SCHOOL OF MATHEMATICAL AND PHYSICAL SCIENCES **Autumn Semester 2024–25**

MAS334 Combinatorics **2 hours 30 minutes**

Attempt all the questions. Give justification for all numerical answers. The allocation of marks is shown in brackets. There are 100 marks in total.

This copy also contains solutions.

No auxiliary material is provided.

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1 Put $U = \{1,2,3,4,5,6,7,8,9\}$, and consider subsets $A \subseteq U$.

(a) How many subsets are there in total? ***(2 marks)***

(b) How many subsets A are there such that $|A| < |A^c|$? ***(2 marks)***

(c) How many subsets A are there such that A contains at least one even number and at least one odd number? ***(2 marks)***

(d) How many nonempty subsets A are there such that $\max(A) = \min(A) + 2$? ***(2 marks)***

Solution. **Part (a) is standard, the rest is similar to questions that have been seen.**

(a) The total number of subsets is $2^9 = 512$. [2]

(b) Put $Q = \{A \mid |A| < |A^c|\}$ and $R = \{A \mid |A| > |A^c|\}$. As $|U|$ is odd we have $PU = Q \amalg R$. The map $A \mapsto A^c$ gives a bijection $Q \to R$ so $|Q| = |PU|/2 = 256$. Alternatively, we have $|A| < |A^c|$ iff $|A| \leq 4$ so $|Q| = \sum_{k=0}^{4} \binom{9}{k} = 1 + 9 + 36 + 84 + 126 = 256$. [2]

(c) To choose a set A of the indicated type, we choose a nonempty subset $B \subseteq \{2,4,6,8\}$ and a nonempty subset $C \subseteq \{1,3,5,7,9\}$ and put $A = B \amalg C$. There are $2^4 - 1 = 15$ choices for B and $2^5 - 1 = 31$ choices for C giving $15 \times 31 = 465$ choices for A. [2]

(d) To form a subset A with with $\max(A) - \min(A) = 2$, we first choose what $\min(A)$ should be. If $\min(A) = i$ then $\max(A) = i + 2$, and both i and $i + 2$ must lie in $\{1, \ldots, 9\}$, so $i \in \{1, \ldots, 7\}$. There are thus 7 choices for i. Having chosen i, the only question is whether A should be $\{i, i+2\}$ or $\{i, i+1, i+2\}$, which gives another factor of two. Thus, there are 14 possible choices for A. [2]

2 Show that for every natural number n we have

$$\sum_{k=1}^{n} k\binom{n}{k} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n\,2^{n-1}.$$

Hint: You may wish to consider the set P of pairs (A, a), where A is a subset of the set $N = \{1, \ldots, n\}$ and $a \in A$. ***(5 marks)***

Solution. **Similar questions have been seen.** To choose an element $(A, a) \in P$, we can first choose the size of A (which will be a number k with $1 \leq k \leq n$), then choose the set A itself (for which there are $\binom{n}{k}$ choices), then choose the element $a \in A$ (for which there are k choices). This gives $|P| = \sum_{k=1}^{n} k\binom{n}{k}$. Alternatively, we can first choose an element $a \in N$ (for which there are n choices), then choose a subset $B \subseteq N \setminus \{a\}$ (for which there are 2^{n-1} choices) then put $A = B \amalg \{a\}$. This shows that $|P| = n\,2^{n-1}$. Equating these two expressions for $|P|$ gives the required result. [5]

3 Find the number of integer solutions for each of the following questions:

(a) $x_0 + \cdots + x_9 = 111$ with $x_0, \ldots, x_9 \geq 1$. ***(2 marks)***

(b) $x_0 + \cdots + x_9 = 111$ with $x_0, \ldots, x_9 \geq 10$. ***(2 marks)***

(c) $x_0 + \cdots + x_9 = 111$ with $x_0, \ldots, x_9 \geq 10$ and $\max(x_0, \ldots, x_9) = 20$. ***(3 marks)***

(d) $x_0 \times \cdots \times x_9 = 111$ with $x_0, \ldots, x_9 \geq 0$. ***(3 marks)***

You can leave your answers as binomial coefficients or similar expressions, you do not need to calculate the actual numbers. For (d), you may wish to find the prime factorisation of the relevant numbers.

Solution.

(a) **Standard.** The number of variables is $k = 10$, the right hand side of the equation is $n = 111$ and the variables are positive integers. By the standard method explained in the notes, the number of solutions is $\binom{n-1}{k-1} = \binom{110}{9} = 4643330358810$. [2]

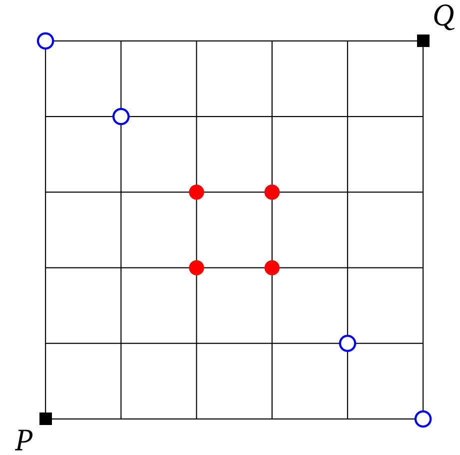
(b) **Standard.** We can rewrite this in terms of the variables $y_i = x_i - 10$. These satisfy $y_i \geq 0$ with $y_0 + \cdots + y_9 = 11$. The number of variables is $k = 10$, the right hand side is $m = 11$, so the number of solutions is $\binom{m+k-1}{k-1} = \binom{20}{9} = 167960$. [2]

(c) **Similar problems seen.** In terms of the variables y_i we have $y_0 + \ldots + y_9 = 11$ with $\max(y_0, \ldots, y_9) = 10$. Clearly one variable y_i must be equal to 10, then a different variable y_j must be equal to 1, and all the other variables must be zero. There are 10 choices for i and then 9 choices for j, giving 90 solutions altogether. [3]

(d) **Essentially the same as in two recent past papers.**

The prime factorisation of 111 is 3×37. A factor of 3 must appear in one of the variables x_i, and a factor of 37 must appear in one of the variables x_j (where j might be equal to i). There are 10 choices for i and 10 choices for j giving 100 solutions altogether. [3]

4 This question concerns routes in the grid illustrated:



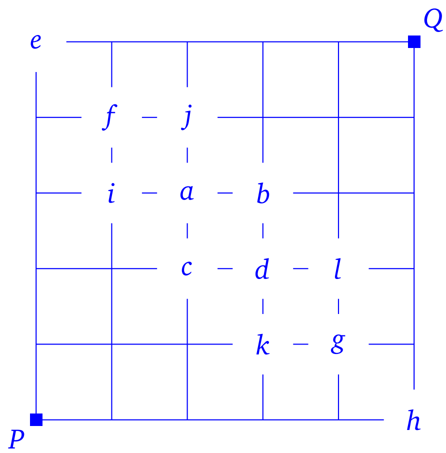
The points P and Q are marked with squares, four other points are marked with empty circles, and another four points are marked with filled discs.

(a) How many routes are there from P to Q along the lines of the grid (always moving up or to the right, as usual)? Give a brief reason for your answer.
(3 marks)

(b) How many of these routes do not pass through any of the filled discs? (To answer this, you may also want to think about the unfilled circles.)
(5 marks)

(c) How many of the routes in (a) pass through precisely one of the filled discs?
(5 marks)

Solution. We will refer to various points by the following letters:



(a) **Bookwork.** The number of possible routes from P to Q is $\binom{10}{5} = 252$. [1]Indeed, we need to take 5 horizontal steps and 5 vertical steps, and the route is specified by choosing which 5 of the 10 steps are to be horizontal. [2]

(b) **Unseen, partially similar to recent exams.** Any route from P to Q must cross the diagonal line from e to h in precisely one place [2]. The route avoids the filled discs

4 (continued)

at a, b, c and d if and only if it crosses the diagonal at e, f, g or h. There is only one route from P to e and only one route from e to Q, so there is one route from P to Q via e. Similarly, there is one route from P to Q via h. There are 5 routes from P to f and 5 from f to Q giving 25 routes via f. There are also 25 routes via g, giving 52 routes from P to Q avoiding the four filled discs. [3]

(c) **Unseen.** The only way that a route can involve precisely one filled disc is if it goes via (i, a, j) or via (k, d, l). There are 4 routes from P to i, or from P to k, or from j to Q, or from l to Q. This gives $4 \times 4 + 4 \times 4 = 32$ routes from P to Q of the required type. [5]

5 (a) Let A and B be finite sets with $|A| < |B|$. One of the following statements is equivalent to the Pigeonhole Principle (and so is true). The other statements are false. Identify which statement is true, and give counterexamples for all the other statements. ***(6 marks)***

(A) Every map $f : A \to B$ is injective

(B) Every map $f : A \to B$ is surjective

(C) Every map $g : B \to A$ is injective

(D) Every map $g : B \to A$ is non-injective

(E) Every map $g : B \to A$ is non-surjective

(b) Let $A \subseteq \mathbb{N}$ be a set of natural numbers, such that $|A| = 10$ and $a \geq 1000$ for all $a \in A$. For every subset $X \subseteq A$ let $\sum X$ denote the sum of all the numbers in X. Show that there are two subsets $B, C \subseteq A$ with $B \neq C$ and $B \cap C = \varnothing$ such that $\sum B = \sum C \pmod{1000}$. ***(5 marks)***

Solution.

(a) Statement (D) is a version of the Pigeonhole Principle [2](**Bookwork**). For the counterexamples, we take $A = \{0, 1\}$ and $B = \{0, 1, 2\}$, and we define maps $A \xrightarrow{f} B \xrightarrow{g} A$ by $f(0) = f(1) = 0$ and $g(0) = 0$ and $g(1) = g(2) = 1$. Then f is neither injective nor surjective, so it gives a counterexample to both (A) and (B). The map g is not injective, and so gives a counterexample to (C). The map g is surjective, so it is not non-surjective, so it is also a counterexample to (D) [4]. (**Unseen**)

(b) **A slight adjustment of an example in the notes.** Let P be the set of nonempty subsets of A, so $|P| = 2^{10} - 1 = 1023$ [1]. Put $N = \{0, \dots, 999\}$, so $|N| = 1000$ [1]. Define $f : P \to N$ by taking $f(A)$ to be the residue of $\sum A$ modulo 1000, or equivalently, the last three digits in $\sum A$ [1]. As $|P| > |N|$ there must exist $B_0, C_0 \in P$ with $f(B_0) = f(C_0)$ [1]. Put $D = B_0 \cap C_0$ and $B = B_0 \setminus D$ and $C = C_0 \setminus D$. We find that $B \neq C$ and $B \cap C = \varnothing$. We also have $f(B_0) = f(B) + f(D) \pmod{1000}$ and $f(C_0) = f(C) + f(D) \pmod{1000}$ so from $f(C_0) = f(D_0)$ we can deduce $f(C) = f(D)$ [1].

6 (a) State the Inclusion/Exclusion Principle. ***(3 marks)***

(b) How many of the numbers in the set $N = \{0, 1, \ldots, 23099\}$ are coprime with 231? ***(6 marks)***

Solution.

(a) Consider a finite set B, with a list of subsets $B_1, \ldots, B_n \subseteq B$. For $I \subseteq \{1, \ldots, n\}$ put $B_I = \bigcap_{i \in I} B_i$, with the convention that $B_\varnothing = B$. The IEP says that

$$|B_1 \cup \cdots \cup B_n| = \sum_{I \neq \varnothing} (-1)^{|I|-1} |B_I|, [3]$$

or equivalently

$$|B \setminus (B_1 \cup \cdots \cup B_n)| = \sum_I (-1)^{|I|} |B_I|.$$

Bookwork. Full marks will be given for either of the equivalent forms. Versions with ellipses instead of summation notation will be accepted if they are sufficiently clear.

(b) Put $n = 231 = 3 \times 7 \times 11$ and $N = 100n$ and $X = \{0, 1, \ldots, N-1\}$. For $p \in \{3, 7, 11\}$ put $X_p = \{x \in X \mid x = 0 \pmod{p}\}$ [1]. We need to find the size of the set $X^* = X \setminus (X_3 \cup X_7 \cup X_{11})$. As p divides N we just have $|X_p| = N/p$ [1]. Similarly, if p and q are two distinct primes taken from $\{3, 7, 11\}$ then $X_p \cap X_q$ is the set of multiples of pq in X and we have $|X_p \cap X_q| = N/(pq)$ [1]. In the same way, we have $|X_3 \cap X_7 \cap X_{11}| = N/(3 \times 7 \times 11) = 100$. The IEP now gives

$$\begin{aligned}
|X^*| &= |X| - |X_3| - |X_7| - |X_{11}| + |X_3 \cap X_7| + |X_3 \cap X_{11}| + |X_7 \cap X_{11}| - |X_3 \cap X_7 \cap X_{11}| [1] \\
&= N\left(1 - 3^{-1} - 7^{-1} - 11^{-1} + (3 \times 7)^{-1} + (3 \times 11)^{-1} + (7 \times 11)^{-1} - (3 \times 7 \times 11)^{-1}\right) \\
&= N\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{7}\right)\left(1 - \frac{1}{11}\right) = N \times \frac{2 \times 6 \times 10}{3 \times 7 \times 11} \\
&= 23100 \times \frac{120}{231} = 12000. [2]
\end{aligned}$$

Standard application of IEP.

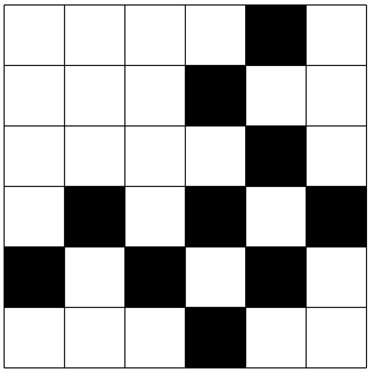
7 (a) Let B be part of an $n \times n$ board, and let C and D be subsets of B.

(i) Explain what it means to say that B is the fully disjoint union of C and D. ***(3 marks)***

(ii) If B is the fully disjoint union of C and D, what is the relationship between the corresponding rook polynomials? ***(1 mark)***

(b) Give the formula for $c_n(B)$ in terms of the rook coefficients for the complementary board $\overline{B}$. ***(2 marks)***

(c) Consider the following board B:



(i) Draw the complementary board $\overline{B}$, and divide it into two fully disjoint subsets. ***(3 marks)***

(ii) Calculate the rook polynomial of $\overline{B}$. ***(3 marks)***

(iii) Calculate $c_6(B)$. ***(2 marks)***

Solution.

(a) (i) **Bookwork.** We say that B is the fully disjoint union of C and D if

- $B = C \cup D$
- No row meets both C and D
- No column meets both C and D. [3]

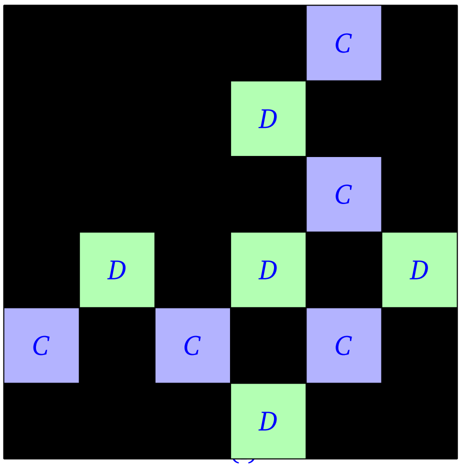
(ii) If so, then $r_B(x) = r_C(x) r_D(x)$. [1]

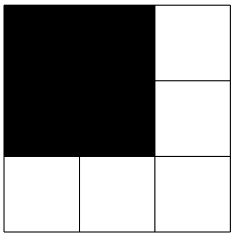
(b) **Bookwork.** $c_n(B) = \sum_{k=0}^{n} (-1)^k (n-k)! c_k(\overline{B})$. [2]

(c) **All standard.**

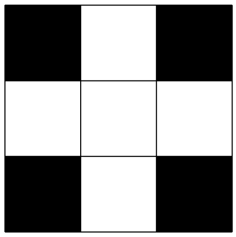
The complementary board $\overline{B}$ can be factored as the fully disjoint union of boards C and D as shown below. [3]

7 (continued)





C



D

(ii) By inspection, we have $r_C(x) = r_D(x) = 1 + 5x + 4x^2$. It follows that

$$r_{\overline{B}}(x) = r_C(x)r_D(x) = (1 + 5x + 4x^2)^2 = 1 + 10x + 33x^2 + 40x^3 + 16x^4.\text{[3]}$$

(iii) This then gives

$$\begin{aligned} c_6(B) &= 6!c_0(\overline{B}) - 5!c_1(\overline{B}) + 4!c_2(\overline{B}) - 3!c_3(\overline{B}) + 2!c_4(\overline{B}) \\ &= 720 \times 1 - 120 \times 10 + 24 \times 33 - 6 \times 40 + 16 \times 1 \\ &= 720 - 1200 + 792 - 20 + 32 = 104.\text{[2]} \end{aligned}$$

8 (a) State Landau's theorem on scores in tournaments. ***(4 marks)***

(b) Does there exist a tournament with score sequence $(7, 6, 6, 3, 3, 1, 1, 1)$? Justify your answer. ***(2 marks)***

(c) Consider tournaments with four players in which there are two players with a score of one and no players with a score of zero. What are the possible score sequences? For each possible score sequence, give an example of a corresponding tournament. ***(5 marks)***

Solution.

(a) **Bookwork.** Landau's theorem is as follows. Consider a list $s_1, \dots, s_n$ of nonnegative integers with $\sum_i s_i = \binom{n}{2}$ [1]. Then the following conditions are equivalent:

(1) There is an n-player tournament in which player i wins s_i games for all i [1]

(2) The sum of any k of the terms s_i is at least $\binom{k}{2}$ [1]

(3) The sum of any k of the terms s_i is at most $\binom{k}{2} + k(n-k)$. [1]

 Question 8 continued on next page

8 (continued)

(b) **Standard.** In the sequence $(7,6,6,3,3,1,1,1)$ the sum of the last 5 scores is 9, which is less than $\binom{5}{2} = 10$, so Landau's criterion is not satisfied. Thus, this cannot be the score sequence of a tornament. [2]

(c) **Similar problems have been seen.** The score sequence must be of the form $(x, y, 1, 1)$ with $x \geq y \geq 1$ and $x + y + 1 + 1 = \binom{4}{2} = 6$, or equivalently $x + y = 4$. The only possible sequences satisfying these criteria are $(3,1,1,1)$ and $(2,2,1,1)$ [2]. Corresponding tournaments are shown below [3].

	0	1	2	3
0		W	W	W
1	L		W	L
2	L	L		W
3	L	W	L	

scores $(3,1,1,1)$

	0	1	2	3
0		W	W	L
1	L		W	W
2	L	L		W
3	W	L	L	

scores $(2,2,1,1)$

Full credit will be given for correct examples constructed by any means.

9 Find numbers $a, \ldots, x$ such that the following matrix becomes a latin square:

(7 marks)

$$\left[\begin{array}{cccc|cc} \mathbf{a} & 3 & 2 & 4 & \mathbf{b} & \mathbf{c} \\ 2 & 4 & 6 & 5 & \mathbf{d} & \mathbf{e} \\ 4 & 6 & \mathbf{f} & 3 & \mathbf{g} & \mathbf{h} \\ 6 & 5 & 4 & 2 & \mathbf{i} & \mathbf{j} \\ \hline \mathbf{k} & \mathbf{m} & \mathbf{n} & \mathbf{p} & \mathbf{q} & \mathbf{r} \\ \mathbf{s} & \mathbf{t} & \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{x} \end{array}\right]$$

Solution. **Standard. Full marks will be given for finding a correct matrix by any means.**

First consider the top left 4×4 block. We need to fill in a and f to make a 4×4 latin square that is extendable to a 6×6 latin square. By the standard criterion, this means that the number of occurrences of each entry in the 4×4 block should be at least $4 + 4 - 6 = 2$ [1]. The numbers of occurrences of $2, \ldots, 6$ are $3, 2, 4, 2, 3$, but 1 does not occur at all. Thus, the only way to satisfy the criterion is to take $a = f = 1$. [1]

This gives a 4×4 latin square, in which 1, 3 and 5 occur twice, and everything else occurs at least three times. We can draw this square, together with the options for columns 5 and 6, as follows:

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 5/6 & 6/5 \\ 2 & 4 & 6 & 5 & 1/3 & 3/1 \\ 4 & 6 & 1 & 3 & 2/5 & 5/2 \\ 6 & 5 & 4 & 2 & 1/3 & 3/1 \end{bmatrix} \text{[1]}$$

We need to fill in column 5 in such a way that the extendibility criterion is still satisfied. This means that we need one extra occurrence of 1, 3 and 5, so these numbers must appear somewhere in column 5 [1]. Once we have filled in column 5, there will be a unique way to fill in column 6. There are four possibilities for columns 5 and 6, as follows:

$$\begin{bmatrix} 5 & 6 \\ 1 & 3 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} \text{[1]} \qquad \begin{bmatrix} 5 & 6 \\ 3 & 1 \\ 2 & 5 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 6 & 5 \\ 1 & 3 \\ 5 & 2 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 6 & 5 \\ 3 & 1 \\ 5 & 2 \\ 1 & 3 \end{bmatrix}$$

It is sufficient to find any one of these, which is easily done by trial and error. For the rest of this solution, we will just use the first possibility.

We now have a 4×6 latin rectangle, which we need to extend to 6×6. The options for the last two rows can be displayed as follows:

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 1 & 3 \\ 4 & 6 & 1 & 3 & 2 & 5 \\ 6 & 5 & 4 & 2 & 3 & 1 \\ 3/5 & 1/2 & 3/5 & 1/6 & 4/6 & 2/4 \\ 5/3 & 2/1 & 5/3 & 6/1 & 6/4 & 4/2 \end{bmatrix} \text{[1]}$$

It turns out that no backtracking is needed here, we can just fill in row 5 by using the first option that has not already been used, giving $(3, 1, 5, 6, 4, 2)$, and then row 6 is forced to be

9 (continued)

$(5,2,3,1,6,4)$. We end up with the following latin square:

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 1 & 3 \\ 4 & 6 & 1 & 3 & 2 & 5 \\ 6 & 5 & 4 & 2 & 3 & 1 \\ 3 & 1 & 5 & 6 & 4 & 2 \\ 5 & 2 & 3 & 1 & 6 & 4 \end{bmatrix} [1]$$

10 (a) In a (v,b,r,k,λ)-block design, the number of varieties is v and the number of blocks is b. Explain the meaning of each of the other parameters. ***(3 marks)***

(b) State two equations and two inequalities reating the parameters v, b, r, k and λ. ***(4 marks)***

(c) There are precisely two lists (v,b,r,k,λ) of positive integers with $v = 16$ and $\lambda = 1$ that can be the parameters of a block design (and so satisfy the the above equations and inequalities). Find these lists. ***(5 marks)***

Solution.

(a) **Bookwork.** The number of blocks of the design each variety appears in is r [1]. The number of varieties per block is k [1]. The number of blocks of the design that each pair of varieties appears in is λ [1].

(b) **Bookwork.**

$$vr = bk[1] \qquad r(k-1) = \lambda(v-1)[1]$$

$$\lambda < r[1] \qquad v \leq b.[1]$$

(c) **Unseen.** With $v = 16$ and $\lambda = 1$, the above constraints become

$$16r = bk \qquad r(k-1) = 15$$

$$1 < r \qquad 16 \leq b.[1]$$

The equations can be rearranged as $k = 15/r + 1$ and

$$b = \frac{16r}{k} = \frac{16r}{15/r+1} = \frac{16r^2}{15+r}.$$

The equation $r(k-1) = 15$ means that r must be a divisor of 15 [1], but we also know that $r > 1$, so $r \in \{3,5,15\}$. If $r = 3$ then the above equations give $k = 6$ and $b = 8$ but this contradicts the relation $v \leq b$ [1]. If $r = 5$ then the above equations give $(v,b,r,k,\lambda) = (16,20,5,4,1)$ [1], and if $r = 15$ they give $(v,b,r,k,\lambda) = (16,120,15,2,1)$ [1]. In both these cases, the constraints $\lambda < r$ and $v \leq b$ are also satisfied.

End of Question Paper

Total question marks: 100