Rational global Balmer spectra

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Context: rational equivariance for a single group

- For finite G, let $\mathbb{Q}\,\mathrm{Sp}_G$ be the category of rational genuine G-spectra.
- Put $\mathcal{A}G = [\mathrm{Orb}_G^\times, \mathrm{Vect}] \simeq \prod_{(H)} \mathrm{Mod}_{\mathbb{Q}[W_G H]}$; a semisimple category (product over conjugacy classes of subgroups; $W_G H = (N_G H)/H$)
- Using the geometric fixed point functors $\phi^H \colon \mathbb{Q}\,\mathrm{Sp}_G \to \mathbb{Q}\,\mathrm{Sp}_{W_G H}$ we obtain an equivalence $\mathbb{Q}\,\mathrm{Sp}_G \to D(\mathcal{A}G) \simeq \mathrm{Gr}(\mathcal{A}G)$
- In this triangulated category, an object is compact iff strongly dualisable iff it has finite total dimension.
- Every prime thick ideal in the compact subcategory $\mathbb{Q}\,\mathrm{Sp}_G^c$ is the kernel of ϕ^H for some H (unique up to conjugacy).
- The Balmer spectrum $\mathrm{Spc}(\mathbb{Q}\,\mathrm{Sp}_G^c)$ is the (finite, discrete) set of conjugacy classes of subgroups.

Global equivariance

- Let $\mathcal{G}$ be the category of finite groups and conjugacy classes of surjective homomorphisms.
- A $\mathcal{G}$-globally equivariant spectrum X is a compatible system of G-spectra $X_G \in \mathrm{Sp}_G$ for all $G \in \mathcal{G}$; category of such objects is $\mathrm{Sp}_{\mathcal{G}}$ (Schwede)
- Examples: S, KU, KO, MU, MO, H
- The category $\mathrm{Sp}_{\mathcal{G}}^c$ is not rigid: the only dualisable objects are in the essential image of $\mathrm{Sp}^c \to \mathrm{Sp}_{\mathcal{G}}^c$, but there are many more compact objects.
- Put $\mathcal{A}(\mathcal{G}) = [\mathcal{G}^{\mathrm{op}}, \mathrm{Vect}]$. This is a symmetric monoidal abelian category with some unusual properties. It is not semisimple.
- Schwede proved $\mathbb{Q}\,\mathrm{Sp}_{\mathcal{G}} \simeq D(\mathcal{A}\mathcal{G})$ as homotopy categories. This can be improved to an equivalence of ∞-categories.
- Again, $\mathbb{Q}\,\mathrm{Sp}_{\mathcal{G}}^c$ is not rigid, so most techniques for studying Balmer spectra (e.g. nilpotence) are not applicable.
- Much remain true for suitable full subcategories $\mathcal{U} \subseteq \mathcal{G}$, such as:
  - $\mathcal{E}[p] = \{\text{elementary abelian } p\text{-groups}\}$
  - $\mathcal{G}\langle r\rangle = \{G \in \mathcal{G} \mid G \text{ can be generated by a set of size } \leq r\}$
  - $\mathcal{Z}[p] = \{\text{abelian } p\text{-groups}\}$
  - $\mathcal{Z}[p]\langle r\rangle = \mathcal{Z}[p] \cap \mathcal{G}\langle r\rangle$.
- Let $\widehat{\mathcal{U}}$ be the category of finitely generated profinite groups for which a cofinal sequence of finite quotients lie in $\mathcal{U}$.

The bounded rank theorem

- **Theorem A:** The Balmer spectrum $\mathrm{Spc}(\mathbb{Q}\,\mathrm{Sp}^c_{\mathcal{G}\langle r\rangle}) = \mathrm{Spc}(D(\mathcal{A}\mathcal{G}\langle r\rangle))$ is the profinite space $\pi_0\widehat{\mathcal{G}}\langle r\rangle$.
- More generally, for $\mathcal{U} \subseteq \mathcal{G}\langle r\rangle$ satisfying mild conditions we have $\mathrm{Spc}(\mathbb{Q}\,\mathrm{Sp}^c_{\mathcal{U}}) = \pi_0\widehat{\mathcal{U}}$.
- In particular: put $D[r] = \{d \in \mathbb{N}^r_\infty \mid d_1 \geq \cdots \geq d_r\}$ and for $d \in D[r]$ put $G_d = \prod_i \mathbb{Z}_p/p^{d_i}$ (where p^∞ means 0).
- For $\mathcal{U} = \mathcal{Z}[p]\langle r\rangle$: every object is isomorphic to G_d for a unique d, so $\mathrm{Spc}(\mathbb{Q}\,\mathrm{Sp}^c_{\mathcal{U}}) = D[r]$. (Also: $\mathcal{A}\mathcal{U}$ is locally noetherian in this case.)
- In these cases:
  - Finitely generated thick ideals in $\mathrm{Sp}^c_{\mathcal{U}}$ biject with clopen subsets of $\pi_0\widehat{\mathcal{U}}$
  - Arbitrary thick ideals in $\mathrm{Sp}^c_{\mathcal{U}}$ biject with open subsets of $\pi_0\widehat{\mathcal{U}}$
  - Prime thick ideals in $\mathrm{Sp}^c_{\mathcal{U}}$ biject with complements of points
  - Thomason subsets ($=$ unions of complements of compact open sets) are the same as open sets; so the topology is Hochster self-dual.
- For $X \in \mathcal{A}\mathcal{U}$ and $G \in \widehat{\mathcal{U}}$ let $X(G) \in \mathrm{Vect}$ be the colimit of $X(G/N)$ for N open normal in G with $G/N \in \mathcal{U}$.
- The prime ideal $\mathfrak{p}_G \in \mathrm{Spc}(D(\mathcal{A}\mathcal{U})^c)$ corresponding to $[G] \in \pi_0\widehat{\mathcal{U}}$ is $\mathfrak{p}_G = \{X \in D(\mathcal{A}\mathcal{U})^c \mid H_*(X)(G) = 0\}$.

The elementary abelian theorem

- Recall: $\mathcal{E}[p] = \{$ elementary abelian p-groups$\} = \{C_p^r \mid r \in \mathbb{N}\}$.
- For $r \in \mathbb{N}$ put $\mathfrak{p}_r = \{X \in D(\mathcal{A}\mathcal{E}[p])^c \mid H_*(X)(C_p^r) = 0\}$.
  This is clearly a prime thick ideal.
- Put $\mathfrak{p}_\infty = \{0\} = \bigcap_{r\in\mathbb{N}} \mathfrak{p}_r$.
  It is true but not obvious that this is also prime.
- **Theorem B:** $\mathrm{Spc}(\mathrm{Sp}^c_{\mathcal{E}[p]}) = \{\mathfrak{p}_r \mid r \in \mathbb{N}_\infty\} \simeq \mathbb{N}_\infty$
- Consider a subset $U \subseteq \mathbb{N}_\infty$:
  - U is open iff ($U \subseteq \mathbb{N}$ or $U = \mathbb{N}_\infty$)
  - U is compact open iff (($U \subseteq \mathbb{N}$ and $|U| < \infty$) or $U = \mathbb{N}_\infty$)
  - U is Thomason iff ($U = \mathbb{N}_\infty \setminus F$ for some finite $F \subset \mathbb{N}$) or $U = \emptyset$
  - Thick ideals biject with Thomason subsets,
    and they are all finitely generated.
- The methods for Theorem B are orthogonal to those for Theorem A. Some are special to the elementary abelian case but many are more general.
- Future goal: combine methods for Theorems A and B to prove a general result without bounds on the number of generators.
- In this direction:
  Theorem C: if $\mathcal{U}$ is closed under products, subgroups and quotients then the zero ideal in $\mathrm{Sp}^c_{\mathcal{U}}$ is prime.

Reflective filtrations

- For $G \in \mathcal{Z} = \{$ finite abelian groups $\}$ put $N_n(G) = \{g^{n!} \mid g \in G\}$ and $q_n(G) = G/N_n(G)$. This is left adjoint to the inclusion $\mathcal{Z}[n] = \{G \in \mathcal{Z} \mid N_n(G) = 1\} \to \mathcal{Z}$.
- For $G \in \mathcal{P} = \{$finite p-groups$\}$ let ΦG be generated by pth powers and commutators. Put $N_n(G) = \Phi^n(G)$ and $q_n(G) = G/N_n(G)$. This is left adjoint to the inclusion $\mathcal{P}[n] = \{G \in \mathcal{P} \mid N_n(G) = 1\} \to \mathcal{Z}$.
- For $G \in \mathcal{G}$ let $N_n(G)$ be the intersection of all subgroups of index at most n and put $q_n(G) = G/N_n(G)$. This is again a left adjoint.
- This kind of structure is called a *reflective filtration*; it exists automatically in many cases.
- There are some wrinkles in the story about adjoints, because morphisms in $\mathcal{G}$ are conjugacy classes of surjective homomorphisms, and (co)limits do not generally exist.
- In the first two examples, any morphism $\alpha \colon G \to H$ has $\alpha(N_n(G)) = N_n(H)$, but this does not hold in the third example.
- Usually q_n extends to give $q_n \colon \widehat{\mathcal{U}} \to \mathcal{U}[n]$.
- The lattice of thick ideals in $\mathbb{Q}\,\mathrm{Sp}_{\mathcal{U}}^c = D(\mathcal{A}\mathcal{U})^c$ is the colimit of the corresponding lattices for $\mathcal{U}[n]$, so the Balmer spectrum is the inverse limit.

Reduction to the finite case

- If $\pi_0\mathcal{U}$ is finite, we can prove that the natural map $\pi_0\mathcal{U} \to \mathrm{Spc}(D(\mathcal{A}\mathcal{U})^c)$ is bijective and that the topology is discrete.
- For $G \in \mathcal{G}$ recall that $N_n(G)$ is the intersection of all subgroups of index at most n, or the common kernel of all homomorphisms to Σ_n. Also $\mathcal{G}[n] = \{G \mid N_n(G) = 1\}$.
- If G can be generated by a set of size at most r then $|\mathrm{Hom}(G, \Sigma_n)| \leq n!^r$. This gives an upper bound on the index of $N_n(G)$.
- Using this we see that $\pi_0(\mathcal{G}[n] \cap \mathcal{G}\langle r\rangle)$ is finite so $\mathrm{Spc}(D(\mathcal{A}(\mathcal{G}[n] \cap \mathcal{G}\langle r\rangle))) = \pi_0(\mathcal{G}[n] \cap \mathcal{G}\langle r\rangle)$.
- By passing to the limit we see that $\mathrm{Spc}(D(\mathcal{A}\mathcal{G}\langle r\rangle))$ is the inverse limit of the finite discrete sets $\pi_0(\mathcal{G}[n] \cap \mathcal{G}\langle r\rangle)$.
- The functors $q_n \colon \widehat{\mathcal{G}}\langle r\rangle \to \mathcal{G}[n] \cap \mathcal{G}\langle r\rangle$ can be used to identify $\pi_0\widehat{\mathcal{G}}\langle r\rangle$ with the same inverse limit.
- The same line of argument works for many other subcategories $\mathcal{U}$ with $\mathcal{U} \subseteq \mathcal{G}\langle r\rangle$.

Hypotheses on $\mathcal{U}$

- For the rest of this talk, assume that $\mathcal{U}$ is closed under products, subgroups and quotients, and consists of abelian groups.
- The nonabelian case is similar but requires fiddly bookkeeping of conjugacies.
- Closure under products means that we can have no bound on the number of generators. This makes an important qualitative difference in some places.

Tensor structure

- Projective generator $e_G \in \mathcal{AU} = [\mathcal{U}^{\mathrm{op}}, \mathrm{Vect}]$ given by $e_G(T) = \mathbb{Q}\mathcal{U}(T, G)$.
- Say $W \leq G \times H$ is *wide* if projections to G and H are both surjective iff there exists $N \leq G$, $M \leq H$, $\alpha \colon G/N \xrightarrow{\simeq} H/M$ with $W = \{(g, h) \mid \alpha(gN) = hM\}$.
- Example: $G \times H$ is wide in $G \times H$, diagonal $\Delta \leq G \times G$ is also wide.
- There is an easy isomorphism $e_G \otimes e_H = \bigoplus_W e_W$.
- Put $DX = \underline{\mathrm{Hom}}(X, \mathbb{1})$ so X is strongly dualisable iff $DX \otimes X \to \underline{\mathrm{Hom}}(X, X)$ is iso.
- Suppose that $X \neq 0$ but $X(1) = 0$. Then $(DX \otimes X)(1) = 0$ but $\underline{\mathrm{Hom}}(X, X)(1) = \mathcal{AU}(X, X) \neq 0$ so X is not strongly dualisable.
- Thus e_G is not strongly dualisable unless $G = 1$.
  In fact X is only strongly dualisable if it is constant and finite-dimensional.
- If $|C| = p$ then $\underline{\mathrm{Hom}}(e_C, e_C) = e_{C^2} \oplus (2p-1)e_C \oplus (p-1)\mathbb{1}$ but $D(e_C) \otimes e_C = e_{C^2} \oplus pe_C$.

More about duals and internal homs

- We can write down an isomorphism $\bigoplus_{N \leq G} e_{G/N} \xrightarrow{\simeq} D(e_G)$.
- We will show later that $\mathbb{1}$ is injective.
  Also any X is flat, and it follows that DX is injective.
  As e_G is a retract of $D(e_G)$, it is also injective.
  It follows that all projectives are injective.
- However, $t_G(K) = \mathrm{Map}(\mathcal{U}(G, K), \mathbb{Q})$ is injective but not projective.
- A *virtual homomorphism* from G to H is a pair (A, A') where
  $A' \leq A \leq G \times H$ and A is wide and $A' \cap (1 \times H) = 1$ and $A/A' \in \mathcal{U}$.
- $\underline{\mathrm{Hom}}(e_G, e_H)$ has a natural filtration with associated graded $\bigoplus_{(A,A')} e_{A/A'}$.
  As $e_{A/A'}$ is projective, the filtration splits and $\underline{\mathrm{Hom}}(e_G, e_H)$ is projective.
  We do not know whether the filtration splits naturally.

Asymptotic freedom

- Let $F_{nm} \in \mathcal{U}$ be the quotient of the free group on n generators by the intersection of all normal subgroups N with quotient in $\mathcal{U}_{\leq m}$.
- Given morphisms $F_{nm} \xrightarrow{\phi} H \xleftarrow{\alpha} G$ in $\mathcal{U}$ with $|G| \leq \min(n, m)$, we can choose $\psi \colon F_{nm} \to G$ in $\mathcal{U}$ with $\alpha\psi = \phi$.
  (Some care is needed to ensure that ψ is surjective.)
- We can choose a tower $G_0 \leftarrow G_1 \leftarrow G_2 \leftarrow \cdots$ in $\mathcal{U}$ such that G_n gets rapidly larger and freer as $n \to \infty$.
- We then find that

$$\varinjlim_{G \in \mathcal{U}^{\mathrm{op}}} X(G) = \varinjlim_{n} X(G_n)_{\mathrm{Out}(G_n)},$$

  and this is an exact functor of X (because we work over $\mathbb{Q}$).
- $\mathcal{AU}(X, \mathbb{1})$ is hom from the above colimit to $\mathbb{Q}$; so $\mathbb{1}$ is injective.
- As mentioned previously: it follows that $D(e_G)$ is injective, then that e_G is injective, then that all projectives are injective.
- Using this: any object of finite projective dimension is projective.

Rates of growth

- If n is large, the proportion of n-tuples in G^n that generate G is close to 1 (theorem of Lynne Butler, 1994).
- Using this plus nearly free groups as on the previous slide:
  if X is a nontrivial summand of e_G, then an appropriate lim sup of $\dim(X(T))/|G|^{\delta(T)}$ is nonzero and finite, where $\delta(T)$ is the minimal size of a generating set.
- We can define Serre subcategories and then quotient categories using rates of growth. We have not yet exploited this fully.
- This approach show that monomorphisms between projective objects split, even for some $\mathcal{U}$ where projectives are not injective.

The order filtration

- For a $\mathbb{Q}[\mathrm{Out}(G)]$-module V, put

  $$e_{G,V}(K) = V \otimes_{\mathbb{Q}[\mathrm{Out}(G)]} e_G(K)$$

  This is projective. Every indecomposable projective has the form $e_{G,S}$ for some indecomposable $\mathbb{Q}[\mathrm{Out}(G)]$-module S. We define the order of $e_{G,S}$ to be the order of G.
- We say that X is *pure of order k* if it is isomorphic to a sum of indecomposable projectives of order k.
- The subcategory of such objects is equivalent to the semisimple category $\mathcal{AU}_k = [\mathcal{U}_k^{\mathrm{op}}, \mathrm{Vect}]$.
- If X is pure of order k, and Y is pure of order $m > k$, then $\mathcal{AU}(X, Y) = 0$.
- Let $(L_{\leq m}X)(G)$ be the sum of all $\alpha^*(X(H)) \leq X(G)$ for $H \in \mathcal{U}_{\leq m}$ and $\alpha \in \mathcal{U}(G, H)$.
- Put $L_m X = L_{\leq m}X / L_{<m}X$.
- If P is projective, then $P \simeq \bigoplus_k P_k \simeq \prod_k P_k$, where P_k is pure of order k. It follows that $L_{\leq m}X = \bigoplus_{k\leq m} P_k$ and $L_m X \simeq P_m$ so the filtration splits.

The derived category

- Let $\mathcal{PU}$ be the subcategory of projectives in $\mathcal{AU}$.
- There is an additive functor $P_0 = I_! I^* \colon \mathcal{AU} \to \mathcal{PU}$ with a surjective natural transformation $\epsilon\colon P_0(X) \to X$, where I is the inclusion $\mathcal{U}^\times \to \mathcal{U}$.
- If $X(G) = 0$ for $|G| < n$ then $\ker(\epsilon)(G) = 0$ for $|G| \leq n$. This also works when $|\mathcal{U}| < \infty$ and is a key step in the proof that $\mathrm{Spc}(D(\mathcal{AU})^c) = \pi_0\mathcal{U}$.
- Using P_0 we can define an additive functor $P\colon \mathrm{Ch}(\mathcal{AU}) \to \mathrm{Ch}(\mathcal{PU})$ with a natural surjective quasiisomorphism $P(X) \to X$.
- From this and other results:

$$\mathrm{Ch}(\mathcal{AU})[\mathrm{we}^{-1}] = \mathrm{hCh}(\mathcal{PU}) := \mathrm{Ch}(\mathcal{PU})/(\text{chain homotopy}).$$

  (For general abelian categories, the story is more subtle.)
- If $X, Y \in \mathrm{Ch}(\mathcal{PU})$ then $X \otimes Y$, $\underline{\mathrm{Hom}}(X, Y) \in \mathcal{PU}$.
- Say $X \in \mathrm{Ch}(\mathcal{PU})$ is *thin* if for every $m > 0$, the differential on $L_m X$ is 0, i.e. the differential on X involves only maps $e_{G,S} \to e_{H,T}$ with $|H| < |G|$.
- Every homotopy type has an essentially unique thin representative. (But thin $\otimes$ thin and $\underline{\mathrm{Hom}}(\text{thin}, \text{thin})$ need not be thin.)
- A thin complex X is compact iff $\bigoplus_n X_n$ is finitely generated.

Supports and thick ideals

- For compact X (represented as a thin complex), several notions of support:
- $\mathrm{hsupp}(X) = \{G \mid H_*(X)(G) \neq 0\}$
- $\mathrm{esupp}(X) = \{G \mid X(G) \neq 0\}$
- $\mathrm{eqsupp}(X) = \{G \mid \text{ some } e_{G,S} \text{ is a retract of some } X_d\}$.
- It is easy to see that $\mathrm{esupp}(X)$ is the upwards closure of $\mathrm{eqsupp}(X)$.
- True but less obvious: $\mathrm{esupp}(X)$ is the upwards closure of $\mathrm{hsupp}(X)$.
- Conjecture: $\mathrm{thickid}\langle X\rangle \subseteq \mathrm{thickid}\langle Y\rangle$ iff $\mathrm{hsupp}(X) \subseteq \mathrm{hsupp}(Y)$.
- This holds in all the cases that we understand.
- The obvious prime ideals are $\mathfrak{p}_G = \{X \mid H_*(X)(G) = 0\}$.
- If X is thin and n is largest with $L_nX \neq 0$, then $X = H_*(X) = L_nX$ mod terms of slower growth.
- Using this: we prove Theorem C: the zero ideal is also prime.
- We have various partial results and examples, especially conditions under which $e_G \in \mathrm{thickid}\langle Y\rangle$.
- Given X, Y with $\mathrm{hsupp}(X) \subseteq \mathrm{hsupp}(Y)$, and a large integer $N > 0$, we can show that $\mathrm{thickid}\langle X\rangle \subseteq \mathrm{thickid}\langle \{Y\} \cup \{e_G \mid |G| > N\}\rangle$.
- This work is ongoing.