. We then have $T \otimes T^{-1} = (\mu(1 \times \chi))_\bullet(T \times_X T) = 0_\bullet(T \times_X T) = G$, so T^{-1} is an inverse for T. Finally, we need to show that the twist map $\tau\colon T \otimes T \to T \otimes T$ is the identity. As the map $q\colon T \times_X T \to T \otimes T$ is a regular epimorphism, it suffices to show that $\tau q = q$, and clearly $\tau q(a, b) = q(b, a)$ so we need to show that $q(a, b) = q(b, a)$. In the case $T = G$ we have $T \otimes T = G$ and the map q is just $q(a, b) = ab$, so the claim holds. For general T, we just pull back along a faithfully flat map p such that $p^*T \simeq p^*G$ and use the fact that p^* is faithful. □

Proposition 6.26. Let $\mathbb{G}_m$ denote the multiplicative group, which is defined by $\mathbb{G}_m(R) = R^\times$. Then the functor $L \mapsto \mathbb{A}(L)^\times$ (as in Definition 2.55 and Remark 4.43) is an equivalence from the category of line bundles over X and isomorphisms, to the category of $\mathbb{G}_m$-torsors over X. Moreover, this equivalence respects tensor products.

Proof. Let L be a line bundle over X. For any $x \in X(R)$, we have a rank one projective module L_x over R, and clearly $R^\times = \mathbb{G}_m(R)$ acts on the set of bases for L_x (even though this set may be empty). If L is free then it is clear that $\mathbb{A}(L)^\times \simeq \mathbb{A}(\mathcal{O})^\times = \mathbb{G}_m \times X$, and thus that $\mathbb{A}(L)^\times$ is a torsor. In general, we know from Proposition 4.55 that L is fpqc-locally isomorphic to $\mathcal{O}$, so $\mathbb{A}(L)^\times$ is fpqc-locally isomorphic to $\mathbb{A}(\mathcal{O})^\times = \mathbb{G}_m \times X$, and thus is a torsor.

In the opposite direction, let T be a $\mathbb{G}_m$-torsor over X. Define a formal scheme A over X by the coequaliser

$$\mathbb{A}^1 \times \mathbb{G}_m \times T \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \mathbb{A}^1 \times T \to A,$$

where $\lambda(a, u, t) = (au, t)$ and $\rho(a, u, t) = (a, ut)$. Locally in the flat topology we may assume that $T = \mathbb{G}_m \times X$, and it is easy to check that $\mathbb{A}^1 \times X$ is the split coequaliser of the fork. Thus Proposition 2.69 tells us that A is the strong coequaliser of the original fork. Also, we can make $\mathbb{A}^1 \times \mathbb{G}_m \times T$ and $\mathbb{A}^1 \times T$ into modules over the ring scheme $\mathbb{A}^1$. As the functor $\mathbb{A}^1 \times (-)$ preserves our coequaliser, the formal scheme A is also a module over $\mathbb{A}^1$. This means that if we define L_x to be the preimage of $x \in X(R)$ under the map $A(R) \to X(R)$, then L_x is an R-module. Locally on X we have $T \simeq \mathbb{G}_m \times X$ and thus $A \simeq \mathbb{A}^1$ and thus $L_x \simeq R$. One can deduce that L is a line bundle over X, with $\mathbb{A}(L) = A$ and thus $\mathbb{A}(L)^\times = T$.

We leave it to the reader to check that this gives an equivalence of categories, which preserves tensor products. □

7. Ordinary formal groups

Recall that an *ordinary formal group* over a scheme X is a formal group G over X that is isomorphic to $X \times \widehat{\mathbb{A}}^1$ as a formal scheme over X. In particular, G is a pointed formal curve over X, so we can choose a normalised coordinate x on G giving an isomorphism $G \simeq \widehat{\mathbb{A}}^1 \times X$ in $\mathrm{Based}\widehat{\mathcal{X}}_X$. However, for the usual reasons it is best to proceed as far as possible in a coordinate-free way. Lazard's book [17] gives an account in this spirit, but in a somewhat different framework.

If we do choose a coordinate x on G then we have a function $(g, h) \mapsto x(g + h)$ from $G \times_X G$ to $\widehat{\mathbb{A}}^1$. As $G \times_X G \simeq \widehat{\mathbb{A}}^2 \times X$, we see that this can be written uniquely in the form $x(g + h) = \sum_{i,j} a_{ij} x(g)^i x(h)^j = F_x(x(g), x(h))$ for some power series $F_x(s, t) \in \mathcal{O}_X[\![s, t]\!]$. It is easy to see that this is a formal group law (Example 2.6), so we get a map $X \to \mathrm{FGL}$. This construction gives a canonical map $\mathrm{Coord}(G) \to \mathrm{FGL}$. We can let the group scheme IPS act on FGL as in Example 2.9, and on $\mathrm{Coord}(G)$ by $f.x = f(x)$. It is easy to see that the map $\mathrm{Coord}(G) \to \mathrm{FGL}$ is IPS-equivariant.

Definition 7.1. Let G be a formal group over an affine scheme X. Let I be the ideal in $\mathcal{O}_X$ of functions $g\colon X \to \mathbb{A}^1$ such that $g(0) = 0$.

Define $\omega_G = \omega_{G/X} = I/I^2$, and let $d_0(g)$ denote the image of g in $\omega_{G/X}$. We also define

$$\mathrm{Prim}(\Omega^1_{G/X}) = \{\alpha \in \Omega^1_{G/X} \mid \sigma^*\alpha = \pi_0^*\alpha + \pi_1^*\alpha \in \Omega_{G \times_X G/X}\}.$$

Here $\pi_0, \pi_1\colon G \times_X G \to G$ are the two projections, and $\sigma\colon G \times_X G \to G$ is the addition map.

We now give a formal version of the fact that left-invariant differential forms on a Lie group biject with elements of the cotangent space at the identity element.

Proposition 7.2. $\omega_{G/X}$ is a free module on one generator over $\mathcal{O}_X$. Moreover, there are natural isomorphisms $\omega_{G/X} \simeq \mathrm{Prim}(\Omega^1_{G/X})$ and $\Omega^1_{G/X} = \mathcal{O}_G \otimes_{\mathcal{O}_X} \omega_{G/X}$.

Proof. Let x be a normalised coordinate on G. We then have $\mathcal{O}_G = \mathcal{O}_X[\![x]\!]$, and it is easy to check that $I = (x)$ so $I^2 = (x^2)$ so $\omega_{G/X}$ is freely generated over $\mathcal{O}_X$ by $d_0(x)$.

Now let K be the ideal in $\mathcal{O}_{G\times_X G}$ of functions k such that $k(0,0) = 0$. In terms of the usual description $\mathcal{O}_{G\times_X G} = \mathcal{O}_X[\![x', x'']\!]$, this is just the ideal generated by x' and x''. Given $g \in I$, we define $\delta(g)(u,v) = g(u+v) - g(u) - g(v)$. We claim that $\delta(g) \in K^2$. Indeed, we clearly have $\delta(g)(0,v) = 0$, so $\delta(g)$ is divisible by x'. We also have $\delta(g)(u,0) = 0$, so $\delta(g)$ is divisible by x''. It follows easily that $\delta(g)$ is divisible by $x'x''$ and thus that it lies in K^2 as claimed.

Next, let J be the ideal of functions on $G \times_X G$ that vanish on the diagonal (so we have $\Omega^1_{G/X} = J/J^2$). For any function $g \in I$ we define $\lambda(g) \in J$ by $\lambda(g)(u,v) = g(u-v)$. As $g(0) = 0$ we see that $\lambda(g) \in J$, so λ induces a map $\omega_{G/X} \to \Omega^1_{G/X}$. We claim that $\lambda(g) \in \mathrm{Prim}(\Omega^1_{G/X})$. To make this more explicit, let L be the ideal of functions l on G^4_X such that $l(s,s,u,u) = 0$. The claim is that $\sigma^*\lambda(g) - \pi_0^*\lambda(g) - \pi_1^*\lambda(g) = 0$ in L/L^2, or equivalently that the function

$$k\colon (s,t,u,v) \mapsto \lambda(g)(s+u, t+v) - \lambda(g)(s,t) - \lambda(g)(u,v)$$

lies in L^2. To see this, note that $k = \delta(g) \circ \theta$, where $\theta(s,t,u,v) = (s-t, u-v)$. It is clear that $\theta^*K \subset L$ and thus that $\theta^*K^2 \subset L^2$, and we have seen that $\delta(g) \in K^2$ so $k \in L^2$ as claimed. Thus, we have a map $\lambda\colon \omega_{G/X} \to \mathrm{Prim}(\Omega^1_{G/X})$.

Next, given a function $h(u,v)$ in J, we have a function $\mu(h)(u) = h(u,0)$ in I. It is clear that μ induces a map $\Omega^1_{G/X} \to \omega_{G/X}$ with $\mu \circ \lambda = 1$. Now suppose that h gives an element of $\mathrm{Prim}(\Omega^1_{G/X})$ and that $\mu(h) \in I^2$. Define $k(s,t,u,v) = h(s+u, t+v) - h(s,t) - h(u,v)$. The primitivity of h means that $k \in L^2$. Define $\phi\colon G\times_X G \to G \times_X G \times_X G \times_X G$ by $\phi(s,t) = (t,t,s-t,0)$. One checks that $\phi^*L \subseteq J$ and that

$$h(s,t) = k(t,t,s-t,0) + h(t,t) + h(s-t,0).$$

Noting that $h(t,t) = 0$, we see that $h = \phi^*k + \psi^*\mu(h)$, where $\psi(u,v) = u - v$. As $\mu(h) \in I^2$ and $k \in L^2$ we conclude that $h \in J^2$. This means that μ is injective on $\mathrm{Prim}(\Omega^1_{G/X})$. As $\mu\lambda = 1$, we conclude that λ and μ are isomorphisms.

Finally, we need to show that the map $f \otimes \alpha \mapsto f\lambda(\alpha)$ gives an isomorphism $\mathcal{O}_G \otimes_{\mathcal{O}_X} \omega_{G/X} \to \Omega^1_{G/X}$. As $\Omega^1_{G/X}$ is freely generated over $\mathcal{O}_G$ by $d(x)$, we must have $\lambda(d_0(x)) = u(x)d(x)$ for some power series u. As $\omega_{G/X}$ is freely generated over $\mathcal{O}_X$ by $d_0(x)$, it will suffice to check that u is invertible, or equivalently that $u(0)$ is a unit in $\mathcal{O}_X$. To see this, observe that $\mu(f\,d(g)) = f(0)d_0(g)$, so that $d_0(x) = \mu\lambda(d_0(x)) = \mu(u(x)d(x)) = u(0)d_0(x)$, so $u(0) = 1$. ☐

More explicitly, let F be the formal group law such that $x(a+b) = F(x(a), x(b))$, and define $H(s) = D_2F(s,0)$, where D_2F is the partial derivative with respect to the second variable. We observe that $H(0) = 1$, so H is invertible in $R[\![s]\!]$. We then define $\alpha = H(x)^{-1}dx \in \Omega^1_{G/X}$. One can check that, in the notation of the above proof, we have $\alpha = \lambda(d_0(x))$, and thus that α generates $\mathrm{Prim}(\Omega^1_{G/X})$.

7.1. **Heights.**

Proposition 7.3. Let G and H be ordinary formal groups over an affine scheme X, and let $s\colon G \to H$ be a homomorphism. Suppose that the induced map $s^*\colon \omega_H \to \omega_G$ is zero.

(a) If X is a scheme over $\mathrm{spec}(\mathbb{Q})$, then $s = 0$.
(b) If X is a scheme over $\mathrm{spec}(\mathbb{F}_p)$ for some prime p then there is a unique homomorphism $s'\colon F^*_X G \to H$ of formal groups over X such that $s = s' \circ F_{G/X}$.

Proof. It follows from the definitions that our identification of $\omega_{G/X}$ with $\mathrm{Prim}(\Omega_{G/X})$ is natural for homomorphisms. Thus, if $\alpha \in \mathrm{Prim}(\Omega_{H/X})$ then $s^*\alpha = 0$. We also know that $\Omega_{H/X} = \mathcal{O}_H \otimes_{\mathcal{O}_X} \omega_{H/X}$, so any element of $\Omega_{H/X}$ can be written as $f\alpha$ with $f \in \mathrm{Prim}(\Omega_{H/X})$. Thus $s^*(f\alpha) = (f \circ s).s^*\alpha = 0$. Thus, Proposition 5.32 applies to s. If X lies over $\mathrm{spec}(\mathbb{Q})$ then we conclude that s is constant on each fibre. As it is a homomorphism, it must be the zero map. Suppose instead that X lies over $\mathrm{spec}(Fp)$. In that case we know that there is a unique map $s' \colon G' = F_X^* G \to H$ such that $s = s' \circ F_{G/X}$, and we need only check that this is a homomorphism. In other words, we need to check that the map $t'(u,v) = s'(u+v) - s'(u) - s'(v)$ (from $G' \times_X G'$ to H) is zero. Because s and $F_{G/X}$ are homomorphisms, we see that $t' \circ F_{G \times_X G/X} = 0 \colon G \times_X G \to H$. Using the uniqueness clause in Proposition 5.32, we conclude that $t' = 0$ as required. □

Corollary 7.4. Let G and H be ordinary formal groups over an affine scheme X, which lies over $\mathrm{spec}(\mathbb{F}_p)$. Let $s \colon G \to H$ be a homomorphism. Then either $s = 0$ or there is an integer $n \geq 0$ and a homomorphism $s' \colon (F_X^n)^* G \to H$ such that $s = s' \circ F_{G/X}^n$ and $(s')^*$ is nonzero on $\omega_{H/X}$.

Proof. Suppose that there is a largest integer n (possibly 0) such that s can be factored in the form $s = s' \circ F_{G/X}^n$. Write $G' = (F_X^n)^* G$, so that $s' \colon G' \to H$. If $(s')^* = 0$ on $\omega_{H/X}$ then the proposition gives a factorisation $s' = s'' \circ F_{G'/X}$ and thus $s = s'' \circ F_{G/X}^{n+1}$ contradicting maximality. Thus $(s')^* \neq 0$ as claimed. On the other hand, suppose that there is no largest n. Choose coordinates x and y on G and H, so there is a series g such that $y(s(u)) = g(x(u))$ for all points u of G. As s is a homomorphism we have $g(0) = 0$. If s factors through $F_{G/X}^n$ we see that $g(x) = h(x^{p^n})$ for some series h. As this happens for arbitrarily large n, we see that g is constant. As $g(0) = 0$ we conclude that $g = 0$ and thus $s = 0$. □

Definition 7.5. Let G and H be ordinary formal groups over an affine scheme X, which lies over $\mathrm{spec}(\mathbb{F}_p)$. Let $s \colon G \to H$ be a homomorphism. If $s = 0$, we say that s has *infinite height*. Otherwise, the *height* of s is defined to be the integer n occurring in Corollary 7.4. The height of the group G is defined to be the height of the endomorphism $p_G \colon G \to G$ (which is just p times the identity map).

Definition 7.6. Let G be an ordinary formal group over an affine scheme X. Let X_n be the largest closed subscheme of X on which G has height at least n, and write $G_n = G \times_X X_n$. We then have a map $s_n \colon H_n = (F_X^n)^* G_n \to G_n$ such that $p_{G_n} = s_n \circ F_{G/X}^n$, and thus a map $s_n^* \colon \omega_{G_n} \to \omega_{H_n}$ of trivialisable line bundles over X_n. If we trivialise these line bundles then s_n^* becomes an element $u_n \in \mathcal{O}_{X_n}$, which is well-defined up to multiplication by a unit, and $X_{n+1} = V(u_n) = \mathrm{spec}(\mathcal{O}_{X_n}/u_n)$. Note also that $u_0 = p$.

We say that G is *Landweber exact* if for all p and n, the element u_n is not a zero-divisor in $\mathcal{O}_{X_n}$. Because $X_0 = X$ and $u_0 = p$, this implies in particular that $\mathcal{O}_X$ is torsion-free.

7.2. **Logarithms.**

Definition 7.7. A *logarithm* for an ordinary formal group G is a map of formal schemes $u \colon G \to \widehat{\mathbb{A}}^1$ satisfying $u(g+h) = u(g) + u(h)$, or in other words a homomorphism $G \to \widehat{\mathbb{G}}_a$. A logarithm for a formal group law F over a ring R is a power series $f(s) \in R[\![s]\!]$ such that $f(F(s,t)) = f(s) + f(t) \in R[\![s,t]\!]$. Clearly, if x is a coordinate on G and F is the associated formal group law then logarithms for F biject with logarithms for G by $u(g) = f(x(g))$. It is also clear that when u is a logarithm, the differential du lies in ω_G. We thus have a map $d \colon \mathrm{Hom}(G, \widehat{\mathbb{G}}_a) \to \mathbb{A}(\omega_G)$.

Proposition 7.8. If $\mathcal{O}_X$ is a $\mathbb{Q}$-algebra then the map $d \colon \mathrm{Hom}(G, \widehat{\mathbb{G}}_a) \to \mathbb{A}(\omega_G)$ is an isomorphism.

Proof. If $u = f(x)$ is a logarithm and $du = f'(x)dx = 0$ then f is constant (because $\mathcal{O}_X$ is rational so we can integrate) but $f(0) = 0$ (because $u(0) = u(0+0) = u(0) + u(0)$) so $f = 0$ so $u = 0$. Thus d is injective. Conversely, suppose that $\alpha = g(x)dx \in \omega_G$. Let f be the integral of g with $f(0) = 0$, so $u = f(x) \colon G \to \widehat{\mathbb{A}}^1$ and $du = \alpha$. Consider the function $w(g,h) = u(g+h) - u(g) - u(h)$, so $w \colon G \times_X G \to \widehat{\mathbb{A}}^1$ and $dw = \sigma^*\alpha - \pi_1^*\alpha - \pi_2^*\alpha = 0$. Thus w is constant and $w(0,0) = 0$ so $u(g+h) = u(g) + u(h)$ as required. □

Corollary 7.9. Any ordinary formal group over a scheme X over $\mathrm{spec}(\mathbb{Q})$ is isomorphic to the additive group $\mathbb{A}^1 \times X$. □

7.3. **Divisors.** An ordinary formal group G over X is in particular a pointed formal curve over X, so it makes sense to consider the schemes $\mathrm{Div}_n^+(G) = G_X^n/\Sigma_n$ and so on. Moreover, Proposition 6.13 tells us that $\mathrm{Div}^+(G) = M^+(G)$ and so on.

Proposition 7.10. The formal scheme $\mathrm{Div}^+(G)$ has a natural structure as a commutative semiring object in the category $\widehat{\mathcal{X}}_X$.

Proof. Everywhere in this proof, products really mean fibre products over X.

We define a map $\nu_{i,j}\colon G^i \times G^j \to G^{ij}$ by

$$\nu_{i,j}(a_1,\dots,a_i,b_1,\dots,b_j) = (a_1+b_1,\dots,a_i+b_j).$$

Using the fact that the colimits involved are strong, we see that there is a unique map $\mu_{i,j}\colon G^i/\Sigma_i \times G^j/\Sigma_j \to G^{ij}/\Sigma_{ij}$ that is compatible with the maps $\nu_{i,j}$ in the evident sense. We can use the isomorphisms $\mathrm{Div}_i^+(G) = G^i/\Sigma_i$ and $\mathrm{Div}^+(G) = \coprod_i \mathrm{Div}_i^+(G)$ to piece these maps together, giving a map $\mu\colon \mathrm{Div}^+(G) \times \mathrm{Div}^+(G) \to \mathrm{Div}^+(G)$. Given two divisors D and E we write $D * E = \mu(D,E)$. The above discussion really just shows that the definition $(\sum_i [a_i]) * (\sum_j [b_j]) = \sum_{i,j}[a_i+b_j]$ makes sense. It is easy to check (although tedious to write out in detail) that the operation $*$ is associative and commutative, and that the divisor $[0]$ is a unit for it, and that it distributes over addition. Thus, $\mathrm{Div}^+(G)$ is a semiring object in $\widehat{\mathfrak{X}}_X$ as claimed. □

Remark 7.11. One can also interpret and prove the statement that $\mathrm{Div}^+(G)$ is a graded λ-semiring object in $\widehat{\mathfrak{X}}_X$, with

$$\lambda^k(\sum_{i=1}^n [a_i]) = \sum_{i_1<\dots<i_k} [a_{i_1} + \dots + a_{i_k}].$$

Proposition 7.12. The formal scheme $\mathrm{Div}(G)$ has a natural structure as a commutative ring object in the category $\widehat{\mathfrak{X}}_X$.

Proof. We know that $\mathrm{Div}(G) = M(G)$ is a group under addition. It thus makes sense to define a map $\mu(n,m)\colon \mathrm{Div}^+(G) \times_X \mathrm{Div}^+(G) \to \mathrm{Div}(G)$ by

$$\mu(n,m)(D,E) = D * E - mE - nD + nm[0].$$

It is easy to check that

$$\mu(n+i,m+j)(D+i[0],E+j[0]) = \mu(n,m)(D,E).$$

Recall that $\mathrm{Div}(G) = \varinjlim_n \mathrm{Div}^+(G)$, where the maps in the diagram are of the form $D \mapsto D + i[0]$. This is a filtered colimit and thus a strong one, so $\mathrm{Div}(G) \times_X \mathrm{Div}(G) = \varinjlim_{m,n} \mathrm{Div}^+(G) \times_X \mathrm{Div}^+(G)$, where the maps have the form $(D,E) \mapsto (D+i[0], E+j[0])$. It follows that the maps $\mu(n,m)$ fit together to give a map $\mu\colon \mathrm{Div}(G) \times_X \mathrm{Div}(G) \to \mathrm{Div}(G)$. We leave it to the reader to check that this product makes $\mathrm{Div}(G)$ into a ring object. □

8. Formal schemes in algebraic topology

In this section, we show how suitable cohomology theories give rise to functors from suitable categories of spaces to formal schemes. In particular, the space $\mathbb{C}P^\infty$ gives rise to a formal group G. We show how vector bundles over spaces give rise to divisors on G over the corresponding formal schemes, and we investigate the schemes arising from classifying spaces of Abelian Lie groups. We then give a related construction that associates informal schemes to ring spectra. Using this we relate the Thom isomorphism to the theory of torsors, and maps of ring spectra to homomorphisms of formal groups.

8.1. Even periodic ring spectra.

In this section, we define the class of cohomology theories that we wish to study. We would like to restrict attention to commutative ring spectra, but unfortunately that would exclude some examples that we really need to consider. We therefore make the following *ad hoc* definition, which should be ignored at first reading.

Definition 8.1. Let E be an associative ring spectrum, with multiplication $\mu\colon E \wedge E \to E$. A map $Q\colon E \to \Sigma^d E$ is a *derivation* if we have

$$Q \circ \mu = \mu \circ (1 \wedge Q + Q \wedge 1).$$

A ring spectrum E is *quasi-commutative* if there is a derivation Q of odd degree d and a central element $v \in \pi_{2d}E$ such that $2v = 0$ and

$$\mu - \mu \circ \tau = v\mu \circ (Q \wedge Q).$$

Note that if 2 is invertible in $\pi_* E$ then $v = 0$ and E is actually commutative.

Definition 8.2. An *even periodic ring spectrum* is a quasi-commutative ring spectrum E such that

(1) $\pi_1 E = 0$
(2) $\pi_2 E$ contains a unit.

This implies that $\pi_{\text{odd}}(E) = 0$. Thus, the derivation Q in Definition 8.1 acts trivially on E_*, so E_* is a commutative ring. Similarly, if X is any space such that $E^1 X = 0$ then $E^0 X$ is commutative.

Example 8.3. The easiest example is $E^*X = H^*(X; \mathbb{Z}[u^{\pm 1}])$, where we give u degree 2. This is represented by the even periodic ring spectrum

$$HP = \bigvee_{k \in \mathbb{Z}} \Sigma^{2k} H.$$

Example 8.4. The next most elementary example is the complex K-theory spectrum KU. This is an even periodic ring spectrum, by the Bott periodicity theorem. If p is a prime then we can smash this with the mod p Moore space to get a spectrum KU/p. It is true but not obvious that this is a ring spectrum. It is commutative when $p > 2$, but only quasi-commutative when $p = 2$. The derivation Q in Definition 8.1 is just the Bockstein map $\beta\colon KU/2 \to \Sigma KU/2$.

Example 8.5. Let MP be the Thom spectrum associated to the tautological virtual bundle over $\mathbb{Z} \times BU$. It is more usual to consider the connected component $BU = 0 \times BU$ of $\mathbb{Z} \times BU$, giving the Thom spectrum MU. It turns out that $MP = \bigvee_{k \in \mathbb{Z}} \Sigma^{2k} MU$, and that this is an even periodic ring spectrum. Moreover, a fundamental theorem of Quillen tells us that $MP_0 = L = \mathcal{O}_{\text{FGL}}$.

Example 8.6. It turns out [5, 28] that given any ring E_0 that can be obtained from $MP_0[\frac{1}{2}]$ by inverting some elements and killing a regular sequence, there is a canonical even periodic ring spectrum E with $\pi_0 E = E_0$. If we work over MP_0 rather than $MP_0[\frac{1}{2}]$ then things are more complicated, but typically not too different in cases of interest, except that we only have quasi-commutativity rather than commutativity. Because $MP_0 = \mathcal{O}_{\text{FGL}}$, the theory of formal group laws provides us with many naturally defined rings E_0 to which we can apply this result.

8.2. **Schemes associated to spaces.** Let E be an even periodic ring spectrum. We write $S_E = \operatorname{spec}(E^0)$.

Example 8.7. As mentioned above, Quillen's theorem tells us that $S_{MP} = \text{FGL}$. Less interestingly, we have $S_{HP} = S_K = 1 = \operatorname{spec}(\mathbb{Z})$, the terminal scheme.

If Z is a finite complex, we write $Z_E = \operatorname{spec}(E^0 Z) \in \mathcal{X}_{S_E}$. This is a covariant functor of Z. If Z is an arbitrary space, we write $\Lambda(Z)$ for the category of pairs (W, w), where W is a finite complex and w is a homotopy class of maps $W \to Z$.

Lemma 8.8. The category $\Lambda(Z)$ is filtered and essentially small.

Proof. It is well-known that every finite CW complex is homotopy equivalent to a finite simplicial complex, and that there are only countably many isomorphism types of finite simplicial complexes. It follows easily that $\Lambda(Z)$ is essentially small. If (W, w) and (V, v) are objects of $\Lambda(Z)$ then there is an evident map $u\colon U = V \amalg W \to Z$ whose restrictions to V and W are v and w. Thus $(U, u) \in \Lambda(Z)$, and there are maps $(V, v) \to (U, u) \leftarrow (W, w)$ in $\Lambda(Z)$.

On the other hand, suppose we have a parallel pair of maps $f_0, f_1\colon (V, v) \to (W, w)$ in $\Lambda(Z)$. Let U be the space $(W \amalg V \times I)/\sim$, where $(x, t) \sim f_t(x)$ whenever $x \in V$ and $t \in \{0, 1\}$. Let $g\colon W \to U$ be the evident inclusion, so clearly $gf_0 \simeq gf_1$. We are given that wf_0 and wf_1 are homotopic to v. A choice of homotopy between wf_0 and wf_1 gives a map $u\colon U \to X$ with $ug = w$. Thus g is a map $(W, w) \to (U, u)$ in $\Lambda(Z)$ with $gf_0 = gf_1$. This proves that $\Lambda(Z)$ is filtered. □

Remark 8.9. Let Z be a space with a given CW structure, and let $\Lambda_{\text{CW}}(Z)$ be the directed set of finite subcomplexes of Z. Then there is an evident functor $\Lambda_{\text{CW}}(Z) \to \Lambda(Z)$, which is easily seen to be cofinal. We can also define $\Lambda_{\text{stable}}(Z)$ to be the filtered category of finite spectra W equipped with a map $w\colon W \to \Sigma^\infty Z_+$. There is an evident stabilisation functor $\Lambda(Z) \to \Lambda_{\text{stable}}(Z)$, and one checks that this is also cofinal.

Remark 8.10. Given two spaces Y and Z, there is a functor $\Lambda(Y) \times \Lambda(Z) \to \Lambda(Y \times Z)$ given by $((V,v),(W,w)) \mapsto (V \times W, v \times w)$. This is always cofinal, as one can see easily from the previous remark (for example).

Definition 8.11. For any space Z, we write

$$Z_E = \varinjlim_{(W,w)\in\Lambda(Z)} \operatorname{spec}(E^0W) \in \widehat{\mathcal{X}}_{S_E}.$$

We also give E^0Z the linear topology defined by the ideals $I_{(W,w)} = \ker(E^0Z \xrightarrow{w^*} E^0W)$. Thus

$$\operatorname{spf}(E^0Z) = \varinjlim_{\Lambda(Z)} \operatorname{spec}(\operatorname{image}(E^0Z \to E^0W)).$$

We write $\widehat{E}^0Z$ for the completion of E^0Z. There is an evident map $Z_E \to \operatorname{spf}(E^0Z)$. Also, if Y is another space then the projection maps $Y \leftarrow Y \times Z \to Z$ give rise to a map $(Y \times Z)_E \to Y_E \times_{S_E} Z_E$.

Remark 8.12. We know from [1] that the map $E^0Z \to \varprojlim_{\Lambda(Z)} E^0W$ is surjective; the kernel is the ideal of phantom maps. It is clear that the map $E^0(Z)/I_{(W,w)} \to E^0W$ is injective, so the same is true of the map

$$\varprojlim E^0(Z)/I_{(W,w)} \to \varprojlim E^0W.$$

It follows by diagram chasing that $\widehat{E}^0Z = \varprojlim E^0(Z)/I_{(W,w)} = \varprojlim E^0W$, and that this is a quotient of E^0Z. From this we see that E^0Z is complete if and only if there are no phantom maps $Z \to E$.

Definition 8.13. We say that Z is *tolerable* (relative to E) if $Z_E = \operatorname{spf}(E^0Z)$ and $(Y \times Z)_E = Y_E \times_{S_E} Z_E$ for all finite complexes Y.

Proposition 8.14. If Z is tolerable and Y is arbitrary then

$$(Y \times Z)_E = Y_E \times_{S_E} Z_E.$$

If Y is also tolerable then so is $Y \times Z$, and $\widehat{E}^0(Y \times Z) = \widehat{E}^0(Y)\widehat{\otimes}_{E^0}\widehat{E}^0(Z)$. Of course if E^0Y, E^0Z and $E^0(Y \times Z)$ are complete this means that $E^0(Y \times Z) = E^0Y\widehat{\otimes}_{E^0}E^0Z$.

Proof. If we fix $V \in \Lambda(Y)$ then the functor from $\Lambda(Z)$ to $\Lambda(V \times Z)$ given by $W \mapsto V \times W$ is clearly cofinal, so $\varinjlim_W (V \times W)_E = (V \times Z)_E$, and this is the same as $V_E \times_{S_E} Z_E$ because Z is tolerable and V is finite. If we now take the colimit over V and use the fact that filtered colimits of formal schemes commute with finite limits, we find that $\varinjlim_{V,W} (V \times W)_E = Y_E \times_{S_E} Z_E$. It follows from Remark 8.10 that $(Y \times Z)_E = \varinjlim_{V,W} (V \times W)_E$, so the first claim follows.

Now suppose that Y is tolerable. Then

$$\begin{aligned}(Y \times Z)_E &= Y_E \times_{S_E} Z_E \\ &= \operatorname{spf}(E^0Y) \times_{S_E} \operatorname{spf}(E^0Z) \\ &= \operatorname{spf}(\widehat{E}^0Y) \times_{S_E} \operatorname{spf}(\widehat{E}^0Z) \\ &= \operatorname{spf}(\widehat{E}^0Y\widehat{\otimes}_{E^0}\widehat{E}^0Z).\end{aligned}$$

It follows that $\widehat{E}^0(Y \times Z) = \mathcal{O}_{(Y\times Z)_E} = \widehat{E}^0Y\widehat{\otimes}_{E^0}\widehat{E}^0Z$ as claimed. It also follows that $(Y \times Z)_E$ is solid, and thus that $(Y \times Z)_E = \operatorname{spf}(E^0(Y \times Z))$.

Now let X be a finite complex. We need to show that $(X \times Y \times Z)_E = X_E \times_{S_E} (Y \times Z)_E = X_E \times_{S_E} Y_E \times_{S_E} Z_E$. In fact, we have $(X \times Y)_E = X_E \times_{S_E} Y_E$ because Y is tolerable, and $((X \times Y) \times Z)_E = (X \times Y)_E \times_{S_E} Z_E$ because Z is tolerable, and the claim follows. ☐

Definition 8.15. A space Z is *decent* if H_*Z is a free Abelian group, concentrated in even degrees.

Example 8.16. The spaces $\mathbb{C}P^\infty$, $BU(n)$, $\mathbb{Z} \times BU$, BSU and ΩS^{2n+1} are all decent.

Proposition 8.17. Let Z be a decent space. Then Z is tolerable for any E, and Z_E is coalgebraic over S_E. Moreover, for any map $E \to E'$ of even periodic ring spectra, the resulting diagram

$$\begin{array}{ccc} Z_{E'} & \longrightarrow & Z_E \\ \downarrow & & \downarrow \\ S_{E'} & \longrightarrow & S_E \end{array}$$

is a pullback.

Proof. We may assume that Z is connected (otherwise treat each component separately). As $H_1 Z = 0$ we see that $\pi_1 Z$ is perfect, so we can use Quillen's plus construction to get a homology equivalence $Z \to Z^+$ such that $\pi_1(Z^+) = 0$. By the stable Whitehead theorem, this map is a stable equivalence, so $E^0(Y \times Z^+) = E^0(Y \times Z)$ for all Y. We may thus replace Z by Z^+ and assume that $\pi_1 Z = 0$. This step is not strictly necessary, but it seems the cleanest way to avoid trouble from the fundamental group. Given this, it is well-known that Z has a CW structure in which all the cells have even dimension. It follows that the Atiyah-Hirzebruch spectral sequence collapses and that E_*Z is a free module over E_*, with one generator e_i for each cell. As E_* is two-periodic, we can choose these generators in degree zero. Similarly, $E_*(Z \times Z)$ is free on generators $e_i \otimes e_j$ and thus is isomorphic to $E_*(Z) \otimes_{E_*} E_*(Z)$, so we can use the diagonal map to make E_*Z into a coalgebra over E_*. By periodicity, $E_0(Z \times Z) = E_0(Z) \otimes_{E_0} E_0(Z)$ and $E_0 Z$ is a coalgebra over E_0, and is freely generated as an E_0-module by the e_i.

If W is a finite subcomplex of Z, it is easy to see that $E_0 W$ is a standard subcoalgebra of $E_0 Z$ (in the language of Definition 4.58). Moreover, any finite collection of cells lies in a finite subcomplex, so it follows that any finitely generated submodule of $E_0 Z$ lies in a standard subcoalgebra. It follows that $\{e_i\}$ is a good basis for $E_0 Z$, so that $E_0 Z \in \mathcal{C}'_{S_E}$.

It follows from the above in the usual way that $E \wedge Z_+$ is equivalent as an E-module spectrum to a wedge of copies of E (one for each cell), and thus that $E^*Z = \mathrm{Hom}_{E^*}(E_*Z, E_*)$. Using the periodicity we conclude that $E^0 Z = \mathrm{Hom}_{E^0}(E_0 Z, E^0)$. It follows that $\mathrm{spf}(E^0 Z) = \mathrm{sch}_{S_E}(E_0 Z)$ is a solid formal scheme, which is coalgebraic over S_E. It is also easy to check that $\mathrm{spf}(E^0 Z)$ is the colimit of the schemes $\mathrm{spec}(E^0 W)$ as W runs over the finite subcomplexes. It follows from Remark 8.9 that $\mathrm{spf}(E^0 Z) = Z_E$.

Now let Y be another space. Let W be a finite subcomplex of Z, and let (V, v) be an object of $\Lambda(Y)$. The usual Künneth arguments show that $E^0(W \times V) = E^0 W \otimes_{E^0} E^0 V$, and thus that $(W \times V)_E = W_E \times_{S_E} V_E$. Using Remark 8.10 we conclude that

$$(Z \times Y)_E = \varinjlim_{W,V} W_E \times_{S_E} V_E = (\varinjlim_W W_E) \times_{S_E} (\varinjlim_V V_E) = Z_E \times_{S_E} Y_E.$$

This proves that Z is tolerable. We leave it to the reader to check that a map $E \to E'$ gives an isomorphism $Z_{E'} = Z_E \times_{S_E} S_{E'}$. ☐

Example 8.18. It follows from the proposition that the spaces $\mathbb{C}P^\infty$, $BU(n)$, $\mathbb{Z} \times BU$, BSU and ΩS^{2n+1} are all tolerable, and the corresponding schemes are coalgebraic over S_E. The case of $\mathbb{C}P^\infty$ is particularly important. We note that $\mathbb{C}P^\infty = BS^1 = K(\mathbb{Z}, 2)$ is an Abelian group object in the homotopy category, so $G_E = \mathbb{C}P^\infty{}_E$ is an Abelian formal group over S_E. Because $H^*\mathbb{C}P^\infty = \mathbb{Z}[\![x]\!]$, the Atiyah-Hirzebruch spectral sequence tells us that $E^0\mathbb{C}P^\infty = E^0[\![x]\!]$ (although this does not give a canonical choice of generator x). This means that $G_E \simeq \widehat{\mathbb{A}}^1 \times S_E$ in Based$\widehat{\mathcal{X}}_{S_E}$, so that G_E is an ordinary formal group.

We next recall that for $n > 0$ there is a quasicommutative rings spectrum $P(n) = BP/I_n$ with $P(n)^* = \mathbb{F}_p[v_k \mid k \geq 0]$, where v_k has degree $-2(p^k - 1)$. The cleanest construction now available is given in [5, 28], although of course there are much older constructions using Baas-Sullivan theory. We also have $P(0) = BP$, with $P(0)^* = \mathbb{Z}_{(p)}[v_k \mid k > 0]$.

Definition 8.19. Let E be an even periodic ring spectrum. We say that E is an *exact $P(n)$-module* (for some $n \geq 0$) if it is a module-spectrum over $P(n)$, and the sequence $(v_n, v_{n+1}, \ldots)$ is regular on E_*.

Proposition 8.20. Let E be an exact $P(n)$-module. Let Z be a CW complex of finite type such that $K(m)_*Z$ is concentrated in even degrees for infinitely many m. If $n = 0$, assume that $H^s(Z;\mathbb{Q}) = 0$ for $s \gg 0$. Then Z is tolerable for E.

Remark 8.21. When combined with Proposition 8.14 this gives a useful Künneth theorem.

The proof will follow after Corollary 8.27. Many spaces are known to which this applies: simply connected finite Postnikov towers of finite type, classifying spaces of many finite groups and compact Lie groups, the spaces QS^{2m}, BO, ImJ and $BU\langle 2m\rangle$ for example. See [24] for more details. The proof of our proposition will also rely heavily on the results of that paper.

We next need some results involving the pro-completion of the category of graded Abelian groups, which we denote by $\mathrm{Pro}(\mathrm{Ab}_*)$. It is necessary to distinguish this carefully from the category $\mathrm{Pro}(\mathrm{Ab})_*$ of graded systems of pro-groups. A tower of graded groups can be regarded as an object in either category, but the morphisms are different. A tower $\{A_{0*} \leftarrow A_{1*} \leftarrow \cdots\}$ in $\mathrm{Pro}(\mathrm{Ab}_*)$ is pro-trivial if for all j, there exists $k > j$ such that the map $A_{k*} \to A_{j*}$ is zero. It is pro-trivial in $\mathrm{Pro}(\mathrm{Ab})_*$ if for all j and d there exists k such that the map $A_{kd} \to A_{jd}$ is zero. Because k is allowed to depend on d, this is a much weaker condition than triviality in $\mathrm{Pro}(\mathrm{Ab}_*)$. Note also that if $R_* \to R'_*$ is a map of graded rings, and $\{M_{\alpha *}\}$ is a pro-system of R_*-modules that is trivial in $\mathrm{Pro}(\mathrm{Ab}_*)$, then the same is true of $R'_* \otimes_{R_*} M_*$. However, the corresponding statement for $\mathrm{Pro}(\mathrm{Ab})_*$ is false.

Remark 8.22. If E is an exact $P(n)$-module, we know from work [29] of Yagita that the functor $M \mapsto E^* \otimes_{P(n)^*} M$ is an exact functor on the category of $P(n)^*P(n)$-modules that are finitely presented as modules over $P(n)^*$. (This category is Abelian, because the ring $P(n)^*$ is coherent.) It follows that $E^*Z = E^* \otimes_{P(n)^*} P(n)^*Z$ for all finite complexes Z.

The following lemma is largely a paraphrase of results in [24].

Lemma 8.23. Fix $n \geq 0$. Suppose that Z is a CW complex of finite type, and write Z^r for the r-skeleton of Z. If $n = 0$ we also assume that $H^s(Z;\mathbb{Q}) = 0$ for $s \gg 0$. Let $F^{r+1} = \ker(P(n)^*Z \to P(n)^*Z^r)$ denote the $(r+1)$'st Atiyah-Hirzebruch filtration in $P(n)^*Z$. Then the tower $\{P(n)^*Z^r\}_{r\geq 0}$ is isomorphic to $\{P(n)^*(Z)/F^{r+1}\}_{r\geq 0}$ in $\mathrm{Pro}(\mathrm{Ab}_*)$, and thus is Mittag-Leffler. Moreover, the groups $P(n)^*(Z)/F^{r+1}$ are finitely presented modules over $P(n)^*$, and their inverse limit is $P(n)^*Z$.

Proof. Write $P = P(n)$ for brevity. Write $A_r = P^*Z^r$ and

$$B_r = P^*(Z)/F^{r+1} = \mathrm{image}(P^*Z \to A_r).$$

We then have an inclusion of towers $\{B_r\} \to \{A_r\}$, for which we need to provide an inverse in the Pro-category. We claim that for each r, there exists $m(r) > r$ such that the image of the map $A_{m(r)} \to A_r$ is precisely B_r. We will deduce the lemma from this before proving it. Define $m_0 = 0$ and $m_{k+1} = m(m_k) > m_k$. By construction, the map $A_{m_{k+1}} \to A_{m_k}$ factors through $B_{m_k} \subseteq A_{m_k}$. One checks that the resulting maps $A_{m_{k+1}} \to B_{m_k}$ are compatible as k varies, and that they provide the required inverse. We also know that P^* is a coherent ring, so the category of finitely presented modules is Abelian and closed under extensions. It follows in the usual way that A_r is finitely presented for all r, and thus that $B_r = \mathrm{image}(A_{m(r)} \to A_r)$ is finitely presented.

We now need to show that $m(r)$ exists. By the basic setup of the Atiyah-Hirzebruch spectral sequence, it suffices to show that for large m, the first $r+1$ columns in the spectral sequence for P^*Z^m are the same as in the spectral sequence for P^*Z. This is Lemma 4.4 of [24]. (When $n = 0$, we need to check that we are in the case $P(0) = BP$ of their Definition 1.5. This follows from our assumption that $H^s(Z;\mathbb{Q}) = 0$ for $s \gg 0$.)

Finally, we need to show that $P^*Z = \varprojlim_r P^*(Z)/F^{r+1}$. This is essentially [24, Corollary 4.8]. $\square$

Corollary 8.24. Let Z and n be as in the Lemma, and let E be an exact $P(n)$-module. Then E^0Z is complete, and $Z_E = \mathrm{spf}(E^0Z)$, and $E^*Z = E^* \widehat{\otimes}_{P(n)^*} P(n)^*Z$. Moreover we have isomorphisms

$$\begin{aligned}\{E^*(Z^r)\} \simeq \{E^*(Z)/F^{r+1}\} &\simeq \{E^* \otimes_{P(n)^*} P(n)^*Z^r\} \\ &\simeq \{E^* \otimes_{P(n)^*} (P(n)^*(Z)/F^{r+1})\}\end{aligned}$$

in $\mathrm{Pro}(\mathrm{Ab}_*)$.

Proof. We reuse the notation of the previous proof. We also define $A'_r = E^* \otimes_{P^*} A_r$ and $B'_r = E^* \otimes_{P^*} B_r$. As Z^r is finite we see that $A'_r = E^* Z^r$. Next recall that for any r we can choose $m > r$ such that $B_r = \text{image}(A_m \to A_r)$. As the functor $E^* \otimes_{P^*} (-)$ is exact on finitely presented comodules, we see that B'_r is the image of the map $A'_m \to A'_r$ and in particular that the map $B'_r \to A'_r$ is injective. Next, the map $E^* \otimes_{P^*} P^*Z \to E^* \otimes_{P^*} P^*Z^m = E^*Z^m = A'_m$ clearly factors through E^*Z, so our epimorphism $P^*Z \to A_m \to B_r$ gives an epimorphism $E^* \otimes_{P^*} P^*Z \to A'_m \to B'_r$ which factors through E^*Z, so the map $E^*Z \to B'_r$ is surjective. Thus $B'_r = \text{image}(E^*Z \to E^*Z^r) = E^*(Z)/F^{r+1}$. We can now apply the functor $E^* \otimes_{P^*} (-)$ to the pro-isomorphisms in the Lemma to get the pro-isomorphisms in the present corollary. This makes it clear that the tower $\{E^*Z^r\}$ is Mittag-Leffler so the Milnor sequence tells us that

$$E^*Z = \varprojlim_r E^*Z^r = \varprojlim_r E^*(Z)/F^{r+1}.$$

This means in particular that E^0Z is complete with respect to the linear topology generated by the ideals F^{r+1}, which is easily seen to be the same as the topology in Definition 8.11. Moreover, we have an isomorphism $\{A'_r\} \simeq \{B'_r\}$ in the Pro category of groups, and it is easy to see from the construction that this is actually an isomorphism in the Pro category of rings as well, so by applying $\text{spec}(-)$ we get an isomorphism in the Ind category of schemes, which is just the category of formal schemes. From the definitions we have $Z_E = \varinjlim_r \text{spec}(A'_r)$ and $\text{spf}(E^0Z) = \varinjlim_r \text{spec}(B'_r)$, so we conclude that $Z_E = \text{spf}(E^0Z)$. ☐

Lemma 8.25. Let E and Z be as in Corollary 8.24, and suppose that $K(m)^*Z$ is concentrated in even degrees for infinitely many m. Then the ring E^*Z is Landweber exact over $P(n)^*$, so the function spectrum $F(Z_+, E)$ is an exact $P(n)$-module.

Proof. We know from [24, Lemma 5.3] that $P(m)^*Z$ is concentrated in even degrees for all m, and from [24, Corollary 4.6] that the tower $\{P(m)^*Z^r\}$ has the Mittag-Leffler property. It follows that the tower $\{P(m)^{\text{odd}}Z^r\}$ is pro-trivial. Next, consider the cofibration $\Sigma^{2(p^m-1)}P(m) \xrightarrow{v_m} P(m) \to P(m+1) \to \Sigma^{2p^m-1}P(m)$. This gives a pro-exact sequence of towers

$$0 \to \{P(m)^{\text{ev}}Z^r\} \xrightarrow{v_m} \{P(m)^{\text{ev}}Z^r\} \to \{P(m+1)^{\text{ev}}Z^r\} \to 0.$$

It follows that the sequence $(v_n, v_{n+1}, \ldots)$ acts regularly on the tower $\{P(n)^*Z^r\}$. Next, for any spectrum X we have a map $P(m)\wedge X \to P(m)\wedge BP\wedge X$ which makes $P(m)_*X$ a comodule over $P(m)_*BP = BP_*BP/I_n$. Moreover, we have $P(m)_*X \otimes_{P(m)_*} P(m)_*BP = P(m)_*X \otimes_{BP_*} BP_*BP$ so this actually makes $P(m)_*X$ into a comodule over BP_*BP. One can check from this construction that the maps $\Sigma^{2(p^m-1)}P(m) \xrightarrow{v_m} P(m) \to P(m+1) \to \Sigma^{2p^m-1}P(m)$ give rise to maps of comodules, so our whole diagram of towers is a diagram of finitely-presented comodules over $P(n)_*BP$. The functor $E^* \otimes_{P(n)^*} (-)$ is exact on this category. It is easy to conclude by induction that $\{E^* \otimes_{P(n)^*} P(m)^*Z^r\} \simeq \{E^*(Z^r)/I_m\}$, that the odd dimensional part of these towers is pro-trivial, the towers are Mittag-Leffler, and the sequence $(v_n, v_{n+1}, \ldots)$ is regular on the tower $\{E^*(Z^r)\}$. We can now pass to the inverse limit (using the Mittag-Leffler property to show that the $\varprojlim^1$ terms vanish) to see that the sequence $(v_n, v_{n+1}, \ldots)$ is regular on $E^*(Z)$. ☐

Our next few results are closely related to those of [24, Section 9], although a precise statement of the relationship would be technical.

Lemma 8.26. Let Z be a CW complex of finite type such that $K(m)^*Z$ is concentrated in even degrees for infinitely many m. If $n = 0$ we also assume that $H^s(Z; \mathbb{Q}) = 0$ for $s \gg 0$. Then for any finite spectrum W we have pro-isomorphisms

$$\begin{aligned}\{P(n)^*(Z^r \times W)\} &\simeq \{P(n)^*Z^r \otimes_{P(n)^*} P(n)^*W\} \\ &\simeq \{P(n)^*(Z)/F^{r+1} \otimes_{P(n)^*} P(n)^*W\},\end{aligned}$$

and these towers are Mittag-Leffler. Moreover, we have isomorphisms

$$P(n)^*(Z \times W) = P(n)^*Z \otimes_{P(n)^*} P(n)^*W = P(n)^*Z \widehat{\otimes}_{P(n)^*} P(n)^*W.$$

Proof. Write $P = P(n)$ for brevity. The usual Landweber exactness argument shows that $W \mapsto P^*Z \otimes_{P^*} P^*W$ is a cohomology theory and thus that it coincides with $P^*(Z\times W)$. We can also do the same argument with pro-groups. We saw in the proof of the previous lemma that the sequence $(v_n, v_{n+1}, \ldots)$ acts regularly on the pro-group $\{P^*Z^r\}$, so the pro-group $\{\mathrm{Tor}_1^{P^*}(P(m)^*, P^*Z^r)\}$ is trivial for all $m \geq n$. Any finitely presented comodule M^* has a finite Landweber filtration whose quotients have the form $P(m)^*$ for $m \geq n$, and we see by induction on the length of the filtration that $\{\mathrm{Tor}_1^{P^*}(M^*, P^*Z^r)\}$ is trivial. This implies that the construction $M^* \mapsto \{M^* \otimes_{P^*} P^*Z^r\}$ gives an exact functor from finitely presented comodules to $\mathrm{Pro}(\mathrm{Ab}_*)$, so that $W \mapsto \{P^*W \otimes_{P^*} P^*Z^r\}$ is a $\mathrm{Pro}(\mathrm{Ab}_*)$-valued cohomology theory on finite complexes. The construction $W \mapsto \{P^*(W \times Z^r)\}$ gives another such cohomology theory, and we have a natural transformation from the first to the second that is an isomorphism when W is a sphere, so it is an isomorphism in general. Thus $\{P^*(Z^r \times W)\} = \{P^*Z^r \otimes_{P^*} P^*W\}$, as claimed. We have seen that the tower $\{P^*Z^r\}$ is pro-isomorphic to $\{P^*(Z)/F^{r+1}\}$, so it follows that $\{P^*Z^r \otimes_{P^*} P^*W\} \simeq \{P^*(Z)/F^{r+1} \otimes_{P^*} P^*W\}$. The second of these is a tower of isomorphisms, so all three of our towers are Mittag-Leffler as claimed. As $Z \times W$ is the homotopy colimit of the spaces $Z^r \times W$, the Milnor sequence gives an isomorphism $P^*(Z \times W) = \varprojlim_r P^*(Z)/F^{r+1} \otimes_{P^*} P^*W$, and the right hand side is by definition $P^*(Z)\widehat{\otimes}_{P^*}P^*W$, which completes the proof. □

Corollary 8.27. Let E be an exact $P(n)$-module. Let Z be a CW complex of finite type such that $K(m)^*Z$ is concentrated in even degrees for infinitely many m. If $n = 0$ we also assume that $H^s(Z;\mathbb{Q}) = 0$ for $s \gg 0$. Then for any finite spectrum W we have pro-isomorphisms

$$\{E^0(Z^r \times W)\} \simeq \{E^0Z^r \otimes_{E^0} E^0W\} \simeq \{E^0(Z)/F^{r+1} \otimes_{E^0} E^0W\},$$

and these towers are Mittag-Leffler. Moreover, we have isomorphisms

$$E^0(Z \times W) = E^0Z \otimes_{E^0} E^0W = E^0Z\widehat{\otimes}_{E^0}E^0W.$$

Proof. If we apply the functor $E^* \otimes_{P(n)^*} (-)$ to the pro-isomorphisms in the lemma, we get the pro-isomorphisms in the corollary. We deduce in the same way as in the lemma that $E^0(Z\times W) = E^0Z\widehat{\otimes}_{E^0}E^0W$. On the other hand, we see from Lemma 8.25 that

$$E^*(Z \times W) = [W_+, F(Z_+, E)]^* = E^*Z \otimes_{P(n)^*} P(n)^*W = E^*Z \otimes_{E^*} (E^* \otimes_{P(n)^*} P(n)^*W) = E^*Z \otimes_{E^*} E^*W.$$

Thus $E^0(Z \times W) = E^0Z \otimes_{E^0} E^0W$ as claimed. □

Proof of Proposition 8.20. Corollary 8.24 shows that $Z_E = \mathrm{spf}(E^0Z)$. Write

$$F^{r+1} = \ker(E^0Z \to E^0Z^r),$$

so $Z_E = \varinjlim_r V(F^r)$. Let W be a finite complex. We then have

$$\begin{aligned} Z_E \times_{S_E} W_E &= \varinjlim_r V(F^r) \times_{S_E} W_E \\ &= \varinjlim_r \mathrm{spec}(E^0(Z)/F^r \otimes_{E^0} E^0(W)) \\ &= \mathrm{spf}(E^0(Z)\widehat{\otimes}_{E^0}E^0(W)) \\ &= \mathrm{spf}(E^0(Z \times W)), \end{aligned}$$

where we have used Corollary 8.27. We can apply Lemma 8.23 to $Y \times Z$ and conclude that $\mathrm{spf}(E^0(Y\times Z)) = (Y \times Z)_E$, giving the required isomorphism $(Y \times Z)_E = Y_E \times_{S_E} Z_E$. □

8.3. **Vector bundles and divisors.** Let V be a complex vector bundle of rank n over a tolerable space Z. We write $P(V)$ for the space of pairs (z, W), where $z \in Z$ and W is a line (i.e. a one-dimensional subspace) in V_z. This is clearly a fibre bundle over Z with fibres $\mathbb{C}P^{n-1}$. We write $\mathbb{D}(V) = P(V)_E$. There is a tautological line bundle L over $P(V)$, whose fibre over a pair (z, W) is W. This is classified by a map $P(V) \to \mathbb{C}P^\infty$. By combining this with the projection to Z, we get a map $P(V) \to \mathbb{C}P^\infty \times Z$ and thus a map $\mathbb{D}(V) \to G \times_S Z_E$. The well-known theorem on projective bundles translates into our language as follows.

Proposition 8.28. The above map is a closed inclusion, making $\mathbb{D}(V)$ into an effective divisor of degree n on G.

Proof. Choose an orientation x of E, so $x \in \tilde{E}^0\mathbb{C}P^\infty$. We also write x for the image of x under the map $P(V) \to \mathbb{C}P^\infty$, which is just the Euler class of L. We claim that $E^*P(V)$ is freely generated over E^*Z by $\{1, x, \dots, x^{n-1}\}$, which will prove the claim. This is clear when V is trivialisable. For the general case, we may assume that Z is a regular CW complex. The claim holds when Z is a finite union of subcomplexes on which V is trivialisable, by a well-known Mayer-Vietoris argument. It thus holds when Z is a finite complex, and the general case follows by passing to colimits. □

Proposition 8.29. If V and W are two vector bundles over a tolerable space Z then $\mathbb{D}(V \oplus W) = \mathbb{D}(V) + \mathbb{D}(W)$.

Proof. Choose an orientation, and let x be the Euler class of the usual line bundle over $P(V \oplus W)$. The polynomial $f_{\mathbb{D}(V\oplus W)}(t)$ is the unique one of degree $\dim(V \oplus W)$ of which x is a root, so it suffices to check that $f_{\mathbb{D}(V)}(x)f_{\mathbb{D}(W)}(x) = 0$. There are evident inclusions $P(V) \to P(V \oplus W) \leftarrow P(W)$ with $P(V) \cap P(W) = \emptyset$. Write $A = P(V \oplus W) \setminus P(V)$ and $B = P(V \oplus W) \setminus P(W)$, so that $A \cup B = P(V \oplus W)$. By a well-known argument, if $a, b \in E^0P(V \cup W)$ and $a|_A = 0$ and $b|_B = 0$ then $ab = 0$, so it suffices to check that $f_{\mathbb{D}(V)}(x)|_B = 0$ and $f_{\mathbb{D}(W)}(x)|_A = 0$. It is not hard to see that the inclusions $P(V) \to B$ is a homotopy equivalence and thus that $f_{\mathbb{D}(V)}(x)|_B = 0$, and the other equation is proved similarly. □

Proposition 8.30. If M is a complex line bundle over a tolerable space Z, which is classified by a map $u\colon Z \to \mathbb{C}P^\infty$, then $\mathbb{D}(M)$ is the image of the map $(u, 1)_E\colon Z_E \to (\mathbb{C}P^\infty \times Z)_E = G \times_S Z_E$.

Proof. This follows from the definitions, using the obvious fact that $P(M) = Z$. □

Proposition 8.31. There is a natural isomorphism $BU(n)_E = \mathrm{Div}_n^+(G)$.

Proof. This is essentially well-known, but we give some details to illustrate how everything fits together. Let $T(n)$ be the maximal torus in $U(n)$, so that $BT(n) \simeq (\mathbb{C}P^\infty)^n$ and $BT(n)_E = G_S^n$. Thus, the inclusion $i\colon T(n) \to U(n)$ gives a map $G_S^n \to BU(n)_E$. If $\sigma \in \Sigma_n$ is a permutation, then the evident action of σ on $T(n)$ is compatible with the action on $U(n)$ given by conjugating with the associated permutation matrix. This matrix can be joined to the identity matrix by a path in $U(n)$, so the conjugation is homotopic to the identity. Thus, our map $G_S^n \to BU(n)_E$ factors through a map $\mathrm{Div}_n^+(G) = G_S^n/\Sigma_n \to BU(n)_E$. On the other hand, the tautological bundle V_n over $BU(n)$ gives rise to a divisor $\mathbb{D}(V_n)$ over $BU(n)_E$ and thus a map $BU(n)_E \to \mathrm{Div}_n^+(G)$. The composite $G_S^n = BT(n)_E \to BU(n)_E \to \mathrm{Div}_n^+(G) = G_S^n/\Sigma_n$ classifies the divisor $\mathbb{D}(i^*V_n)$. Let $M_1, \dots, M_n$ be the evident line bundles over $BT(n)$, so that $i^*V_n = M_1 \oplus \dots \oplus M_n$. One checks from this and Propositions 8.29 and 8.30 that the composite is just the usual quotient map $G_S^n \to G_S^n/\Sigma_n$, and thus the composite $\mathrm{Div}_n^+(G) \to BU(n)_E \to \mathrm{Div}_n^+(G)$ is the identity.

Next, we take the space of n-frames in $\mathbb{C}^\infty$ as our model for $EU(n)$. There is then a homeomorphism $EU(n)/(S^1 \times U(n-1)) \to P(V_n)$ (sending $(w_1, \dots, w_n)$ to the pair (L, W), where W is the span of $\{w_1, \dots, w_n\}$ and L is the span of w_1). The left hand side is a model for $\mathbb{C}P^\infty \times BU(n-1)$. By induction on n, we may assume that $BU(n-1)_E = G^{n-1}/\Sigma_{n-1}$. This gives a commutative diagram as follows.

$$\begin{tikzcd}
G \times G^{n-1}/\Sigma_{n-1} \arrow[r, "\simeq"] \arrow[d] & P(V_n)_E \arrow[d, two heads] \\
G^n/\Sigma_n \arrow[r, tail] & BU(n)_E
\end{tikzcd}$$

The top horizontal is an isomorphism by induction and the right hand vertical is faithfully flat, and thus a categorical epimorphism. It follows that the bottom map is an epimorphism, but we have already seen that it is a split monomorphism, so it is an isomorphism as required. □

Definition 8.32. Let x be a coordinate on G. If V is a vector bundle of rank n over a tolerable space Z, then we have $\mathbb{D}(V) = \mathrm{spf}(E^0Z[\![x]\!]/f(x))$ for a unique monic polynomial $f(x) = \sum_{i=0}^n c_i(V)x^{n-i}$, with $c_i(V) \in E^0Z$. We call $c_i(V)$ the i'th *Chern class* of V.

Definition 8.33. We write $\mathbb{L}(V)$ for $L(\mathbb{D}(V))$, the Thom sheaf of $\mathbb{D}(V)$, which is a line bundle over Z_E. It is easy to see that $\mathbb{L}(V) = \tilde{E}^0Z^V$, where $Z^V = P(\mathbb{C} \oplus V)/P(V)$ is the Thom space of V.

Remark 8.34. Let E be an even periodic ring spectrum and put $G = G_E = (\mathbb{C}P^\infty)_E$ and $S = S_E = \operatorname{spec}(E^0)$ as usual. Then the Thom spectra $\mathbb{C}P^\infty_{-n}$ form a tower, and there is a natural identification $\mathcal{M}_{G/S} = \varinjlim_n E^0(\mathbb{C}P^\infty_{-n})$. We also have $\omega_{G/S} = \widetilde{E}^0\mathbb{C}P^1 = \widetilde{E}^0 S^2 = \pi_2 E$. The theory of invariant differentials identifies $\mathcal{M}\Omega^1_{G/S}$ with $\mathcal{M}_{G/S} \otimes_{E^0} \omega_{G/S} = \varinjlim_n E^0(\Sigma^2\mathbb{C}P^\infty_{-n})$. The S^1-equivariant Segal conjecture gives an equivalence between $\operatorname{holim}_{\longleftarrow n} \Sigma^2\mathbb{C}P^\infty_{-n}$ and the profinite completion of S^0, and one can show that the resulting map $\mathcal{M}\Omega^1_{G/S} = \varinjlim_n E^0(\Sigma^2\mathbb{C}P^\infty_{-n}) \to E^0$ is just $\operatorname{res}_{G/S}$.

Proposition 8.35. There are natural isomorphisms

$$\begin{aligned}
(\coprod_n BU(n))_E &= M^+(G) = \operatorname{Div}^+(G) \\
BU_E &= N^+(G) = N(G) = \operatorname{Div}_0(G) \\
(\mathbb{Z} \times BU)_E &= M(G) = \operatorname{Div}(G) \\
(\mathbb{Z} \times BU)^E &= \operatorname{Map}_S(G, \mathbb{G}_m).
\end{aligned}$$

Proof. This is well-known, and follows easily from Proposition 8.31 and the remarks following Definition 5.8. The fourth statement follows from the third one by Cartier duality. □

Next, recall that there is a "complex reflection map" $r\colon S^1 \times \mathbb{C}P^{n-1}_+ \to U(n)$, where $r(z, L)$ has eigenvalue z on the line $L < \mathbb{C}^n$ and eigenvalue 1 on $L^\perp$. This gives an unbased map $\mathbb{C}P^{n-1} \to \Omega U(n)$. We can also fix a line $L_0 < \mathbb{C}^n$ and define $\overline{r}(z, L) = r(z, L)r(z, L_0)^{-1}$, giving a map $\overline{r}\colon \mathbb{C}P^{n-1} \to \Omega SU(n)$. Moreover, the Bott periodicity isomorphisms $\Omega U = \mathbb{Z} \times BU$ and $\Omega SU = BU$ give us maps $\Omega U(n) \to \mathbb{Z} \times BU$ and $\Omega SU(n) \to BU$. It is easy to see that $(\mathbb{C}P^{n-1})_E$ is the divisor $D_n = n[0] = \operatorname{spec}(E^0[\![x]\!]/x^n)$ on G_E over S_E.

Proposition 8.36. There are natural isomorphisms

$$\begin{aligned}
(\Omega U(n))_E &= M(D_n) \\
(\Omega SU(n))_E &= N(D_n) \\
(\Omega U(n))^E &= \operatorname{Map}_S(D_n, \mathbb{G}_m) \\
(\Omega SU(n))^E &= \operatorname{BasedMap}_S(D_n, \mathbb{G}_m).
\end{aligned}$$

Under these identifications, the map $\Omega U(n) \to \mathbb{Z} \times BU$ gives the obvious map $M(D_n) \to M(G_E)$ and so on.

Proof. For the second statement, it is enough (by Remark 6.9) to check that $E_*(\Omega SU(n))$ is the symmetric algebra generated by the reduced E-homology of $\mathbb{C}P^{n-1}$. This is well-known for ordinary homology, and it follows for all E by a collapsing Atiyah-Hirzebruch spectral sequence. See [22, 23] for more details. The inclusion $S^1 = U(1) \to U(n)$ and the determinant map $\det\colon U(n) \to S^1$ give a splitting $U(n) = S^1 \times SU(n)$ of spaces and thus $\Omega U(n) = \mathbb{Z} \times \Omega SU(n)$ of H-spaces and the first claim follows in turn using this. The last two statements follow by Cartier duality. □

8.4. **Cohomology of Abelian groups.** Let A be a compact Abelian Lie group, and write A^* for the character group $\operatorname{Hom}(A, S^1)$, which is a finitely generated discrete Abelian group. Let G be an ordinary formal group over a base S. For any point $s \in S(R)$ we write $\Gamma(G_s) = \widehat{\mathcal{X}}_{\operatorname{spec}(R)}(\operatorname{spec}(R), G_s)$ for the associated group of sections. A coordinate gives a bijection between $\Gamma(G_s)$ and $\operatorname{Nil}(R)$, which becomes a homomorphism if we use an appropriate formal group law to make $\operatorname{Nil}(R)$ a group. We define a formal scheme $\operatorname{Hom}(A^*, G)$ by

$$\operatorname{Hom}(A^*, G)(R) = \{(s, \phi) \mid s \in S(R) \text{ and } \phi\colon A^* \to \Gamma(G_s)\}.$$

(If A^* is a direct sum of r cyclic groups then this can be identified with a closed formal subscheme of G^r_S in an evident way, which shows that it really is a scheme.)

Proposition 8.37. For any finite Abelian group A, there is a natural map $BA_E \to \operatorname{Hom}(A^*, G)$. This is an isomorphism if E is an exact $P(n)$-module for some n.

Proof. An element $\alpha \in A^* = \mathrm{Hom}(A, S^1)$ gives a map $BA \to BS^1$ of spaces and thus a map $BA_E \to (BS^1)_E = G$ of formal groups over S. One checks that the resulting map $A^* \to \mathrm{Ab}\,\widehat{\mathfrak{X}}_S(BA_E, G)$ is a homomorphism, so by adjointing things around we get a map $BA_E \to \mathrm{Hom}(A^*, G)$. If A is a torus then $A^* \simeq \mathbb{Z}^r$ and $BA_E = G^r = \mathrm{Hom}(A^*, G)$, so our map is an isomorphism. Moreover, in this case $BA \simeq (\mathbb{C}P^\infty)^r$ which is decent and thus tolerable for any E. If $A = \mathbb{Z}/m$ then there is a well-known way to identify BA with the circle bundle in the line bundle L^m, where L is the tautological bundle over $\mathbb{C}P^\infty$. This gives a long exact Gysin sequence

$$E^* BA \leftarrow E^* \mathbb{C}P^\infty \xleftarrow{[m](x)} E^* \mathbb{C}P^\infty.$$

The second map here is multiplication by $[m](x)$, which is the image of x under the map $G \xrightarrow{\times m} G$. If this map is injective then the Gysin sequence is a short exact sequence and we have $E^0 BA = E^0 \mathbb{C}P^\infty / [m](x)$, and we conclude easily that $\mathrm{spf}(E^0 BA) = \ker(G \xrightarrow{m} G) = \mathrm{Hom}(A^*, G)$. One can apply similar arguments to the skeleta $S^{2k+1}/(\mathbb{Z}/m)$ of BA and find that $\mathrm{spf}(E^0 BA) = BA_E$.

In the case of two-periodic Morava K-theory we recover the well-known calculation showing that $K(n)^* BA$ is concentrated in even degrees for all n. We also have $H^s(BA, \mathbb{Q}) = 0$ for $s > 0$ so Proposition 8.20 tells us that BA_E is tolerable for any E that is an exact $P(n)$-module for any n. Moreover, it is easy to see that $[m](x)$ is not a zero-divisor in this case so $BA_E = \mathrm{Hom}(A^*, G)$. We have just shown this when A^* is cyclic, but it follows easily for all A by Proposition 8.14. □

8.5. **Schemes associated to ring spectra.** If R is a commutative ring spectrum with a ring map $E \to R$, we have a scheme $\mathrm{spec}(\pi_0 R)$ over S_E. If Z is a finite complex we can take $R = F(Z_+, E)$ and we recover the case $Z_E = \mathrm{spec}(E^0 Z) = \mathrm{spec}(\pi_0 R)$. If M is an arbitrary commutative ring spectrum, we can take $R = E \wedge M$. In this case we write $M^E = \mathrm{spec}(E_0 M)$ for the resulting scheme. If Y is a commutative H-space we can take $M = \Sigma^\infty Y_+$, and we write Y^E for $M^E = \mathrm{spec}(E_0 Y)$ in this case. If we have a Künneth isomorphism $E_0 Y^k = (E_0 Y)^{\otimes k}$ then $E_0 Y$ is a Hopf algebra, so Y^E is a group scheme over S. If Y is decent then $E_0 Y$ is a coalgebra with good basis. In this case Proposition 6.19 applies, and we have a Cartier duality $Y^E = D(Y_E) = \mathrm{Hom}_S(Y_E, \mathbb{G}_m)$ and $Y_E = D(Y^E) = \mathrm{Hom}_S(Y^E, \mathbb{G}_m)$.

If $\{R_\alpha\}$ is an inverse system of ring spectra as above, we have a formal scheme $\varinjlim_\alpha \mathrm{spec}(\pi_0 R_\alpha)$. If Z_α runs over the finite subcomplexes of a CW complex Z, then the rings $F(Z_{\alpha+}, E)$ give an example of this, and the associated formal scheme is just Z_E. Another good example is to take the tower of spectra E/p^k, where E is an even periodic ring spectrum such that E^0 is torsion-free. More generally, if E has suitable Landweber exactness properties then we can smash E with a generalised Moore spectrum S/I (see [13, Section 4], for example) and get a new even periodic ring spectrum E/I, and then we can consider a tower of these. There are technicalities about the existence of products on the spectra E/I, which we omit here.

8.6. **Homology of Thom spectra.** Let Z be a space equipped with a map $Z \xrightarrow{z} \mathbb{Z} \times BU$, and let $T(Z, z)$ be the associated Thom spectrum. It is well-known that T is a functor from spaces over $\mathbb{Z} \times BU$ to spectra, which preserves homotopy pushouts. Moreover, if (Y, y) is another space over $\mathbb{Z} \times BU$ then we can use the addition on $\mathbb{Z} \times BU$ to make $(Y \times Z, (y, z))$ into a space over $\mathbb{Z} \times BU$ and we find that $T(Y \times Z, (y, z)) = T(Y, y) \wedge T(Z, z)$.

The above construction really needs an actual map $Z \xrightarrow{z} \mathbb{Z} \times BU$ and not just a homotopy class. However, we do have the following result.

Lemma 8.38. If Z is a decent space then the spectrum $T(Z, z)$ depends only on the homotopy class of z, up to canonical homotopy equivalence. Thus T can be regarded as a functor from the homotopy category of decent spaces over $\mathbb{Z} \times BU$ to spectra. In particular, we can define $T(Z, V)$ when V is a virtual bundle over Z.

Proof. Suppose we have two homotopic maps $z_0, z_1 \colon Z \to \mathbb{Z} \times BU$. We can then choose a map $w \colon Z \times I \to \mathbb{Z} \times BU$ such that $wj_0 = z_0$ and $wj_1 = z_1$, where $j_t(a) = (a, t)$. The maps j_t induce maps of spectra $T(Z, z_t) \xrightarrow{f_t} T(Z \times I, w)$, and the Thom isomorphism theorem implies that these give equivalences in homology so they are weak equivalences. We thus have a weak equivalence $f_1^{-1} f_0 \colon T(Z, z_0) \to T(Z, z_1)$. This much is true even when Z is not decent.

To see that our map is canonical when Z is decent, note that $KU^* Z$ is concentrated in even degrees, so the space F of unpointed maps from Z to $\mathbb{Z} \times BU$ has trivial odd-dimensional homotopy groups with respect to any basepoint. We can think of z_0 and z_1 as points of F, and w as a path between them. If w' is another path then then we can glue w and w' to get a map of S^1 to F, which can be extended to give a map $u\colon D^2 \to F$ because $\pi_1 F = 0$. It follows that we have a commutative diagram as follows:

$$\begin{tikzcd}
T(Z,z_0) \arrow[rr, "f_0"] \arrow[dd, "f'_0"'] \arrow[dr] & & T(Z\times I, w) \arrow[dl] \\
& T(Z\times D^2, u) & \\
T(Z\times I, w') \arrow[ur] & & T(Z,z_1) \arrow[ll, "f'_1"] \arrow[uu, "f_1"'] \arrow[ul]
\end{tikzcd}$$

It follows easily that $f_1^{-1}\circ f_0 = (f'_1)^{-1}\circ f'_0$, as required. □

A coordinate on G_E is the same as a degree zero complex orientation of E, which gives a multiplicative system of Thom classes for all virtual complex bundles. In particular, this gives isomorphisms $E_*T(Y,y) \simeq E_*Y$, which are compatible in the evident way with the isomorphisms $T(Y\times Z,(y,z)) = T(Y,y)\wedge T(Z,z)$.

If $Z \xrightarrow{z} \{n\}\times BU(n)$ classifies an honest n-dimensional bundle V over Z then we have $T(Z,z) = \Sigma^\infty Z^V$. In particular, the inclusion $\mathbb{C}P^\infty = BU(1) \to \{1\}\times BU$ just gives the Thom spectrum $\Sigma^\infty(\mathbb{C}P^\infty)^L$, which is well-known to be the same as $\Sigma^\infty \mathbb{C}P^\infty$ (without a disjoint basepoint).

Now let Z be a decent commutative H-space. Let $z\colon Z \to \mathbb{Z}\times BU$ be an H-map, and write $M = T(Z,z)$. We note that addition gives a map $(Z\times Z,(z,z)) \to (Z,z)$ of spaces over $\mathbb{Z}\times BU$ and thus a map of spectra $M\wedge M \to M$, which makes M into a commutative ring spectrum. Similarly, the diagonal gives a map $(Z,z) \to (Z\times Z,(0,z))$ and thus a map $M \xrightarrow{\delta} \Sigma^\infty Z_+ \wedge M$. Finally, we consider the shearing map $(a,b)\mapsto (a,a+b)$. This is an isomorphism $(Z\times Z,(z,z)) \to (Z\times Z,(0,z))$ over $\mathbb{Z}\times BU$, which gives an isomorphism $M\wedge M \to \Sigma^\infty Z_+\wedge M$ of spectra.

A choice of coordinate gives a Thom isomorphism $E_*M \simeq E_*Z$, which shows that E_*M is free and in even degrees. For the moment we just use this to show that we have Künneth isomorphisms, from which we will recover a more natural statement about the relationship between E_*Z and E_*M.

Recall that we defined define $Z^E = \operatorname{spec}(E_0Z) = \operatorname{spec}(E_0\Sigma^\infty Z_+)$ (which is a commutative group scheme over $S = S_E$) and $M^E = \operatorname{spec}(E_0M)$. Our diagonal map δ gives an action of Z^E on M^E. The shearing isomorphism $M\wedge M = \Sigma^\infty Z_+\wedge M$ shows that the action and projection maps give an isomorphism $Z^E\times_S M^E \to M^E\times_S M^E$.

A choice of coordinate on G gives an isomorphism $E_0M \simeq E_0Y$. One can check (using the multiplicative properties of Thom classes) that this is an isomorphism of E_0Y-comodule algebras, so it gives an isomorphism $Y^E \simeq M^E$ of schemes, compatible with the action of Y^E. This means that M^E is a trivialisable torsor for Y^E.

In the universal case $Y = \mathbb{Z}\times BU$, this works out as follows. As mentioned previously, we have a map $\mathbb{C}P^\infty = \{1\}\times BU(1) \to \mathbb{Z}\times BU$, and the Thom functor gives a map $\Sigma^\infty\mathbb{C}P^\infty \to MP$. In particular, the bottom cell gives a map $S^2 = \mathbb{C}P^1 \to MP$, or an element $u\in\pi_2 MP$. The inclusion $\{-1\}\to\mathbb{Z}\times BU$ also gives an element of $\pi_{-2}MP$, which one checks is inverse to u. Thus, a ring map $E_0MP \to R$ gives an E_0-algebra structure on R, and an E_0-module map $\tilde{E}_0\mathbb{C}P^\infty \to R$, which sends $\tilde{E}_0S^2$ into $R^\times$. In other words, it gives a point $s\in S_E(R)$ together with an element $y \in R\widehat{\otimes}_{E^0}\tilde{E}^0\mathbb{C}P^\infty$. We can identify $R\widehat{\otimes}_{E^0}\tilde{E}^0\mathbb{C}P^\infty$ with the ideal of functions on G_s that vanish at zero, and the extra condition on the restriction to S^2 says that y is a coordinate. This gives a natural map $MP^E \to \operatorname{Coord}(G)$. Well-known calculations show that E_0MP is the symmetric algebra over E_0 on $\tilde{E}_0\mathbb{C}P^\infty$, with the bottom class inverted. This implies easily that the map $MP^E \to \operatorname{Coord}(G)$ is an isomorphism. Recall also that $(\mathbb{Z}\times BU)^E = \operatorname{Map}(G,\mathbb{G}_m)$. Clearly, if $u\colon G\to\mathbb{G}_m$ and x is a coordinate on G, then the product ux is again a coordinate. This gives an action of $\operatorname{Map}(G,\mathbb{G}_m)$ on $\operatorname{Coord}(G)$, which makes $\operatorname{Coord}(G)$ into a torsor over $\operatorname{Map}(G,\mathbb{G}_m)$. One can check that this structure arises from our geometric coaction of $\mathbb{Z}\times BU$ on MP.

8.7. **Homology operations.** Let G be an ordinary formal group over S, and let H be an ordinary formal group over T. Let π_S and π_T be the projections from $S\times T$ to S and T. We write $\operatorname{Hom}(G,H)$ for $\operatorname{Hom}_{S\times T}(\pi_S^*G,\pi_T^*H)$, which is a scheme over $S\times T$ by Proposition 6.15. Recall that $\operatorname{Hom}(G,H)(R)$ is the set of triples (s,t,u) where $s\in S(R)$ and $t\in T(R)$ and $u\colon G_s\to H_t$ is a map of formal groups over spec(R). We write $\mathrm{Iso}(G,H)(R)$ for the subset of triples for which u is an isomorphism. If we choose coordinates x and y on G and H, then for any u we have $y(u(g))=\phi(x(g))$ for some power series $\phi\in R[\![t]\!]$ with $\phi(0)=0$, and u is an isomorphism if and only if $\phi'(0)$ is invertible. It follows that $\mathrm{Iso}(G,H)$ is an open subscheme of $\mathrm{Hom}(G,H)$.

Proposition 8.39. Let E and E' be even periodic ring spectra. Then there is a natural map $S_{E\wedge E'}\to \mathrm{Iso}(G_E,G_{E'})$ of schemes over $S_E\times S_{E'}$. This is an isomorphism if E or E' is Landweber exact over MP.

Proof. We write $S'=S_{E'}$ and $G'=G_{E'}$. The evident ring maps $E\to E\wedge E'\leftarrow E'$ give maps $S\xleftarrow{q} S_{E\wedge E'}\xrightarrow{q'} S'$, and pullback squares

$$\begin{tikzcd} G & G_{E\wedge E'} \arrow[l] \arrow[r] \arrow[d] & G' \arrow[d] \\ S & S_{E\wedge E'} \arrow[l, "q"'] \arrow[r, "q'"'] & S' \arrow[from=1-1, to=2-1, crossing over] \end{tikzcd}$$

This gives an isomorphism $v\colon q^*G\to(q')^*G'$. Using this, we easily construct the required map.

Now consider the case $E'=MP$, so that $S'=\mathrm{FGL}$. Then $\mathrm{Iso}(G,G')(R)$ is the set of triples (s,F,x), where $s\in S(R)$ and F is a formal group law over R and $x\colon G_s\to \mathrm{spec}(R)\times\widehat{\mathbb{A}}^1$ is an isomorphism over spec(R) such that $x(g+h)=F(x(g),x(h))$. In other words, x is a coordinate on G_s and F is the unique formal group law such that $x(g+h)=F(x(g),x(h))$. Thus, we find that $\mathrm{Iso}(G,G')=\mathrm{Coord}(G)=MP^E=\mathrm{spec}(\pi_0MP)$ (see Section 8.6). It follows after a comparison of definitions that our map $S_{E\wedge E'}\to\mathrm{Iso}(G,G')$ is an isomorphism.

Now suppose that E'' is Landweber exact over E', in the sense that there is a ring map $E'\to E''$ which induces an isomorphism $E''_0\otimes_{E'_0}E'_0Z=E''_0Z$ for all spectra Z. We then find that $G''=G'\times_{S'}S''$ and that

$$S_{E\wedge E''}=S_{E\wedge E'}\times_{S'}S''=\mathrm{Iso}(G,G')\times_{S'}S''=\mathrm{Iso}(G,G''),$$

as required. □

Remark 8.40. If there are enough Künneth isomorphisms, then $E_0\Omega^\infty E'$ will be a Hopf ring over E_0 and thus the $*$-indecomposables $\mathrm{Ind}(E_0\Omega^\infty E')$ will be an algebra over E_0 using the circle product. The procedure described in [15] will then give a map $\mathrm{spec}(\mathrm{Ind}(E_0\Omega^\infty E'))\to\mathrm{Hom}(G,G')$, which is an isomorphism in good cases.

Definition 8.41. Let G and G' be formal groups over S and S', respectively. A *fibrewise isomorphism* from G to G' is a square of the form

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & G' \arrow[d] \\ S \arrow[r, "g"'] & S' \end{tikzcd}$$

such that the induced map $G\to f^*G'$ is an isomorphism of formal groups over S.

Definition 8.42. We write OFG for the category of ordinary formal groups over affine schemes and fibrewise isomorphisms, and EPR for the category of even periodic ring spectra. We thus have a functor $\mathrm{EPR}^{\mathrm{op}}\to\mathrm{OFG}$ sending E to G_E. We write LOFG for the subcategory of OFG consisting of Landweber exact formal groups, and LEPR for the category of those E for which G_E is Landweber exact.

Proposition 8.43. If $E\in\mathrm{EPR}$ and $E'\in\mathrm{LEPR}$ then the natural map

$$\mathrm{EPR}(E',E)\to\mathrm{OFG}(G_E,G_{E'})$$

is an isomorphism. Moreover, the functor $\mathrm{LEPR}^{\mathrm{op}}\to\mathrm{LOFG}$ is an equivalence of categories.

Proof. Using [13, Proposition 2.12 and Corollary 2.14], we see that there is a cofibration $P \to Q \to E' \to \Sigma P$, in which P and Q are retracts of wedges of finite spectra with only even cells, and the connecting map $E' \to \Sigma P$ is phantom. If W is an even finite spectrum then we see from the Atiyah-Hirzebruch spectral sequence that $E_1 W = 0$ and $E_0 W$ is projective over E_0 and $[W, E] = \mathrm{Hom}(E_0 W, E_0)$ and $[\Sigma W, E] = 0$. It follows that all these things hold with W replaced by P or Q. Using the cofibration we see that $E_1 E' = 0$, and there is a short exact sequence

$$E_0 P \rightarrowtail E_0 Q \twoheadrightarrow E_0 E'.$$

Now consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & {[E', E]} \arrow[r] \arrow[d, "\alpha_{E'}"'] & {[Q, E]} \arrow[r] \arrow[d, "\alpha_Q"'] & {[P, E]} \arrow[d, "\alpha_P"'] \\ 0 \arrow[r] & \mathrm{Hom}(E_0 E', E_0) \arrow[r] & \mathrm{Hom}(E_0 Q, E_0) \arrow[r] & \mathrm{Hom}(E_0 P, E_0). \end{tikzcd}$$

The short exact sequence above implies that the bottom row is exact. The top row is exact because of our cofibration and the fact that $[\Sigma P, E] = 0$. We have seen that α_P and α_Q are isomorphisms, and it follows that $\alpha_{E'}$ is an isomorphism. Thus, $[E', E]$ is the set of maps of E_0-modules from $E_0 E'$ to E_0. One can check that the ring maps $E' \to E$ biject with the maps of E_0-algebras from $E_0 E'$ to E_0 (using [13, Proposition 2.19]). We see from Proposition 8.39 that these maps biject with sections of $S_{E \wedge E'} = \mathrm{Iso}(G_E, G_{E'})$ over S_E, and these are easily seen to be the same as fibrewise isomorphisms from G_E to $G_{E'}$. Thus $\mathrm{EPR}(E', E) = \mathrm{OFG}(G_E, G_{E'})$, as claimed. This implies that the functor $\mathrm{LEPR}^{\mathrm{op}} \to \mathrm{LOFG}$ is full and faithful, so we need only check that it is essentially surjective. Suppose that G is a Landweber exact ordinary formal group over an affine scheme S. Define a graded ring E_* by putting $E_{2k+1} = 0$ and $E_{2k} = \omega_{G/S}^{\otimes k}$ for all $k \in \mathbb{Z}$, so in particular $E_0 = \mathcal{O}_S$. A choice of coordinate on G gives a formal group law F over $\mathcal{O}_S = E_0$ and thus a map $S \to \mathrm{FGL}$ or equivalently a map $u \colon MP_0 = \mathcal{O}_{\mathrm{FGL}} \to E_0$. If $G_0 = G_{MP}$ is the evident formal group over FGL then one sees from the construction that $S \times_{\mathrm{FGL}} G_0 = G$. Given this, we see that our map u extends to give a map $MP_* \to E_*$. We define a functor from spectra to graded Abelian groups by

$$E_* Z = E_* \otimes_{MP_*} MP_* Z = E_* \otimes_{MU_*} MU_* Z,$$

where we have used the map $MU \to MP$ of ring spectra to regard E_* as a module over MU_*. One can also check that $E_0 Z = E_0 \otimes_{MP_0} MP_0 Z$. The classical Landweber exact functor theorem implies that this is a homology theory, represented by a spectrum E. The refinements given in [13, Section 2.1] show that E is unique up to canonical isomorphism, and that it admits a canonical commutative ring structure, making it an even periodic ring spectrum. It is easy to check that $E^0 \mathbb{C}P^\infty = E_0 \widehat{\otimes}_{MP_0} MP^0 \mathbb{C}P^\infty$ and thus that $G_E = S \times_{\mathrm{FGL}} G_0 = G$, as required. □

References

[1] J. F. Adams. A variant of E. H. Brown's representability theorem. *Topology*, 10:185–198, 1971.

[2] J. F. Adams. *Stable Homotopy and Generalised Homology.* University of Chicago Press, Chicago, 1974.

[3] M. Demazure. *Lectures on p-divisible groups.* Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 302.

[4] M. Demazure and P. Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs.* Masson & Cie, Éditeur, Paris, 1970. Avec un appendice *Corps de classes local* par Michiel Hazewinkel.

[5] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, Modules and Algebras in Stable Homotopy Theory*, volume 47 of *Amer. Math. Soc. Surveys and Monographs.* American Mathematical Society, 1996.

[6] A. Frölich. *Formal Groups*, volume 74 of *Lecture Notes in Mathematics.* Springer–Verlag, 1968.

[7] J. P. C. Greenlees and N. P. Strickland. Varieties and local cohomology for chromatic group cohomology rings. 40 pp., To appear in Topology, 1996.

[8] A. Grothendieck and J. L. Verdier. *Exposé I in Théorie des topos et cohomologie étale des schémas*, volume 269 of *Lecture Notes in Mathematics.* Springer–Verlag, 1972. (SGA 4, second edition).

[9] R. H. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics.* Springer–Verlag, 1977.

[10] P. Henrici. *Applied and computational complex analysis. Vol. 1.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. Power series—integration—conformal mapping—location of zeros, Reprint of the 1974 original, A Wiley-Interscience Publication.

[11] M. J. Hopkins, M. Ando, and N. P. Strickland. Elliptic spectra, the Witten genus, and the theorem of the cube. 65 pp., Submitted to Inventiones, 1997.

[12] M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel. Generalised group characters and complex oriented cohomology theories. Preprint (various editions, since the late 1980's).

[13] M. Hovey and N. P. Strickland. Morava K-theories and localisation. *Mem. Amer. Math. Soc.*, 139(666):104, 1999.

[14] P. T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1982.

[15] T. Kashiwabara, N. Strickland, and P. Turner. The Morava K-theory Hopf ring for BP. In *Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994)*, Progr. Math., 136, pages 209–222. Birkhäuser, Basel, 1996.

[16] N. M. Katz and B. Mazur. *Arithmetic Moduli of Elliptic Curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, 1985.

[17] M. Lazard. *Commutative formal groups*. Springer-Verlag, Berlin, 1975. Lecture Notes in Mathematics, Vol. 443.

[18] H. Matsumura. *Commutative Ring Theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1986.

[19] D. G. Quillen. On the formal group laws of unoriented and complex cobordism. *Bulletin of the American Mathematical Society*, 75:1293–1298, 1969.

[20] D. G. Quillen. Elementary proofs of some results of cobordism theory using Steenrod operations. *Advances in Mathematics*, 7:29–56, 1971.

[21] D. G. Quillen. The spectrum of an equivariant cohomology ring, I and II. *Annals of Mathematics*, 94:549–602, 1971.

[22] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. Academic Press, 1986.

[23] D. C. Ravenel. *Nilpotence and Periodicity in Stable Homotopy Theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, 1992.

[24] D. C. Ravenel, W. S. Wilson, and N. Yagita. Brown-Peterson cohomology from Morava K-theory. *K-Theory*, 15(2):147–199, 1998.

[25] N. P. Strickland. Functorial philosophy for formal phenomena. 59 pp., Preprint, 1994.

[26] N. P. Strickland. Finite subgroups of formal groups. *Journal of Pure and Applied Algebra*, 121:161–208, 1997.

[27] N. P. Strickland. Morava E-theory of symmetric groups. *Topology*, 37(4):757–779, 1998.

[28] N. P. Strickland. Products on MU-modules. Trans. Amer. Math. Soc., posted on March 1, 1999, PII:S 0002-9947(99)02436-8 (31 pp., to appear in print), 1999.

[29] N. Yagita. The exact functor theorem for BP_*/I_n–theory. *Proceedings of the Japan Academy*, 52:1–3, 1976.