# Double subdivision of relative categories

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# Introduction

- Recall: a *quasicategory* is simplicial set with fillers for all inner horns.
- For $n \in \mathbb{N}$ we have a poset $[n] = \{0, \ldots, n\}$. Posets can be regarded as categories, with one morphism from $x$ to $y$ if $x \leq y$, and none otherwise.
- For any category $\mathcal{C}$, we have a simplicial set $N\mathcal{C}$ with $(N\mathcal{C})_n = \mathrm{Cat}([n], \mathcal{C})$.
- Simplicial sets arising this way are precisely those with *unique* fillers for inner horns; so quasicategories are a generalisation of categories.
- For any simplicial set $X$, we have a homotopy category $\mathrm{Ho}(X)$ with $\mathrm{obj}(\mathrm{Ho}(X)) = X_0$, morphisms generated by $X_1$, one relation $d_1(u) = d_0(u) \circ d_2(u)$ for each $u \in X_2$.




- This satisfies $\mathrm{Cat}(\mathrm{Ho}(X), \mathcal{C}) = \mathrm{sSet}(X, N\mathcal{C})$ for all categories $\mathcal{C}$, i.e. $\mathrm{Ho}\colon \mathrm{sSet} \to \mathrm{Cat}$ is left adjoint to $N\colon \mathrm{Cat} \to \mathrm{sSet}$. Also $\mathrm{Ho}(N\mathcal{C}) \simeq \mathcal{C}$.
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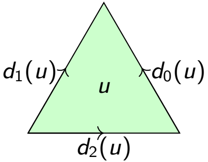


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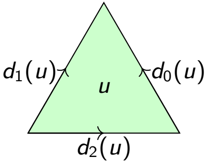


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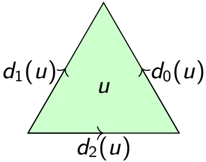


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# The problem

- Problem: construct examples of quasicategories from natural input data.
- One construction is the *coherent nerve* of a simplicial/topological/differential graded category. But that is only appropriate when *all* objects of $\mathcal{C}$ are homotopically well-behaved.
- Often we start with a *relative category*, i.e. a category $\mathcal{C}$ with a class $\text{we} \subseteq \text{mor}(\mathcal{C})$ of *weak equivalences* (containing all identities and closed under composition).
- We want to construct a *relative nerve* $N\mathcal{C}$ which should be a quasicategory with $\text{Ho}(N\mathcal{C}) = \mathcal{C}[\text{we}^{-1}]$.
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## The relative posets $\Xi_n$

- $\Xi_n$ is the set of sets of the form $\theta = \{\sigma_0, \sigma_1, \dots, \sigma_r\}$, where
$$\emptyset \neq \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_r \subseteq [n].$$
- Order this by $\theta \leq \theta'$ iff $\theta \subseteq \theta'$, and so regard $\Xi_n$ as a category.
- Define nondecreasing $\pi \colon \Xi_n \to [n]$ by $\pi(\theta) = \min(\sigma_0) = \min(\bigcap \theta)$.
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- Suppose that $\mathcal{C}$ is discrete, i.e. we $= \{1_c \mid c \in \text{obj}(\mathcal{C})\}$. Then any relative functor $\Xi_n \to \mathcal{C}$ factors uniquely through $\pi \colon \Xi_n \to [m]$, so $N\mathcal{C}$ is just the ordinary nerve.

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$$\emptyset \neq \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_r \subseteq [n].$$
- Order this by $\theta \leq \theta'$ iff $\theta \subseteq \theta'$, and so regard $\Xi_n$ as a category.
- Define nondecreasing $\pi \colon \Xi_n \to [n]$ by $\pi(\theta) = \min(\sigma_0) = \min(\bigcap \theta)$.
- For $\theta \leq \theta'$, declare that $\theta \to \theta'$ is a weak equivalence iff $\pi(\theta) = \pi(\theta')$. This makes $\Xi_n$ a relative category.
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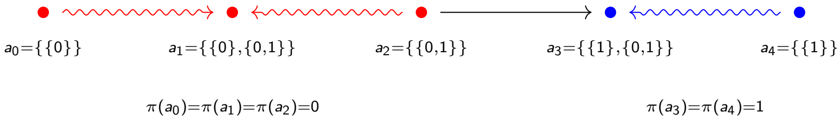


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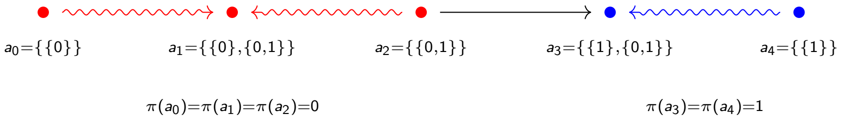


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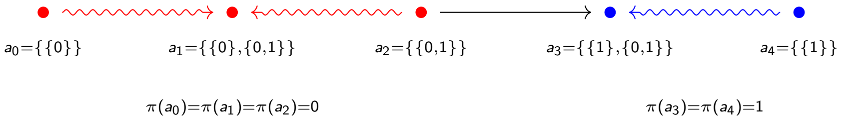


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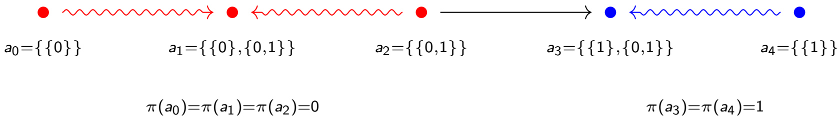


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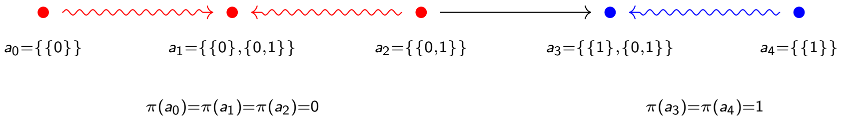


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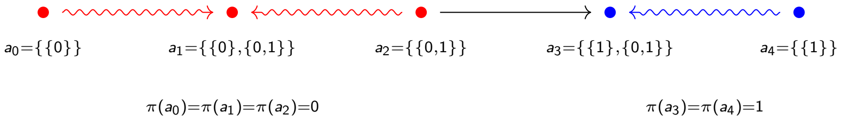


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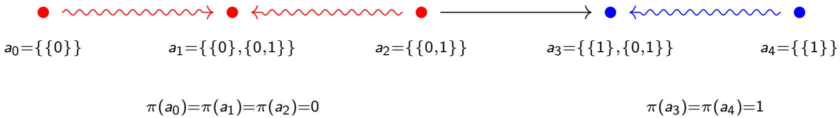


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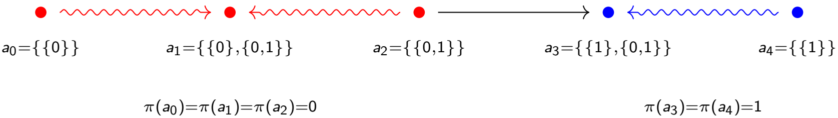


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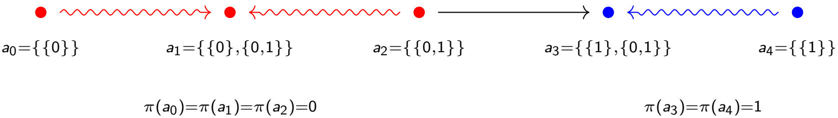


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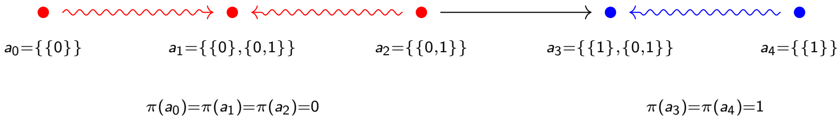


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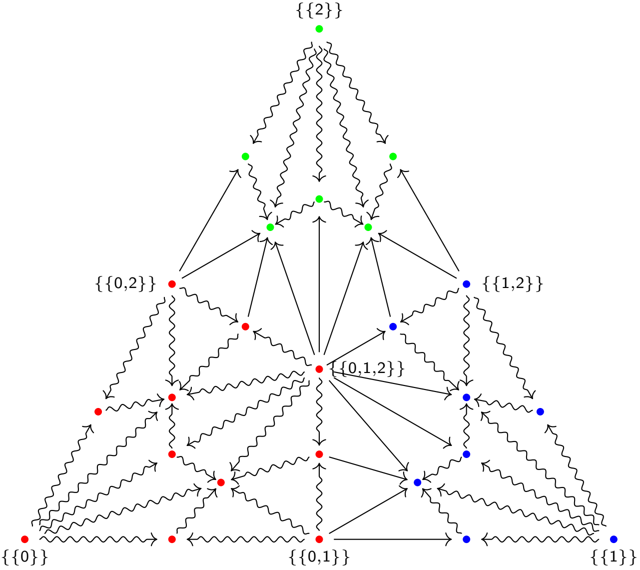

## $\mathrm{Ho}(\Xi_n) \simeq [n]$

- The functor $\pi \colon \Xi_n \to [n]$ induces $\mathrm{Ho}(\Xi_n) \to [n]$. It is easy to guess that this is an equivalence, but not trivial to prove.
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- Any edge $u \in (N\mathcal{C})_1$ gives morphisms $\bullet \xleftarrow{u_0} \bullet \xrightarrow{u_1} \bullet \xleftarrow{u_2} \bullet \xrightarrow{u_3} \bullet$ in $\mathcal{C}$.
- Claim: in $\mathrm{Ho}(N\mathcal{C})$ we have

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- To prove a claim like this about the composite of 4 edges, we need a 4-simplex incorporating those edges.
- Define $g\colon \Xi_4 \to \Xi_1$ as follows.
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## The left adjoint $K\colon \mathsf{sSet} \to \mathsf{RelCat}$

- **Theorem:** There is a functor $K\colon \mathsf{sSet} \to \mathsf{RelCat}$, left adjoint to $N\colon \mathsf{RelCat} \to \mathsf{sSet}$, with $K(X)[\mathrm{we}^{-1}] \simeq \mathrm{Ho}(X)$.
  Moreover, $K(X)$ is actually a poset.
- Morally, $K(X)$ is defined as a certain colimit of $\Xi_n$'s;
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- In this case the final answer is not too bad, although it takes substantial work to prove that.
- Put $\Xi_n^\top = \{\theta \in \Xi_n \mid [n] \in \theta\}$ (the *interior* of $\Xi_n$).
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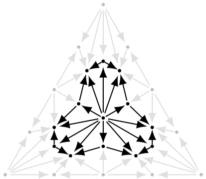

# Extension properties

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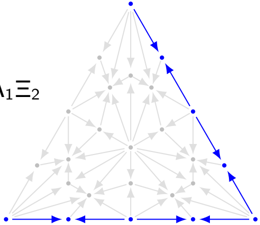

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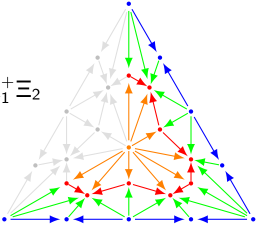

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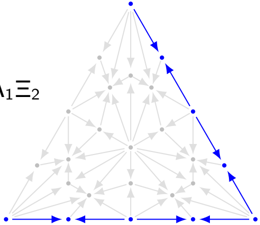

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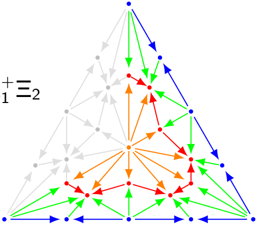

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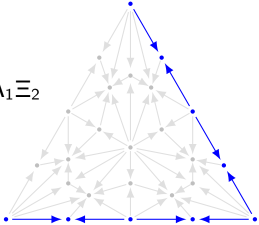

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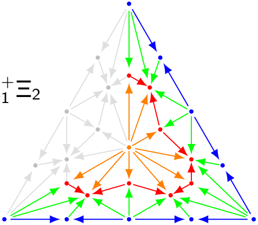

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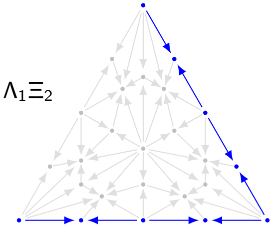


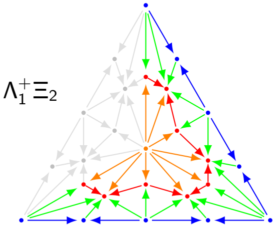


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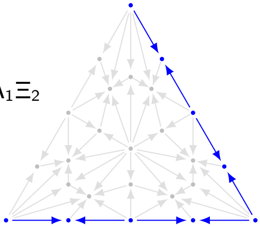

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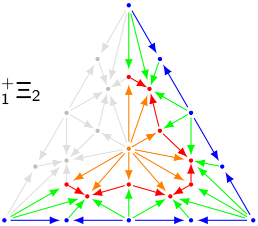

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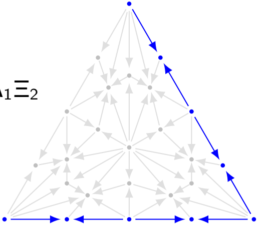

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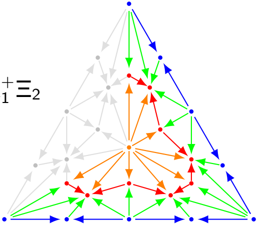

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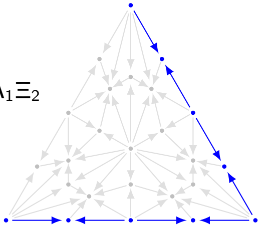

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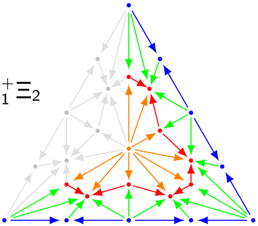

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- Compare this construction with the hammock localisation.
- Compare this construction with the Kan path groupoid.
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# Further topics

- Compare this construction with the coherent nerve.
- Compare this construction with the hammock localisation.
- Compare this construction with the Kan path groupoid.
- Investigate derived functors from this point of view.
- Investigate homotopy (co)limits from this point of view.