

Rational global representation theory

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(with Luca Pol)

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- ▶ Let \mathcal{G} be the category of finite groups and conjugacy classes of surjective homomorphisms.
- ▶ Let \mathcal{U} be a full subcategory of \mathcal{G} , such as finite abelian p -groups.
- ▶ Put $\mathcal{AU} = [\mathcal{U}^{\text{op}}, \text{Vect}]$; then $D(\mathcal{AU})$ is equivalent to the category of globally equivariant spectra with rational homotopy groups, via $X \mapsto (G \mapsto \pi_*(\phi^G(X)))$.
- ▶ For each $G \in \mathcal{U}$ we have a projective generator $e_G(K) = \mathbb{Q}[\mathcal{U}(K, G)]$ (which is 0 if $|K| < |G|$).
- ▶ \mathcal{AU} is closed symmetric monoidal with $(X \otimes Y)(G) = X(G) \otimes Y(G)$ and $\mathbb{1} = e_1 = (G \mapsto \mathbb{Q})$ and $\underline{\text{Hom}}(X, Y)(G) = \mathcal{AU}(e_G \otimes X, Y)$.
- ▶ Goal: study \mathcal{AU} , derived category $D(\mathcal{AU})$, Balmer spectrum of compact objects in $D(\mathcal{AU})$.
- ▶ Problem: projective generators are not strongly dualisable in \mathcal{AU} ; compact objects are not strongly dualisable in $D(\mathcal{AU})$. This blocks most known approaches to the Balmer spectrum.
- ▶ Although duality phenomena are unfamiliar, they are still concrete and tractable.
- ▶ Another unusual feature: all projectives are injective, but not conversely.
- ▶ For abelian p -groups, or elementary abelian p -groups, or cyclic groups: the category \mathcal{AU} is locally noetherian, i.e. subobjects of finitely generated objects are finitely generated.

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- ▶ Example: $G \times H$ is wide in $G \times H$, diagonal $\Delta \leq G \times G$ is also wide.
- ▶ There is an easy isomorphism $e_G \otimes e_H = \bigoplus_W e_W$.
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Adjoint to $\theta_N: e_{G/N} \otimes e_G \rightarrow \mathbb{1}$ we have $\theta_N^\#: e_{G/N} \rightarrow \underline{\text{Hom}}(e_G, \mathbb{1}) = D(e_G)$.

- ▶ Fact: these maps give $\bigoplus_N e_{G/N} \xrightarrow{\simeq} D(e_G)$.
- ▶ We will show later that $\mathbb{1}$ is injective.
Also any X is flat, and it follows that DX is injective.
As e_G is a retract of $D(e_G)$, it is also injective.
It follows that all projectives are injective.
- ▶ However, $t_G(K) = \text{Map}(\mathcal{U}(G, K), \mathbb{Q})$ is injective but not projective.
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We do not know whether the filtration splits naturally.
- ▶ For the nonabelian case, we need to pass to conjugacy classes in the right way, but the details are even more fiddly.

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More about duals and internal homs

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Asymptotic freedom

- ▶ Let F_{nm} be the quotient of the free group on n generators by the intersection of all normal subgroups N with quotient in $\mathcal{U}_{\leq m}$. Under mild conditions on \mathcal{U} we have $F_{nm} \in \mathcal{U}$.
- ▶ Given morphisms $F_{nm} \xrightarrow{\phi} H \xleftarrow{\alpha} G$ in \mathcal{U} with $|G| \leq \min(n, m)$, we can choose $\psi: F_{nm} \rightarrow G$ in \mathcal{U} with $\alpha\psi = \phi$. (Some care is needed to ensure that ψ is surjective.)
- ▶ We can choose a tower $G_0 \leftarrow G_1 \leftarrow G_2 \leftarrow \dots$ in \mathcal{U} such that G_n gets rapidly larger and freer as $n \rightarrow \infty$.
- ▶ We then find that

$$\lim_{G \in \mathcal{U}^{\text{op}}} X(G) = \lim_n X(G_n)_{\text{Out}(G_n)},$$

and this is an exact functor of X (because we work over \mathbb{Q}).

- ▶ $\mathcal{A}\mathcal{U}(X, \mathbb{1})$ is hom from the above colimit to \mathbb{Q} ; so $\mathbb{1}$ is injective.
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Rates of growth

- ▶ If n is large, the proportion of n -tuples in G^n that generate G is close to 1 (theorem of Lynne Butler, 1994).
- ▶ Using this plus nearly free groups as on the previous slide:
if X is a nontrivial summand of e_G , then an appropriate \limsup of $\dim(X(T))/|G|^{\delta(T)}$ is nonzero and finite, where $\delta(T)$ is the minimal size of a generating set.
- ▶ We can define Serre subcategories and then quotient categories using rates of growth. We have not yet exploited this fully.
- ▶ This approach show that monomorphisms between projective objects split, even for some \mathcal{U} where projectives are not injective.

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The order filtration

- ▶ For a $\mathbb{Q}[\text{Out}(G)]$ -module V , put

$$e_{G,V}(K) = V \otimes_{\mathbb{Q}[\text{Out}(G)]} e_G(K)$$

This is projective. Every indecomposable projective has the form $e_{G,S}$ for some indecomposable $\mathbb{Q}[\text{Out}(G)]$ -module S . We define the order of $e_{G,S}$ to be the order of G .

- ▶ We say that X is pure of order k if it is isomorphic to a sum of indecomposable projectives of order k .
- ▶ The subcategory of such objects is equivalent to the semisimple category $\mathcal{AU}_k = [\mathcal{U}_k^{\text{op}}, \text{Vect}]$.
- ▶ If X is pure of order k , and Y is pure of order $m > k$, then $\mathcal{AU}(X, Y) = 0$.
- ▶ Let $(L_{\leq m}X)(G)$ be the sum of all $\alpha^*(X(H)) \leq X(G)$ for $H \in \mathcal{U}_{\leq m}$ and $\alpha \in \mathcal{U}(G, H)$.
- ▶ Put $L_mX = L_{\leq m}X / L_{< m}X$.
- ▶ If P is projective, then $P \simeq \bigoplus_k P_k \simeq \prod_k P_k$, where P_k is pure of order k . It follows that $L_{\leq m}X = \bigoplus_{k \leq m} P_k$ and $L_mX \simeq P_m$ so the filtration splits.

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Noetherian properties

- ▶ We prove that $[\mathcal{U}^{\text{op}}, \text{Vect}]$ is locally noetherian when \mathcal{U} is the category of finite abelian p -groups.
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- ▶ The category $\mathcal{L}_\dagger^{\text{op}}$ and its slice categories are (nonobviously) wqo.
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The derived category

- ▶ Let $\mathcal{P}\mathcal{U}$ be the subcategory of projectives in $\mathcal{A}\mathcal{U}$.
- ▶ There is an additive functor $P_0 = l_!l^* : \mathcal{A}\mathcal{U} \rightarrow \mathcal{P}\mathcal{U}$ with a surjective natural transformation $P_0(X) \rightarrow X$, where l is the inclusion $\mathcal{U}^\times \rightarrow \mathcal{U}$.
- ▶ This extends to give an additive functor $P : \text{Ch}(\mathcal{A}\mathcal{U}) \rightarrow \text{Ch}(\mathcal{P}\mathcal{U})$ with a natural surjective quasiisomorphism $P(X) \rightarrow X$.
- ▶ From this and other results:

$$\text{Ch}(\mathcal{A}\mathcal{U})[we^{-1}] = \text{hCh}(\mathcal{P}\mathcal{U}) := \text{Ch}(\mathcal{P}\mathcal{U})/(\text{chain homotopy}).$$

(For general abelian categories, the story is more subtle.)

- ▶ There is a cofibrantly generated proper stable monoidal model structure, in which everything can be defined explicitly using P and one does not need the small object argument.
- ▶ If $X, Y \in \text{Ch}(\mathcal{P}\mathcal{U})$ then $\underline{\text{Hom}}(X, Y) \in \mathcal{P}\mathcal{U}$.
- ▶ Say $X \in \text{Ch}(\mathcal{P}\mathcal{U})$ is *thin* if for every $m > 0$, the differential on $L_m X$ is 0, i.e. the differential on X involves only maps $e_{G,S} \rightarrow e_{H,T}$ with $|H| < |G|$.
- ▶ Every homotopy type has an essentially unique thin representative. (But $\text{thin} \otimes \text{thin}$ and $\underline{\text{Hom}}(\text{thin}, \text{thin})$ need not be thin.)
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Supports and thick ideals

- ▶ For compact X (represented as a thin complex), several notions of support:
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- ▶ $\text{hsupp}(X) = \{G \mid H_*(X)(G) \neq 0\}$
- ▶ $\text{esupp}(X) = \{G \mid X(G) \neq 0\}$
- ▶ $\text{eqsupp}(X) = \{G \mid \text{some } e_{G,S} \text{ is a retract of some } X_d\}$.
- ▶ It is easy to see that $\text{esupp}(X)$ is the upwards closure of $\text{eqsupp}(X)$.
- ▶ True but less obvious: $\text{esupp}(X)$ is the upwards closure of $\text{hsupp}(X)$.
- ▶ Conjecture: $\text{thickid}\langle X \rangle \subseteq \text{thickid}\langle Y \rangle$ iff $\text{hsupp}(X) \subseteq \text{hsupp}(Y)$.
- ▶ There is a very general method that does most of the work of classifying thick ideals, in cases where all compact objects are strongly dualisable. But that is not applicable here.
- ▶ The obvious prime ideals are $P_G = \{X \mid H_*(X)(G) = 0\}$.
- ▶ If X is thin and n is largest with $L_n X \neq 0$, then $X = H_*(X) = L_n X \bmod$ terms of slower growth. Using this: 0 is also a prime ideal.
- ▶ We have various partial results and examples, especially conditions under which $e_G \in \text{thickid}\langle Y \rangle$.
- ▶ Given X, Y with $\text{hsupp}(X) \subseteq \text{hsupp}(Y)$, and a large integer $N > 0$, we can show that $\text{thickid}\langle X \rangle \subseteq \text{thickid}\langle \{Y\} \cup \{e_G \mid |G| > N\} \rangle$.
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