Questions around the Chromatic Splitting Conjecture

Neil Strickland

April 18, 2022

The Chromatic Splitting Conjecture

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Technical note: throughout this talk, S denotes the p-complete sphere spectrum, and we work in the category of S-modules. Symbols like MU refer to the p-completed versions.

We put $S_n^d = L_n S^d$ and $\widehat{S}_n^d = L_{K(n)}S^d$. Given a ring spectrum R and variables z_i of odd degree d_i and chromatic height n_i, we define

$$E_R[z_1, \dots, z_m] = R \wedge \bigwedge_i (S \vee S_{n_i}^{d_i}).$$

To expand this out, remember that $S_n^i \wedge S_m^j \simeq S_{\min(n,m)}^{i+j}$.

We introduce variables x_{in} for $0 \leq i < n$ height i and degree $1 - 2(n-i)$. The CSC says that there are maps $x_{in} \colon S_i^{1-2(n-i)} \to \alpha_n(S)$ inducing

$$E_{S_{n-1}}[x_{0n}, \dots, x_{n-1,n}] \simeq \alpha_n(S).$$

For example:

$$\begin{aligned} \alpha_3(S) = L_2 L_{K(3)} S &\simeq S_2 \wedge (S \vee S_0^{-5}) \wedge (S \vee S_1^{-3}) \wedge (S \vee S_2^{-1}) \\ &\simeq S_2 \vee S_2^{-1} \vee S_1^{-3} \vee S_1^{-4} \vee S_0^{-5} \vee S_0^{-6} \vee S_0^{-8} \vee S_0^{-9}. \end{aligned}$$

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Generators; Localisation relations

The extended Morava stabiliser group Γ_n of height n acts on $W\mathbb{F}_{p^n}$ with

$$H^*(\Gamma_n; \mathbb{Q} \otimes W\mathbb{F}_{p^n}) = E_{\mathbb{Q}_p}[x_{in} \mid 0 \leq i < n] \qquad x_{in} \in H^{2(n-i)-1}.$$

These elements x_{in} should be related via the $K(n)$-based Adams spectral sequence to the elements x_{in} in the CSC.
Also, $x_{n-1,n} \colon S^{-1}_{n-1} \to L_{n-1} L_{K(n)} S$ should come from the known element $\zeta_n \colon S^{-1} \to L_{K(n)} S$ (defined using $\ker(\det \colon \Gamma_n \to \mathbb{Z}_p^\times)$).

$$L_n L_m = L_{\min(n,m)}$$

$$L_{K(n)} L_m = \begin{cases} L_{K(n)} & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$$

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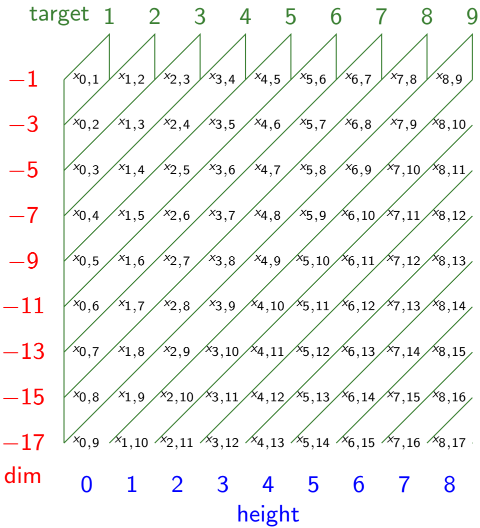
Charts

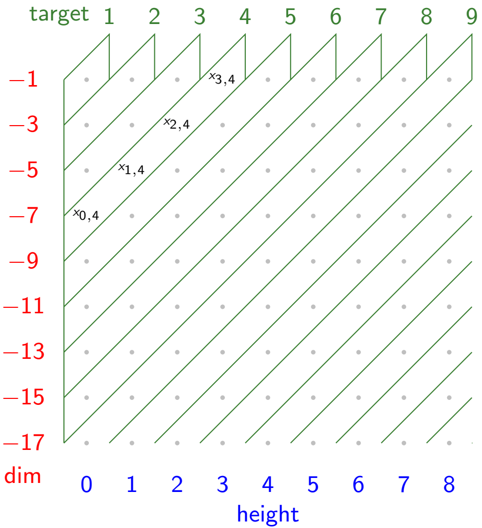


$\alpha_n(S) = E_{S_{n-1}}[x_{0n}, \ldots, x_{n-1,n}]$

x_{in} has height i, target n

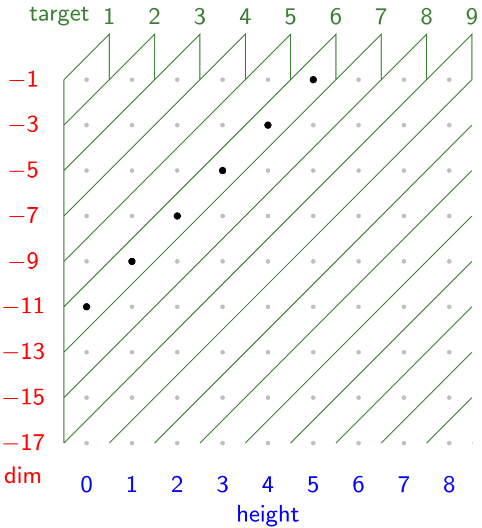
and dimension $1 - 2(n - i)$

Charts



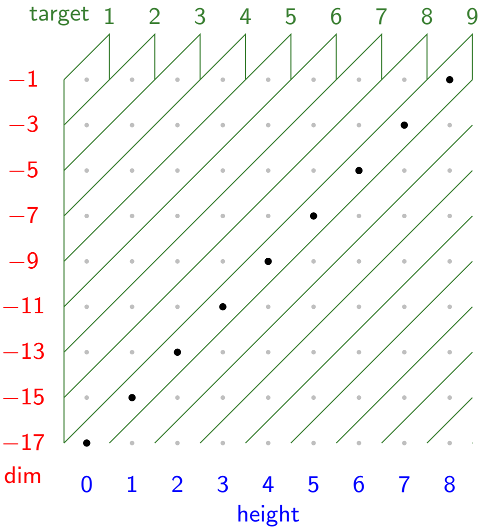
$$\alpha_4(S) = E_{S_3}[x_{04}, x_{14}, x_{24}, x_{34}]$$

Charts



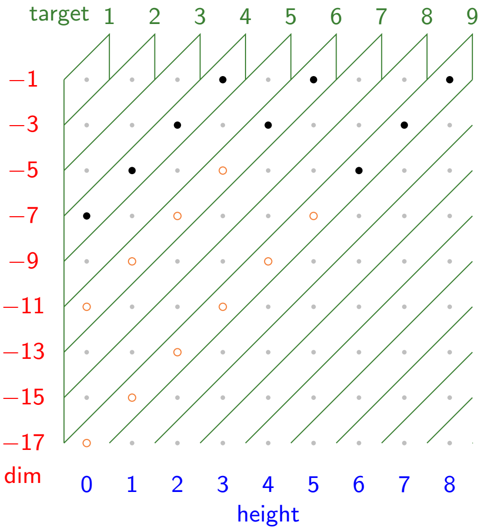
$\alpha_6(S) = E_{S_5}[x_{06}, \ldots, x_{56}]$

Charts



$$\alpha_9(S) = E_{S_8}[x_{09}, \ldots, x_{89}]$$

Charts



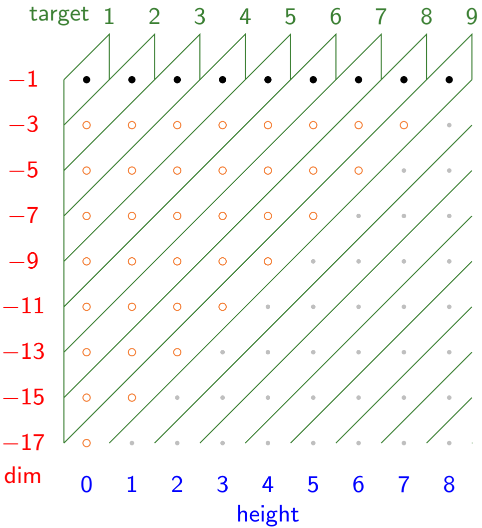
Put $\alpha_{469} = \alpha_4 \circ \alpha_6 \circ \alpha_9$

$$= L_3 L_{K(4)} L_{K(6)} L_{K(9)}$$

$$\begin{tikzcd} & \alpha_4(S) \arrow[d] & \\ & \alpha_{469}(S) & \\ \alpha_6(S) \arrow[ur] & & \alpha_9(S) \arrow[ul] \end{tikzcd}$$

$\alpha_{469}(S)$ is exterior over S_3 on 9 generators indicated in black. Circles are shadowed generators: present but equal to zero.

Charts

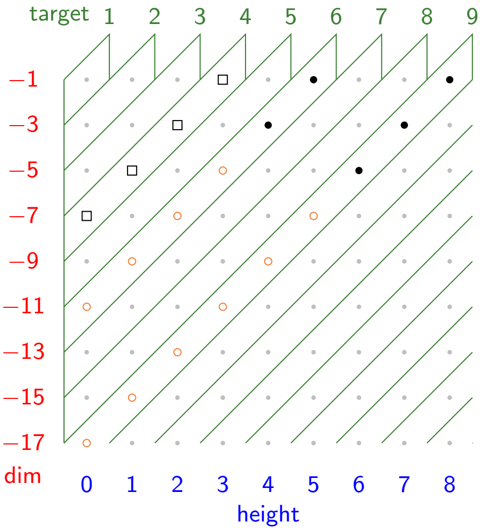


$\alpha_{1\ldots 9} = L_{K(0)} L_{K(1)} \cdots L_{K(9)}$

$\alpha_{1\ldots 9}(S)$ is exterior over S_0 on

$x_{01}, x_{12}, \ldots, x_{89}$ (all degree -1)

Charts



Put $\phi_{469} = L_{K(4)} \circ \alpha_6 \circ \alpha_9$

$$= L_{K(4)} L_{K(6)} L_{K(9)}$$

$$\alpha_{69}(S) \to \phi_{469}(S) \leftarrow \widehat{S}_4$$

$\phi_{469}(S)$ is exterior over $\widehat{S}_4$ on 5 generators marked in black. Circles are shadowed generators: present but equal to zero.

All summands in this exterior algebra are just $\widehat{S}_4^d$.

Chromatic fracture

The following cube of functors is homotopy cartesian (where $\phi_{02} = L_{K(0)}L_{K(2)}$ etc.):

$$\begin{tikzcd}
L_2 \arrow[rr] \arrow[dr] \arrow[dd] & & \phi_0 \arrow[dr] \arrow[dd] & \\
& \phi_1 \arrow[rr] \arrow[dd] & & \phi_{01} \arrow[dd] \\
\phi_2 \arrow[rr] \arrow[dr] & & \phi_{02} \arrow[dr] & \\
& \phi_{12} \arrow[rr] & & \phi_{012}
\end{tikzcd}$$

Homotopy cartesian means:

- L_2 maps by an equivalence to the holim of the rest of the diagram; or
- The total fibre of the cube is zero.

Rules for total fibres:

- tfib(cube) $=$ fib(tfib(face) $\to$ tfib(opposite face))
- tfib(square) $=$ fib(fib(edge) $\to$ fib(opposite edge))

The following cube of functors is homotopy cartesian
(where $\phi_{02} = L_{K(0)}L_{K(2)}$ etc.):

$$\begin{tikzcd}
L_2 \arrow[rr] \arrow[dr] \arrow[dd] & & \phi_0 \arrow[dr] \arrow[dd] & \\
& \phi_1 \arrow[rr] \arrow[dd] & & \phi_{01} \arrow[dd] \\
\phi_2 \arrow[rr] \arrow[dr] & & \phi_{02} \arrow[dr] & \\
& \phi_{12} \arrow[rr] & & \phi_{012}
\end{tikzcd}$$

Homotopy cartesian means:

- L_2 maps by an equivalence to the holim of the rest of the diagram; or
- The total fibre of the cube is zero.

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Aside on spectral sequences

The chromatic fracture cube gives a spectral sequence

$$E^1_{pq} = \prod_{|A|=p} \pi_q(\phi_A(X)) \Longrightarrow 0,$$

where A runs over subsets of $\{0,1,2\}$ and $\phi_\emptyset = L_2$.

For a formally similar situation, take a space $X = U_0 \cup U_1 \cup U_2$, and put $U_{02} = U_0 \cap U_2$ etc. There is a Mayer-Vietoris spectral sequence

$$E_0^{pq} = \prod_{|A|=p} C^q(U_A), \quad E_1^{pq} = \prod_{|A|=p} H^q(U_A) \Longrightarrow 0.$$

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Chromatic splitting and chromatic fracture

$$
\begin{tikzcd}
\alpha_3(S) \arrow[rr] \arrow[dr] \arrow[dd] & & \phi_{03}(S) \arrow[dr] \arrow[dd] & \\
& \phi_{13}(S) \arrow[rr] \arrow[dd] & & \phi_{013}(S) \arrow[dd] \\
\phi_{23}(S) \arrow[rr] \arrow[dr] & & \phi_{023}(S) \arrow[dr] & \\
& \phi_{123}(S) \arrow[rr] & & \phi_{0123}(S)
\end{tikzcd}
$$

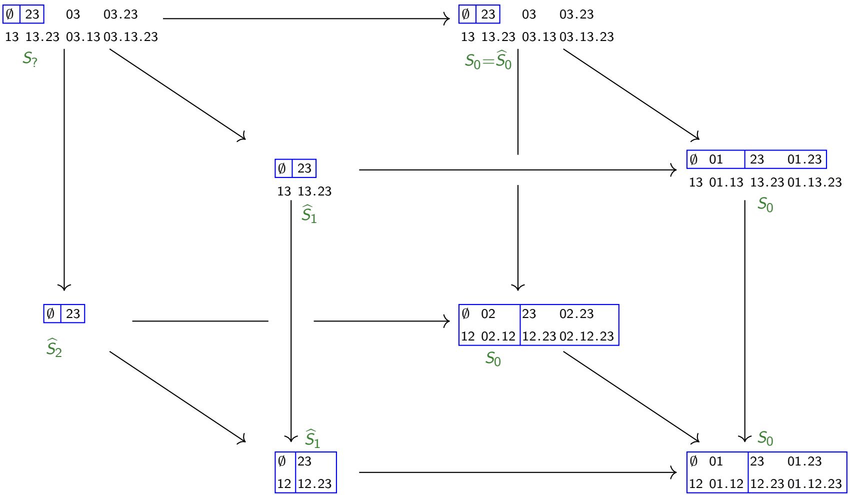
Apply the fracture cube to $\widehat{S}_3$ to get a homotopy cartesian cube as above.
Is this consistent with the Chromatic Splitting Conjecture?

Chromatic splitting and chromatic fracture

$$
\begin{tikzcd}[column sep=small]
\underset{S_?}{\begin{array}{cccc} \emptyset & 23 & 03 & 03.23 \\ 13 & 13.23 & 03.13 & 03.13.23 \end{array}} \arrow[rr] \arrow[dr] \arrow[dd] & & \underset{S_0=\widehat{S}_0}{\begin{array}{cccc} \emptyset & 23 & 03 & 03.23 \\ 13 & 13.23 & 03.13 & 03.13.23 \end{array}} \arrow[dr] \arrow[dd] & \\
& \underset{\widehat{S}_1}{\begin{array}{cc} \emptyset & 23 \\ 13 & 13.23 \end{array}} \arrow[rr] \arrow[dd] & & \underset{S_0}{\begin{array}{cccc} \emptyset & 01 & 23 & 01.23 \\ 13 & 01.13 & 13.23 & 01.13.23 \end{array}} \arrow[dd] \\
\underset{\widehat{S}_2}{\begin{array}{cc} \emptyset & 23 \end{array}} \arrow[rr] \arrow[dr] & & \underset{S_0}{\begin{array}{cccc} \emptyset & 02 & 23 & 02.23 \\ 12 & 02.12 & 12.23 & 02.12.23 \end{array}} \arrow[dr] & \\
& \overset{\widehat{S}_1}{\begin{array}{cc} \emptyset & 23 \\ 12 & 12.23 \end{array}} \arrow[rr] & & \overset{S_0}{\begin{array}{cccc} \emptyset & 01 & 23 & 01.23 \\ 12 & 01.12 & 12.23 & 01.12.23 \end{array}}
\end{tikzcd}
$$

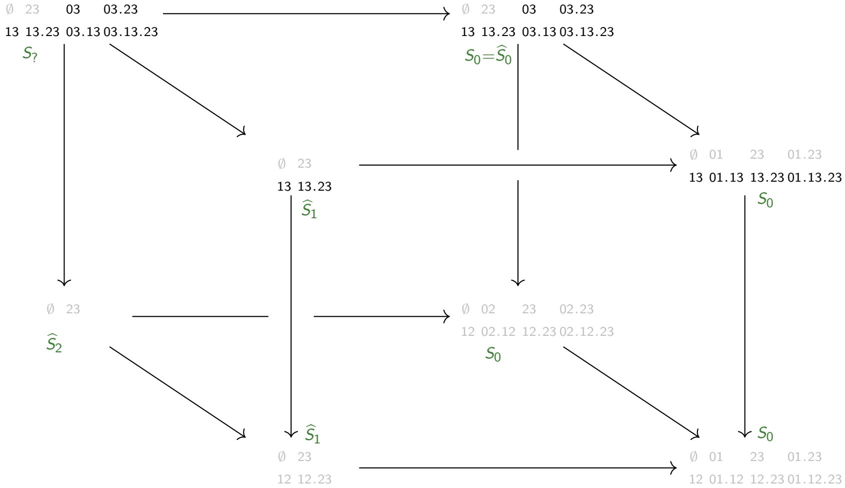
Notation: e.g. $01.13 = x_{01}x_{13}$; also $\emptyset = 1$.
This diagram should be homotopy cartesian.

Chromatic splitting and chromatic fracture



This subdiagram consists of two copies of the fracture cube for S_2 and so is homotopy cartesian.

Chromatic splitting and chromatic fracture



We can remove that subdiagram without changing the total fibre.

Chromatic splitting and chromatic fracture

$$
\begin{tikzcd}
\begin{array}{cccc} \emptyset & 23 & 03 & 03.23 \\ \boxed{13} & \boxed{13.23} & 03.13 & 03.13.23 \\ S_? & & & \end{array} \arrow[rr] \arrow[dr] \arrow[dd] & & \begin{array}{cccc} \emptyset & 23 & 03 & 03.23 \\ \boxed{13} & \boxed{13.23} & 03.13 & 03.13.23 \\ S_0 = \widehat{S}_0 & & & \end{array} \arrow[dr] \arrow[dd] & \\
& \begin{array}{cc} \emptyset & 23 \\ \boxed{13} & \boxed{13.23} \\ \widehat{S}_1 & \end{array} \arrow[rr] \arrow[dd] & & \begin{array}{cccc} \emptyset & 01 & 23 & 01.23 \\ \boxed{13 \; 01.13} & & \boxed{13.23 \; 01.13.23} & \\ & S_0 & & \end{array} \arrow[dd] \\
\begin{array}{cc} \emptyset & 23 \\ \widehat{S}_2 & \end{array} \arrow[rr] \arrow[dr] & & \begin{array}{cccc} \emptyset & 02 & 23 & 02.23 \\ 12 & 02.12 & 12.23 & 02.12.23 \\ & S_0 & & \end{array} \arrow[dr] & \\
& \begin{array}{cc} \widehat{S}_1 & \\ \emptyset & 23 \\ 12 & 12.23 \end{array} \arrow[rr] & & \begin{array}{cccc} & S_0 & & \\ \emptyset & 01 & 23 & 01.23 \\ 12 & 01.12 & 12.23 & 01.12.23 \end{array}
\end{tikzcd}
$$

This subdiagram consists of two copies of the fracture square for S_1 and so is homotopy cartesian.

Chromatic splitting and chromatic fracture

$$
\begin{tikzcd}
\begin{smallmatrix} \emptyset & 23 & 03 & 03.23 \\ 13 & 13.23 & 03.13 & 03.13.23 \\ \color{green}{S_?} & & & \end{smallmatrix} \arrow[rr] \arrow[dr] \arrow[dd] & & \begin{smallmatrix} \emptyset & 23 & 03 & 03.23 \\ 13 & 13.23 & 03.13 & 03.13.23 \\ S_0 = \widehat{S}_0 & & & \end{smallmatrix} \arrow[dr] \arrow[dd] & \\
& \begin{smallmatrix} \emptyset & 23 \\ 13 & 13.23 \\ & \widehat{S}_1 \end{smallmatrix} \arrow[rr] \arrow[dd] & & \begin{smallmatrix} \emptyset & 01 & 23 & 01.23 \\ 13 & 01.13 & 13.23 & 01.13.23 \\ & & S_0 & \end{smallmatrix} \arrow[dd] \\
\begin{smallmatrix} \emptyset & 23 \\ \widehat{S}_2 & \end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix} \emptyset & 02 & 23 & 02.23 \\ 12 & 02.12 & 12.23 & 02.12.23 \\ & S_0 & & \end{smallmatrix} \arrow[dr] & \\
& \begin{smallmatrix} \widehat{S}_1 & \\ \emptyset & 23 \\ 12 & 12.23 \end{smallmatrix} \arrow[rr] & & \begin{smallmatrix} & & S_0 & \\ \emptyset & 01 & 23 & 01.23 \\ 12 & 01.12 & 12.23 & 01.12.23 \end{smallmatrix}
\end{tikzcd}
$$

We can remove that subdiagram without changing the total fibre.

$$\begin{tikzcd}[column sep=small]
\begin{array}{cc|c|c|}\hline \emptyset & 23 & 03 & 03.23 \\ \hline 13 & 13.23 & 03.13 & 03.13.23 \\ \hline \end{array}_{S_?} \arrow[rr] \arrow[dr] \arrow[dd] & & \begin{array}{cc|c|c|}\hline \emptyset & 23 & 03 & 03.23 \\ \hline 13 & 13.23 & 03.13 & 03.13.23 \\ \hline \end{array}_{S_0=\widehat{S}_0} \arrow[dr] \arrow[dd] & \\
& \begin{array}{cc} \emptyset & 23 \\ 13 & 13.23 \end{array}_{\widehat{S}_1} \arrow[rr] \arrow[dd] & & \begin{array}{cccc} \emptyset & 01 & 23 & 01.23 \\ 13 & 01.13 & 13.23 & 01.13.23 \end{array}_{S_0} \arrow[dd] \\
\begin{array}{cc} \emptyset & 23 \end{array}_{\widehat{S}_2} \arrow[rr] \arrow[dr] & & \begin{array}{cccc} \emptyset & 02 & 23 & 02.23 \\ 12 & 02.12 & 12.23 & 02.12.23 \end{array}_{S_0} \arrow[dr] & \\
& \overset{\widehat{S}_1}{\begin{array}{cc} \emptyset & 23 \\ 12 & 12.23 \end{array}} \arrow[rr] & & \overset{S_0}{\begin{array}{cccc} \emptyset & 01 & 23 & 01.23 \\ 12 & 01.12 & 12.23 & 01.12.23 \end{array}}
\end{tikzcd}$$

This subdiagram consists of four copies of the fracture interval for S_0 and so is homotopy cartesian.

Chromatic splitting and chromatic fracture

$$
\begin{tikzcd}[column sep=small]
\begin{smallmatrix}\emptyset & 23 & 03 & 03.23 \\ 13 & 13.23 & 03.13 & 03.13.23\end{smallmatrix}\ S_? \arrow[rr] \arrow[dr] \arrow[dd] & & \begin{smallmatrix}\emptyset & 23 & 03 & 03.23 \\ 13 & 13.23 & 03.13 & 03.13.23\end{smallmatrix}\ S_0 = \widehat{S}_0 \arrow[dr] \arrow[dd] & \\
& \begin{smallmatrix}\emptyset & 23 \\ 13 & 13.23\end{smallmatrix}\ \widehat{S}_1 \arrow[rr] \arrow[dd] & & \begin{smallmatrix}\emptyset & 01 & 23 & 01.23 \\ 13 & 01.13 & 13.23 & 01.13.23\end{smallmatrix}\ S_0 \arrow[dd] \\
\begin{smallmatrix}\emptyset & 23\end{smallmatrix}\ \widehat{S}_2 \arrow[rr] \arrow[dr] & & \begin{smallmatrix}\emptyset & 02 & 23 & 02.23 \\ 12 & 02.12 & 12.23 & 02.12.23\end{smallmatrix}\ S_0 \arrow[dr] & \\
& \begin{smallmatrix}\emptyset & 23 \\ 12 & 12.23\end{smallmatrix}\ \widehat{S}_1 \arrow[rr] & & \begin{smallmatrix}\emptyset & 01 & 23 & 01.23 \\ 12 & 01.12 & 12.23 & 01.12.23\end{smallmatrix}\ S_0
\end{tikzcd}
$$

After removing that subdiagram we see that the original diagram was homotopy cartesian, as required.

Chromatic splitting and chromatic fracture

$$
\begin{tikzcd}
\begin{array}{c}\begin{array}{cccc}\emptyset & 23 & 03 & 03.23\\ 13 & 13.23 & 03.13 & 03.13.23\end{array}\\ S_?\end{array} \arrow[rr] \arrow[dr] \arrow[dd] & & \begin{array}{c}\begin{array}{cccc}\emptyset & 23 & 03 & 03.23\\ 13 & 13.23 & 03.13 & 03.13.23\end{array}\\ S_0=\widehat{S}_0\end{array} \arrow[dr] \arrow[dd] & \\
& \begin{array}{c}\begin{array}{cc}\emptyset & 23\\ 13 & 13.23\end{array}\\ \widehat{S}_1\end{array} \arrow[rr] \arrow[dd] & & \begin{array}{c}\begin{array}{cccc}\emptyset & 01 & 23 & 01.23\\ 13 & 01.13 & 13.23 & 01.13.23\end{array}\\ S_0\end{array} \arrow[dd] \\
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& \begin{array}{c}\widehat{S}_1\\ \begin{array}{cc}\emptyset & 23\\ 12 & 12.23\end{array}\end{array} \arrow[rr] & & \begin{array}{c}S_0\\ \begin{array}{cccc}\emptyset & 01 & 23 & 01.23\\ 12 & 01.12 & 12.23 & 01.12.23\end{array}\end{array}
\end{tikzcd}
$$

Similarly, CSC implies that the chromatic fracture hypercube for $\alpha_A(S) = L_{n-1}(\phi_A(S))$ is a sum of the hypercubes for various S^d_m.

Chromatic splitting and chromatic fracture

$$
\begin{tikzcd}
S_2 \arrow[rr] \arrow[dr] \arrow[dd] & & \widehat{S}_0 \arrow[dr] \arrow[dd] & \\
& \widehat{S}_1 \arrow[rr, dotted] \arrow[dd] & & E_{\widehat{S}_0}[x_{01}] \arrow[dd] \\
\widehat{S}_2 \arrow[rr, dotted] \arrow[dr, dotted] & & E_{\widehat{S}_0}[x_{02}, x_{12}] \arrow[dr] & \\
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$$\begin{tikzcd}
\mathrm{gr}_1(S_2) \arrow[rr] \arrow[dr] \arrow[dd] & & 0 \arrow[dr] \arrow[dd] & \\
& 0 \arrow[rr] \arrow[dd] & & \widehat{S}_0^{-1}.x_{01} \arrow[dd] \\
0 \arrow[rr] \arrow[dr] & & \widehat{S}_0^{-3}.x_{02} \vee \widehat{S}_0^{-1}.x_{12} \arrow[dr] & \\
& \widehat{S}_1^{-1}.x_{12} \arrow[rr] & & \widehat{S}_0^{-1}.x_{01} \vee \widehat{S}_0^{-1}.x_{12}
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& 0 \arrow[rr] \arrow[dd] & & 0 \arrow[dd] \\
0 \arrow[rr] \arrow[dr] & & \widehat{S}_0^{-4}.x_{02}x_{12} \arrow[dr] & \\
& 0 \arrow[rr] & & \widehat{S}_0^{-2}.x_{01}x_{12}
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Associated graded of the filtration of S_n

The associated graded object $\mathrm{gr}_*(S_n)$ is conjecturally as follows:

- For any sequence $u = (u_0 < u_1 < \cdots < u_r = n)$ we have $z_u \colon \widehat{S}_{u_0}^{2(u_0-n)} \to \mathrm{gr}_r(S_n)$.
- There is a fibration $S_n \to S_{n-1} \vee \widehat{S}_n \to \alpha_n(S) \xrightarrow{\delta_n} S_n^1$. Put
$$z'_{ij} = \Sigma^{2j-1}(S_i^{1-2(j-i)} \xrightarrow{x_{ij}} \alpha_j(S) \xrightarrow{\delta_j} S_j^1) \colon S_i^{2i} \to S_j^{2j}.$$
Then z_u is related to the composite
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Euler characteristics

- Put $\chi_n(X) = \dim_{K(n)_*}(K(n)_{\mathrm{even}}(X)) - \dim_{K(n)_*}(K(n)_{\mathrm{odd}}(X))$ (assuming that the dimensions are finite).
- For the X that we have considered: $\chi_n(X)$ is probably 0, occasionally 1.
- Sometimes this is known unconditionally, sometimes it relies on the CSC.
- Some aspects of the previous story can be checked for consistency using these invariants. Often we just get $0 = 0$ which is not very impressive, but in a few cases there are interesting patterns of cancellation.

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Further questions

- For $U \subseteq P\{0, \dots, N\}$ closed upwards, put $\theta_U(X) = \underset{\substack{\longleftarrow \\ A \in U}}{\text{holim}}\, \phi_A(X)$.
- In work with Bellumat we showed that this class of functors contains L_n and $L_{K(n)}$ and is closed under composition and certain homotopy limits.
- We believe that CSC implies a splitting of all $\theta_U(S)$, but have not completed this analysis.
- Ravenel has defined ring spectra
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