



SCHOOL OF MATHEMATICAL AND PHYSICAL SCIENCES

Autumn Semester 2024–25

MAS334 Combinatorics

2 hours 30 minutes

Attempt all the questions. Give justification for all numerical answers. The allocation of marks is shown in brackets. There are 100 marks in total.

No auxiliary material is provided.

**Please leave this exam paper on your desk
Do not remove it from the hall**

Registration number from U-Card (9 digits)
to be completed by student

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1 Put $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and consider subsets $A \subseteq U$.

- (a) How many subsets are there in total? (2 marks)
- (b) How many subsets A are there such that $|A| < |A^c|$? (2 marks)
- (c) How many subsets A are there such that A contains at least one even number and at least one odd number? (2 marks)
- (d) How many nonempty subsets A are there such that $\max(A) = \min(A) + 2$? (2 marks)

2 Show that for every natural number n we have

$$\sum_{k=1}^n k \binom{n}{k} = \binom{n}{1} + 2 \binom{n}{2} + 3 \binom{n}{3} + \cdots + n \binom{n}{n} = n 2^{n-1}.$$

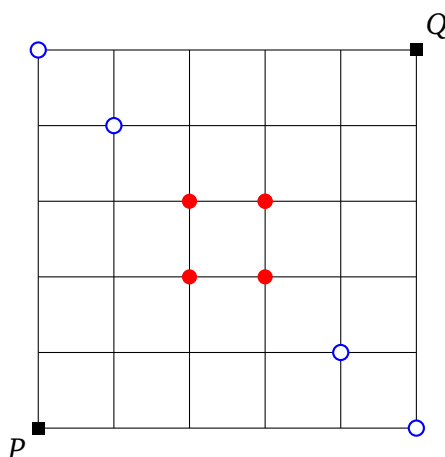
Hint: You may wish to consider the set P of pairs (A, a) , where A is a subset of the set $N = \{1, \dots, n\}$ and $a \in A$. (5 marks)

3 Find the number of integer solutions for each of the following questions:

- (a) $x_0 + \cdots + x_9 = 111$ with $x_0, \dots, x_9 \geq 1$. (2 marks)
- (b) $x_0 + \cdots + x_9 = 111$ with $x_0, \dots, x_9 \geq 10$. (2 marks)
- (c) $x_0 + \cdots + x_9 = 111$ with $x_0, \dots, x_9 \geq 10$ and $\max(x_0, \dots, x_9) = 20$. (3 marks)
- (d) $x_0 \times \cdots \times x_9 = 111$ with $x_0, \dots, x_9 \geq 0$. (3 marks)

You can leave your answers as binomial coefficients or similar expressions, you do not need to calculate the actual numbers. For (d), you may wish to find the prime factorisation of the relevant numbers.

4 This question concerns routes in the grid illustrated:



The points P and Q are marked with squares, four other points are marked with empty circles, and another four points are marked with filled discs.

- (a) How many routes are there from P to Q along the lines of the grid (always moving up or to the right, as usual)? Give a brief reason for your answer.
(3 marks)
- (b) How many of these routes do not pass through any of the filled discs? (To answer this, you may also want to think about the unfilled circles.)
(5 marks)
- (c) How many of the routes in (a) pass through precisely one of the filled discs?
(5 marks)

5 (a) Let A and B be finite sets with $|A| < |B|$. One of the following statements is equivalent to the Pigeonhole Principle (and so is true). The other statements are false. Identify which statement is true, and give counterexamples for all the other statements.
(6 marks)

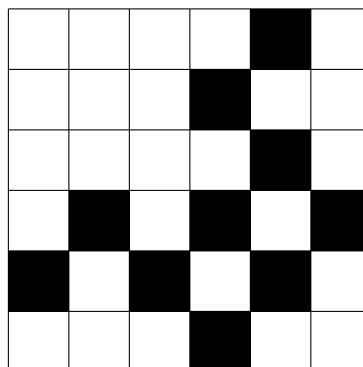
- (A) Every map $f : A \rightarrow B$ is injective
 (B) Every map $f : A \rightarrow B$ is surjective
 (C) Every map $g : B \rightarrow A$ is injective
 (D) Every map $g : B \rightarrow A$ is non-injective
 (E) Every map $g : B \rightarrow A$ is non-surjective

(b) Let $A \subseteq \mathbb{N}$ be a set of natural numbers, such that $|A| = 10$ and $a \geq 1000$ for all $a \in A$. For every subset $X \subseteq A$ let $\sum X$ denote the sum of all the numbers in X . Show that there are two subsets $B, C \subseteq A$ with $B \neq C$ and $B \cap C = \emptyset$ such that $\sum B = \sum C \pmod{1000}$.
(5 marks)

- 6 (a) State the Inclusion/Exclusion Principle. *(3 marks)*
- (b) How many of the numbers in the set $N = \{0, 1, \dots, 23099\}$ are coprime with 231? *(6 marks)*

- 7 (a) Let B be part of an $n \times n$ board, and let C and D be subsets of B .
- (i) Explain what it means to say that B is the fully disjoint union of C and D . *(3 marks)*
- (ii) If B is the fully disjoint union of C and D , what is the relationship between the corresponding rook polynomials? *(1 mark)*
- (b) Give the formula for $c_n(B)$ in terms of the rook coefficients for the complementary board \bar{B} . *(2 marks)*

(c) Consider the following board B :



- (i) Draw the complementary board \bar{B} , and divide it into two fully disjoint subsets. *(3 marks)*
- (ii) Calculate the rook polynomial of \bar{B} . *(3 marks)*
- (iii) Calculate $c_6(B)$. *(2 marks)*

- 8 (a) State Landau's theorem on scores in tournaments. *(4 marks)*
- (b) Does there exist a tournament with score sequence $(7, 6, 6, 3, 3, 1, 1, 1)$? Justify your answer. *(2 marks)*
- (c) Consider tournaments with four players in which there are two players with a score of one and no players with a score of zero. What are the possible score sequences? For each possible score sequence, give an example of a corresponding tournament. *(5 marks)*

- 9 Find numbers a, \dots, x such that the following matrix becomes a latin square: *(7 marks)*

$$\left[\begin{array}{cccc|cc} \mathbf{a} & 3 & 2 & 4 & \mathbf{b} & \mathbf{c} \\ 2 & 4 & 6 & 5 & \mathbf{d} & \mathbf{e} \\ 4 & 6 & \mathbf{f} & 3 & \mathbf{g} & \mathbf{h} \\ 6 & 5 & 4 & 2 & \mathbf{i} & \mathbf{j} \\ \hline \mathbf{k} & \mathbf{m} & \mathbf{n} & \mathbf{p} & \mathbf{q} & \mathbf{r} \\ \mathbf{s} & \mathbf{t} & \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{x} \end{array} \right]$$

- 10 (a) In a (v, b, r, k, λ) -block design, the number of varieties is v and the number of blocks is b . Explain the meaning of each of the other parameters. *(3 marks)*
- (b) State two equations and two inequalities relating the parameters v, b, r, k and λ . *(4 marks)*
- (c) There are precisely two lists (v, b, r, k, λ) of positive integers with $v = 16$ and $\lambda = 1$ that can be the parameters of a block design (and so satisfy the the above equations and inequalities). Find these lists. *(5 marks)*

End of Question Paper