Motivation and examples for ∞-categories

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The path category of a space

- Let X be a topological space.
- For $a, b \in X$ put $P_X(a, b) = \{\text{paths } u\colon [0,1] \to X \text{ from } a \text{ to } b\}$ and $\Pi_X(a, b) = \pi_0(P_X(a, b)) = \{$ pinned homotopy classes of paths $\}$.
- Try to make P_X into a category with composition given by joining paths.
- This does not work because path join is only associative up to pinned homotopy. However, Π_X becomes a category. (In fact, a groupoid.)
- Problem: generalise so that (something like) P_X counts as an ∞-category.
- This can be done, and we get: Spaces $\simeq$ ∞-groupoids $\simeq$ Kan complexes.
- This is one reason why ∞-categories are an excellent framework for homotopy theory.
- Recall: functors $\Pi_X \to \text{Set}$ correspond to covering spaces $E \to X$, or families of sets continuously parametrised by X.
- Fact: ∞-functors $P_X \to \text{Top}$ correspond to fibrations $E \to X$, or families of spaces continuously parametrised by X.
- Similarly: ∞-functors $P_X \to \text{Sp}$ correspond to X-parametrised spectra.

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Rings and bimodules

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- Notation $A \overset{M}{\leftsquigarrow} B$.
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$$M \circ N = M \otimes_B N = (M \otimes_{\mathbb{Z}} N)/(mb \otimes n = m \otimes bn) \colon A \leftsquigarrow C$$

- This almost gives a category Bimod of rings and bimodules except that $M \circ N$ is really only defined by a universal property, and $L \circ (M \circ N)$ is only isomorphic (not equal) to $(L \circ M) \circ N$.
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- If we want to keep track of general homomorphisms of bimodules, that creates many extra difficulties. If we only keep track of isomorphisms of bimodules, we get an ∞-category.

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Cobordism

- Given compact closed topological n-manifolds M and N, a *cobordism* from N to M is an $(n+1)$-manifold W equipped with a specified homeomorphism $\partial W \simeq M \amalg N$.
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- This almost gives a category Cob_n of n-manifolds and cobordisms except that $V \circ W$ is really only defined by a universal property, and $U \circ (V \circ W)$ is only homeomorphic (not equal) to $(U \circ V) \circ W$.
- Problem: generalise so that Cob_n counts as an ∞-category.
- If we want to keep track of general maps between cobordisms, that creates many extra difficulties. If we only keep track of homeomorphisms between cobordisms, we get an ∞-category.

The category of categories

- There will be an ∞-category Cat_∞ of ∞-categories (and functors and natural isomorphisms).
- This works like other ∞-categories, so we can consider (co)limits of diagrams of categories, systems of categories parametrised by a space, and so on. All of these things are much more difficult in traditional category theory.
- We can also consider the ∞-category $\mathrm{Cat}_\infty^{\amalg}$ of ∞-categories that have finite colimits, and functors that preserve them. This is like an additive category.

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- In the category of spaces it is natural to coherent diagrams like

$$\begin{array}{ccc} W & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & h & \downarrow{\scriptstyle r} \\ Y & \xrightarrow[s]{} & Z \end{array}$$

h is a homotopy from rp to sq (and this is part of the data).

- There is a more complicated story about coherent diagrams of more general shape.
- In the category of spaces it is natural to consider homotopy limits.
- Given $f, g \colon X \to Y$ we have

$$\mathrm{eq}(f, g) = \{x \in X \mid f(x) = g(x)\}$$
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- There is a more complicated story about homotopy (co)limits for (coherent) diagrams of more general shape.
- All this applies to any topological category, not just Top.
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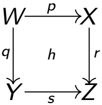

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Other contexts

- There is a well-known analogy between homotopy theory of spaces and chain homotopy theory of chain complexes. There is a good ∞-categorical treatment of this.
- Part of the above story is the Dold-Kan Theorem: the category of nonnegative chain complexes is equivalent to the category of simplicial abelian groups. We can therefore develop homological algebra using simplicial objects instead of chain complexes.
- Simplicial objects also make sense without any group structure, so we can do nonlinear homological algebra a.k.a. homotopical algebra.
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Nerves of categories

- $\mathbf{\Delta}$ is the simplicial category: objects are the sets $[n] = \{0, \dots, n\}$, and morphisms are nondecreasing functions.
- A *simplicial set* is a functor $X \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$.
- Functoriality yields maps $d_i \colon X_n \to X_{n-1}$ and $s_j \colon X_n \to X_{n+1}$.
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- For a category $\mathcal{C}$ we define the nerve $N\mathcal{C}$ by

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Nerves of categories

- $\mathbf{\Delta}$ is the simplicial category: objects are the sets $[n] = \{0, \dots, n\}$, and morphisms are nondecreasing functions.
- A *simplicial set* is a functor $X \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$.
- Functoriality yields maps $d_i \colon X_n \to X_{n-1}$ and $s_j \colon X_n \to X_{n+1}$.
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Horns and fillers

- In $N\mathcal{C}$, the $(2,i)$-horns are diagrams as follows:

(2, 0)-horn (outer)

(2, 1)-horn (inner)

(2, 2)-horn (outer)

- We can always fill a $(2,1)$-horn using $w = vu$. If $\mathcal{C}$ is a groupoid we can fill a $(2,0)$-horn with $v = wu^{-1}$, and fill a $(2,2)$-horn with $u = v^{-1}w$.
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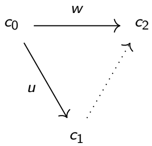

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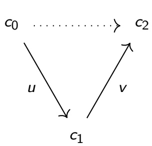

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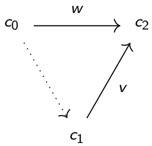

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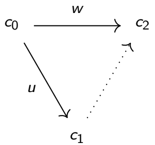

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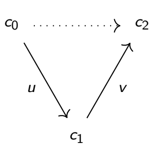

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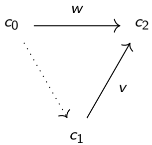

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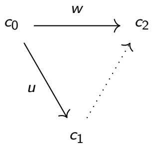

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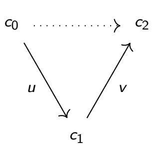

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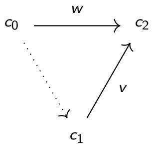

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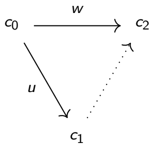

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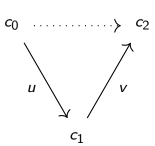

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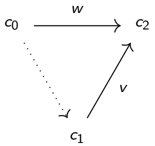

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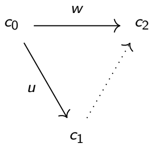

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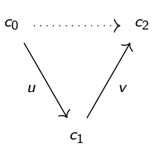

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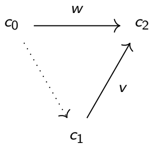

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The singular complex of a space

- Put $\Delta_n = \{(x_0, \ldots, x_n) \in [0,1]^n \mid \sum_i x_i = 1\}$ (the geometric n-simplex).
- These are point, interval, triangle, tetrahedron, ...
- For a space X we put $S_n X = \mathrm{Top}(\Delta_n, X)$.
- For a map $\alpha \colon [n] \to [m]$ in $\mathbf{\Delta}$ we have $\Delta_\alpha \colon \Delta_n \to \Delta_m$ and so $\alpha^* \colon S_m X \to S_n X$ given by $\alpha^*(u) = u \circ \Delta_\alpha$.
  This makes SX into a simplicial set.
- A $(2,i)$-horn in SX is a continuous map from the space $\Lambda_{2i} \subset \Delta_2$ to X.

Λ_{20} Λ_{21} Λ_{22}

- The dashed lines give a retraction $r_{2i} \colon \Delta_2 \to \Lambda_{2i}$ that is the identity on Λ_{2i}.
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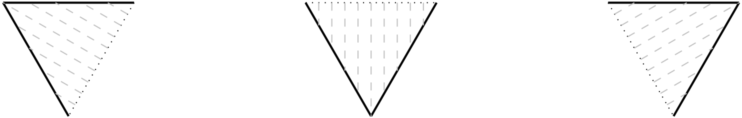

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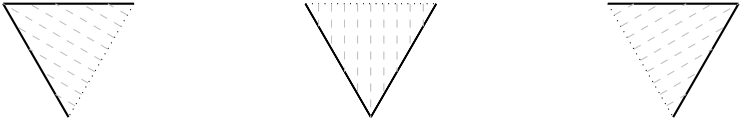

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The singular complex of a space

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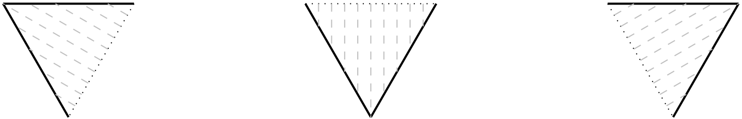

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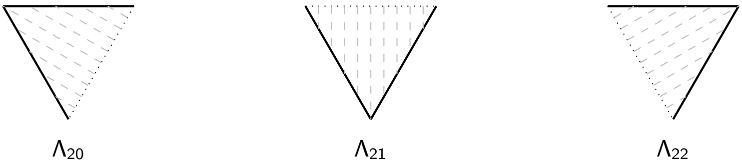

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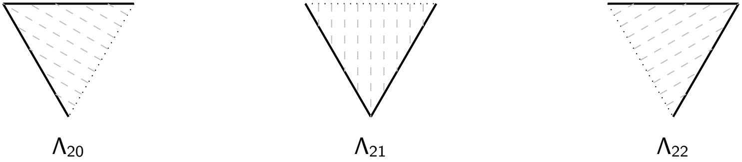

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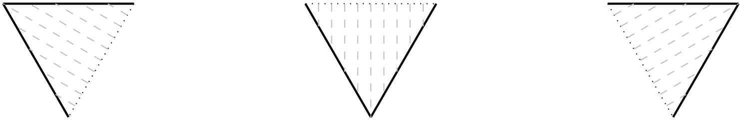

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Coherent nerves

- Let $\mathcal{C}$ be a topological category, so every morphism set $\mathcal{C}(X, Y)$ has a topology and composition is continuous.
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Contractibility

- Say that a simplicial set X is an *acyclic Kan complex* if every $u\colon \partial\Delta_n \to X$ can be extended to give a morphism $\Delta_n \to X$.
- (Here Δ_n is the combinatorial simplex, i.e. the simplicial set with $(\Delta_n)_k = \mathbf{\Delta}([k],[n])$, and $(\partial\Delta_n)_k$ is the subset of non-surjective maps.)
- A standard fact: X is an acyclic Kan complex iff it is an ∞-groupoid and the corresponding space is contractible.
- In an ordinary category $\mathcal{C}$: we say that an object T is terminal iff $\mathcal{C}(X,T)$ is a single point for all X.
- In an ∞-category $\mathcal{C}$: we say that T is terminal iff $\mathcal{C}(X,T)$ is a contractible ∞-groupoid for all X.
- More generally: any definition in category theory involving a unique choice is replaced by a condition involving a contractible space of choices.
- After understanding this principle, we can formulate appropriate definitions of limits, colimits and so on.
- If $\mathcal{C}$ arises as the coherent nerve of a topological category, then these ∞-categorical (co)limits are essentially the same as older notions of homotopy (co)limits.

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- If $\mathcal{C}$ arises as the coherent nerve of a topological category, then these ∞-categorical (co)limits are essentially the same as older notions of homotopy (co)limits.

Contractibility

- Say that a simplicial set X is an *acyclic Kan complex* if every $u\colon \partial\Delta_n \to X$ can be extended to give a morphism $\Delta_n \to X$.
- (Here Δ_n is the combinatorial simplex, i.e. the simplicial set with $(\Delta_n)_k = \mathbf{\Delta}([k],[n])$, and $(\partial\Delta_n)_k$ is the subset of non-surjective maps.)
- A standard fact: X is an acyclic Kan complex iff it is an ∞-groupoid and the corresponding space is contractible.
- In an ordinary category $\mathcal{C}$: we say that an object T is terminal iff $\mathcal{C}(X,T)$ is a single point for all X.
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