Topological Hochschild homology of the dual circle

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October 17, 2024

Introduction

- This talk is about the object $\mathrm{THH}(S^{h\mathbb{Z}}) = \mathrm{THH}(S^{B\mathbb{Z}}) = \mathrm{THH}(D_+(S^1)) = D_+(S^1)^{\otimes \mathbb{T}}$.
- Why should we care about this?
- Hopkins-Devinatz: $L_{K(n)}X = (E \otimes X)^{h\Gamma} = ((E \otimes X)^{h\Gamma_1})^{h\mathbb{Z}_p^\times}$. Here $\mathbb{Z}_p^\times$ is the Galois group of a higher cyclotomic extension of $L_{K(n)}S$.
- There is a parallel higher cyclotomic extension of $L_{T(n)}S$.
- So we are interested in interaction between $(\cdot)^{h\mathbb{Z}_p^\times}$ with other functors.
- $\mathbb{Z}_p^\times$ has a dense subgroup iso to $\mathbb{Z}$ so $(\cdot)^{h\mathbb{Z}}$ is the same in relevant context.
- The key counterexample is of type $K(B^{h\mathbb{Z}})$ so we care about $K(\cdot)$ vs $(\cdot)^{h\mathbb{Z}}$.
- $K(R)$ is well-approximated by $\mathrm{TC}(R)$, which is a functor of $\mathrm{THH}(R)$, so we care about $\mathrm{THH}(\cdot)$ vs $(\cdot)^{h\mathbb{Z}}$.
- It is easy to see that the map $\mathrm{THH}(R^{h\mathbb{Z}}) \to \mathrm{THH}(R)^{h\mathbb{Z}}$ has a large fibre; but does this survive the functor that converts THH to TC?
- Yes, because the fibre generates a large thick subcategory of the category of cyclotomic spectra.
- The initial ring with $\mathbb{Z}$-action is S with trivial action, for which $S^{h\mathbb{Z}} = S^{B\mathbb{Z}} = S^{S^1} = D_+(S^1)$.
- So we care about $\mathrm{THH}(D_+(S^1))$ vs $\mathrm{THH}(S)^{h\mathbb{Z}} = S^{h\mathbb{Z}} = D_+(S^1)$.

Basics about $A = S^{B\mathbb{Z}} = D_+(S^1)$

- In Ho(Sp), we just have $A = S^0 \oplus S^{-1}$, and this is just a square-zero extension of S^0.
- So $\pi_*(A) = \pi_*(S) \otimes E[\epsilon]$ with $|\epsilon| = -1$, $H_*(A) = E[\epsilon]$.
- The monoid $\mathrm{End}(S^1) \ltimes S^1 \simeq \mathbb{Z} \ltimes S^1 \simeq \mathrm{Map}(S^1, S^1)$ acts contravariantly on A with $(m, b)^*(\epsilon) = m\epsilon$.
- As an orthogonal ring spectrum: $A(V) = F(S^1_+, S^V)$.
  The evident pairings $F(S^1_+, S^V) \wedge F(S^1_+, S^W) \to F(S^1_+, S^{V \oplus W})$ make this a strictly commutative ring.
- Given a decomposition $V = \bigoplus_{i \in I} V_i$ and maps $u_i \colon S^1_+ \to S^{V_i}$ we get an element of $A^{\otimes I}(V)$. This is close to a complete description of $A^{\otimes I}(V)$.
- Naively $KU \otimes S^1 \simeq KU \otimes S^{-1}$ so $KU[S^1] \simeq KU \otimes A$.
  In fact there is an equivalence $KU[S^1] \to KU \otimes A$ in $\mathrm{CAlg}(KU)$.
  (Thanks to Maxime Ramzi via Mathoverflow)

The cyclic bar construction

- For a topological monoid M, $B_{\mathrm{cyc}}(M)$ is the space of configurations of finitely many points on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ labelled with elements of M.
- Labels multiply as points collide; points labelled 1 can be freely added or removed.
- There is an evident action of $\mathbb{T}$ by rotation.
- If M is commutative: $B_{\mathrm{cyc}}(M)$ is a commutative monoid with homomorphism $\epsilon\colon B_{\mathrm{cyc}}(M) \to M$ by taking product of labels.
- There is a model for BM as a quotient where labels at 0 are ignored.
- Combining this with the rotation action gives a map $B_{\mathrm{cyc}}(M) \to LB(M) = \mathrm{Map}(\mathbb{T}, BM)$, which is an equivalence when M is a topological group.
- There is a nonequivariant filtration by the number of labelled points, not counting the point $0 \in \mathbb{T}$. The kth filtration quotient is $M_+ \wedge \Sigma^k M^{(k)}$.
- We can make a category $\mathrm{Trans}(\mathbb{R}_+, \mathbb{T})$ with objects $\mathbb{T}$ and morphisms $\sigma_{a,t}\colon a \to a + t$ for $t \in \mathbb{R}_+$.
- For a topological category C, the cyclic bar construction $B_{\mathrm{cyc}}(C)$ is the space of functors $\mathrm{Trans}(\mathbb{R}_+, \mathbb{T}) \to C$ such that nothing happens except at finitely many points of $\mathbb{T}$.

The cyclic bar construction for ring spectra

- In any reasonable model of ring spectra we can define $\mathrm{THH}(R) = B_{\mathrm{cyc}}(R)$ as the geometric realisation of a simplicial object with k-simplices $R^{\otimes(k+1)}$.
- For orthogonal spectra: given points $a_i \in \mathbb{T}$ and a decomposition $V = \bigoplus_i V_i$ and elements $u_i \in R(V_i)$ we get an element of $\mathrm{THH}(R)(V)$. Again labels multiply as points collide, and identity labels can appear or disappear.
- Again $\mathbb{T}$ acts by rotation.
- There is again a filtration with kth quotient $R \otimes \Sigma^k(R/S)^{\otimes k}$, where R/S is the cofibre of the unit $S \to R$.
- For $t \in \mathbb{T}$, define $i_t \colon R \to \mathrm{THH}(R)$ by introducing a label at the point t.
- If R is commutative: $\mathrm{THH}(R)$ is also a commutative ring, and $i_t \colon R \to \mathrm{THH}(R)$ is a ring map.
- In this case $\mathrm{THH}(R) = R^{\otimes \mathbb{T}}$, i.e. for any continuous family of commutative ring maps $\alpha_t \colon R \to U$ there is a unique $\alpha \colon \mathrm{THH}(R) \to U$ with $\alpha i_t = \alpha_t$.
- In particular there is a unique $\epsilon \colon \mathrm{THH}(R) \to R$ with $\epsilon i_t = 1 \colon R \to R$ for all t.

The case of the dual circle

- Now take $A = D_+(S^1)$ and $R = \mathrm{THH}(A) = A^{\otimes \mathbb{T}}$.
- This has a covariant action of the monoid of self-maps of $\mathbb{T}$, which is equivalent to $\mathrm{End}(\mathbb{T}) \ltimes \mathbb{T} = \mathbb{Z} \ltimes \mathbb{T}$.
- It also has a contravariant action of the monoid of self-maps of S^1, which is equivalent to $\mathrm{End}(S^1) \ltimes S^1 = \mathbb{Z} \ltimes S^1$.
- For $q \in \mathrm{Hom}(\mathbb{T}, S^1) \simeq \mathbb{Z}$ we also have a twist map making the following commute:

$$\begin{array}{ccc} D_+(S^1) & \xrightarrow{\tau^*_{q(t)}} & D_+(S^1) \\ {\scriptstyle i_t}\downarrow & & \downarrow{\scriptstyle i_t} \\ R & \xrightarrow[\mathrm{twist}_q]{} & R. \end{array}$$

- Together this gives an action of a monoid Γ which is $\mathrm{End}(\mathbb{T}) \times \mathrm{End}(S^1) \times \mathrm{Hom}(\mathbb{T}, S^1) \times \mathbb{T} \times S^1 \simeq \mathbb{Z}^3 \times (S^1)^2$ as a space.
- The same monoid Γ also acts on the space and ring

$$\mathrm{Map}(\mathbb{T}, S^1) \simeq \mathrm{Hom}(\mathbb{T}, S^1) \times S^1 = S^1 \amalg \coprod_{q \neq 0} \mathbb{T}/\ker(q)$$

$$\widehat{R} = D_+(\mathrm{Hom}(\mathbb{T}, S^1) \times S^1) = D_+(S^1) \times \prod_{q \neq 0} D_+(\mathbb{T}/\ker(q)).$$

- There is a unique ring map $\phi \colon R \to \widehat{R}$ with $\pi_q \circ \phi = \epsilon \circ \mathrm{twist}_q$.

The filtration

- For $A = D_+(S^1) = S^0 \oplus S^{-1}$ we have $\overline{A} = \mathrm{cof}(S \to A) = S^{-1}$
- The kth filtration quotient of $R = \mathrm{THH}(A)$ is $A \otimes \Sigma^k \overline{A}^{\otimes k} = A$.
- It follows that $F_k R \simeq \bigoplus_{i \leq k} A$ and $R \simeq \bigoplus_{\mathbb{N}} A$ as A-modules.
- By comparison with $\mathrm{THH}^H(H \otimes A)$ we find that $H_* R = E[\epsilon] \otimes \mathrm{Tor}^{E[\epsilon]}(\mathbb{Z}, \mathbb{Z}) = E[\epsilon] \otimes DP[x] = E[\epsilon]\{x^{[k]} \mid k \geq 0\}$ with $x^{[j]} x^{[k]} = \binom{j+k}{j} x^{[j+k]}$.
- The ring map $\phi \colon R \to \widehat{R}$ gives a $\mathrm{Hom}(\mathbb{T}, S^1)$-equivariant ring map $\phi_* \colon H_*(R) \to \mathrm{Map}(\mathrm{Hom}(\mathbb{T}, S^1), E[\epsilon]) \simeq \mathrm{Map}(\mathbb{Z}, E[\epsilon])$
- One can write down a homotopy to show that twist_q acts as the identity on filtration quotients.
- (Similar to: the Dehn twist of the torus becomes homotopic to the identity after pinching one circle to a point.)
- Put $b_k(x) = \binom{x}{k}$ and $NP = \{\text{numerical polys}\} = \{f \in \mathbb{Q}[x] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\} = \mathbb{Z}\{b_k \mid k \geq 0\}$.
- Thus $\pi_*(R) = \pi_*(S) \otimes E[\epsilon] \otimes NP \subseteq \mathrm{Map}(\mathbb{Z}, \pi_*(S) \otimes E[\epsilon])$.
- An element $(n, m, q) \in \mathrm{End}(\mathbb{T}) \times \mathrm{End}(S^1) \times \mathrm{Hom}(\mathbb{T}, S^1) = \pi_0(\Gamma)$ acts like $x \mapsto nmx + q$ and $\epsilon \mapsto m\epsilon$.

Interaction with KU

- As mentioned previously: $KU[S^1] \simeq KU \otimes A$
- This gives

$$\begin{aligned} KU \otimes R &= \mathrm{THH}^{KU}(KU \otimes A) = \mathrm{THH}^{KU}(KU[S^1]) \\ &= KU \otimes \mathrm{THH}(S[S^1]) = KU \otimes S[BS^1] = KU[BS^1]. \end{aligned}$$

- It is classical that $\pi_0(KU[BS^1]) = KU_0(BS^1) = KU_0(\mathbb{C}P^\infty) = NP$

Cyclotomic invariants of $\widehat{R}$

In the formulae below we have $q \in \mathrm{Hom}(\mathbb{T}, S^1)$ and $C = C_p < \mathbb{T}$.
We write ζ for any map induced by $p.1_{\mathbb{T}} \colon \mathbb{T} \to \mathbb{T}/C \to \mathbb{T}$.

$$\widehat{R} = D_+(S^1) \times \prod_{q \neq 0} D_+(\mathbb{T}/\ker(q))$$

$$\widehat{R}^{h\mathbb{T}} = D_+(B\mathbb{T} \times S^1) \times \prod_{q \neq 0} D_+(B(\ker(q)))$$

$$\widehat{R}^{hC} = \prod_{p|q} D_+(BC \times S^1) \times \prod_{p \nmid q} D_+(\mathbb{T}/\ker(pq))$$

$$\widehat{R}^{tC} = D_+(S^1)^\wedge_p \times \prod_{p|q \neq 0} D_+(\mathbb{T}/\ker(q))^\wedge_p \simeq \widehat{R}^\wedge_p \text{ via } \zeta$$

$$(\widehat{R}^{tC})^{h\mathbb{T}/C} = D_+(B(\mathbb{T}/C) \times S^1)^\wedge_p \times \prod_{p|q \neq 0} D_+(B(\ker(q)/C))^\wedge_p \simeq (\widehat{R}^\wedge_p)^{h\mathbb{T}} \text{ via } \zeta$$

- The Tate diagonal $\varphi \colon X \to X^\wedge_p \xrightarrow{\simeq} (X^p)^{tC}$ for a space X is induced by the diagonal $\delta \colon X \to X^p$.
- By monoidality, $D_+(X) \xrightarrow{\varphi} D_+(X^p)^{tC} \xrightarrow{\delta^*} D_+(X)^\wedge_p$ is the identity.
- Applying this fibrewise for $p.1_{\mathbb{T}}$ we get $\zeta\varphi = 1 \colon \widehat{R} \to (\widehat{R})^\wedge_p$
- Thus TC is eq $\left(1, \zeta \colon \widehat{R}^{h\mathbb{T}} \to \widehat{R}^{h\mathbb{T}}\right)$.

p-adic completion

- For a profinite space X, Lurie's spherical Witt ring construction gives $WC(X) = WC(X, \mathbb{Z}_p) \in \mathrm{CAlg}(\mathrm{Sp}_p)$ with $\pi_k(WC(X, \mathbb{Z}_p)) = C(X, \pi_k(S)_p^\wedge)$.
- $\mathrm{CAlg}(WC(X, \mathbb{Z}_p), T)$ is the discrete space of ring maps from $C(X, \mathbb{F}_p)$ to $\pi_0(T)/p$ or to $\pi_0^\flat(T) = \{a \in (\pi_0(T)/p)^{\mathbb{N}} \mid a_k = a_{k+1}^p\}$.
- $NC_p^\wedge = C(\mathrm{Hom}(\mathbb{T}, S^1)_p^\wedge) \simeq C(\mathbb{Z}_p)$.
- Using this: in the p-complete category we have $R = WC(\mathrm{Hom}(\mathbb{T}, S^1)_p^\wedge) \otimes A \simeq C(\mathbb{Z}_p) \otimes A$.
- Given an open and closed subspace $U \subseteq \mathbb{Z}_p$ we have an idempotent $\chi_U \in C(\mathbb{Z}_p)$ and thus a commutative ring $R|_U = R[\chi_U^{-1}]$.
- For any $m \geq 0$ this gives $R \simeq R|_{p^m\mathbb{Z}_p} \times \prod_{k<m} R|_{p^k\mathbb{Z}_p^\times} \sim A \times \prod_k R|_{p^k\mathbb{Z}_p^\times}$
- In the category of abelian groups: the colimit of the rings $C(p^m\mathbb{Z}_p)$ is $\mathbb{Z}_p \oplus$ (a rational vector space).
- In the p-complete category the colimit is $\mathbb{Z}_p$, so the colimit of $R|_{p^m\mathbb{Z}_p}$ is A.
- The element $p^m.1_{\mathbb{T}} \in \mathrm{End}(\mathbb{T})$ induces $f \mapsto (x \mapsto f(p^m x))$ on $C(\mathbb{Z}_p)$.
- This kills $1 - \chi_{p^m\mathbb{Z}_p}$ and inverts $\chi_{p^m\mathbb{Z}_p}$, so $R|_{p^m\mathbb{Z}_p} = R[\chi_{p^m\mathbb{Z}_p}^{-1}] \xrightarrow{\simeq} R$.

Cyclotomic invariants of R

On this slide everything is implicitly completed at p.
We write ζ for any map induced by $p.1_{\mathbb{T}}\colon \mathbb{T} \to \mathbb{T}/C \to \mathbb{T}$.

$$R \sim D_+(S^1) \times \prod_{m\geq 0} WC(p^m\mathbb{Z}_p^\times) \otimes D_+(\mathbb{T}/C_{p^m})$$

$$R^{h\mathbb{T}} \sim D_+(B\mathbb{T} \times S^1) \times \prod_{m>0} WC(p^m\mathbb{Z}_p^\times) \otimes D_+(BC_{p^m})$$

$$R^{hC} \sim WC(p\mathbb{Z}_p) \otimes D_+(BC \times S^1) \times WC(\mathbb{Z}_p^\times) \otimes D_+(\mathbb{T}/C_p)$$

$$R^{tC} \sim D_+(S^1) \times \prod_{m>0} WC(p^m\mathbb{Z}_p^\times) \otimes D_+(\mathbb{T}/C_{p^m}) \simeq R \text{ via } \zeta$$

$$(R^{tC})^{h\mathbb{T}/C} \sim D_+(B(\mathbb{T}/C) \times S^1) \times \prod_{m>0} WC(p^m\mathbb{Z}_p^\times) \otimes D_+(BC_{p^m}) \simeq (R_p^\wedge)^{h\mathbb{T}} \text{ via } \zeta$$

- Again φ is inverse to ζ, so $\mathrm{TC}(A) = (R^{h\mathbb{T}})^{\langle\zeta\rangle}$.
- Equivariant Atiyah duality gives $D_+(\mathbb{T}/C_{p^m}) = \Sigma_+^{-1}(\mathbb{T}/C_{p^m})$.
- Put $L = \bigoplus_{m\geq 0} \mathbb{T}/C_{p^m} \in \mathrm{CycSp}$ so $\mathrm{fib}(R \xrightarrow{\epsilon} A) = \Sigma^{-1}C(\mathbb{Z}_p^\times) \otimes L$.

The cyclotomic spectrum L

- Recall $L = \bigoplus_{m \geq 0} S[\mathbb{T}/C_{p^m}]$ (everything implicitly p-complete)
- So $L^{tC_p} \simeq L$
- Also $\mathrm{Sp}^{B\mathbb{T}}(L, X) = \prod_m X^{hC_{p^m}}$
- So the pullback square

$$\begin{array}{ccc} \mathrm{CycSp}(L, X) & \longrightarrow & \mathrm{Sp}^{B\mathbb{T}}(L, X) \\ \downarrow & & \downarrow \\ \mathrm{Sp}^{B\mathbb{T}}(L^{tC_p}, X^{tC_p}) & \longrightarrow & \mathrm{Sp}^{B\mathbb{T}}(L, X^{tC_p}) \end{array}$$

  involves only $\prod_m X^{hC_{p^m}}$ and $\prod_m (X^{tC_p})^{hC_{p^m}}$
- After identifying the maps we find $\mathrm{CycSp}(L, X) = \mathrm{TR}(X)$.
- We also have $\mathrm{CycSp}(S, X) = \mathrm{TC}(X) = \mathrm{fib}(1 - F \colon \mathrm{TR}(X) \to \mathrm{TR}(X))$ so the cofibre of the endomorphism of L corresponding to $1 - F$ is S.
- So S lies in the thick subcategory of CycSp generated by $\mathrm{fib}(\mathrm{THH}(A) \to A)$.
- It is also known that if $X \in \mathrm{CycSp}_{[a,b]}$ then $L \otimes X \in \mathrm{CycSp}_{[a,b+4]}$.