MAS61015 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 11 — Solutions

Please hand in Exercises 2 and 4 by the Wednesday lecture of Week 5. I would prefer paper, but if that is not possible for some reason, then you can send me a scan by email.

Exercise 1. Give an elementary proof (just using real analysis, not algebraic topology) of the case $n = 1$ of the Brouwer Fixed Point Theorem.

Solution: The theorem says that if $f\colon B^1 \to B^1$ is continuous, then there is a point $x \in B^1$ with $f(x) = x$. Note here that B^1 is just the interval $[-1, 1]$. Define $g\colon [-1, 1] \to \mathbb{R}$ by $g(x) = f(x) - x$ (so g is again continuous). Now $g(-1) = f(-1) + 1$ with $-1 \leq f(-1) \leq 1$ so $0 \leq g(-1) \leq 2$. Similarly we have $g(1) = f(1) - 1$ with $-1 \leq f(1) \leq 1$ so $-2 \leq g(1) \leq 0$. As $g(1) \leq 0 \leq g(-1)$, the Intermediate Value Theorem tells us that there exists $x \in [-1, 1]$ such that $g(x) = 0$. As $g(x) = f(x) - x$, this means that $f(x) = x$ as required.

Exercise 2. Consider B^2 as a subset of $\mathbb{C}$, so $B^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$. Check that the following formulae define continuous maps $f_k\colon B^2 \to B^2$, and find their fixed points.

$$f_1(z) = -z \qquad f_2(z) = \overline{z} \qquad f_3(z) = \frac{2z-1}{2-z} \qquad f_4(z) = |z|z + 1 - |z|$$

(Note that in particular, you need to show that $f_i(z) \in B^2$ whenever $z \in B^2$. For $f_3(z)$, you can give a direct argument or you can recall some relevant theory from Complex Analysis. To understand $f_4(z)$, think about the same expression with $|z|$ replaced by an arbitrary real number t.)

Solution: It is clear that $|-z| = |\overline{z}| = |z|$ for all z, so f_1 and f_2 send B^2 to B^2. The map f_3 is a Möbius transformation, and standard complex analysis shows that such maps send circles to circles or straight lines. You can pick any three points $z \in S^1$ (say $z = 1, -1, i$) and check that $f_3(z) \in S^1$. This proves that $f_3(S^1)$ is a circle that meets S^1 in three places and so is the same as S^1. We also have $f_3(0) = -1/2 \in OB^2$ and it follows that f_3 sends B^2 to itself. Alternatively, if $z = x + iy \in B^2$ one can expand everything out to check that

$$|2-z|^2 - |2z-1|^2 = 3(1 - x^2 - y^2) \geq 0,$$

so $|2 - z| \geq |2z - 1|$, so $|f_3(z)| \leq 1$. Finally, note that if $z \in B^2$ and $0 \leq t \leq 1$ then the point $tz + (1 - t)$ lies on the straight line joining z to 1, and so lies in B^2. Taking $t = |z| \in [0, 1]$, we get the point $f_4(z)$, so f_4 also sends B^2 to itself.

It is clear that $z = -z$ iff $z = 0$, so the fixed set of f_1 is $\{0\}$. It is also clear that $z = \overline{z}$ iff z is real, so the fixed set of f_2 is $B^2 \cap \mathbb{R} = [-1, 1]$. Next, we have $f_3(z) = z$ iff $2z - 1 = (2 - z)z$, which simplifies to $z^2 = 1$, so the fixed set of f_3 is $\{1, -1\}$. Finally, we have $f_4(z) - z = (|z| - 1)(z - 1)$, so $f_4(z) = z$ iff $|z| = 1$ or $z = 1$. Moreover, if $z = 1$ then $|z| = 1$ so we do not need to consider the second case separately. We find that the fixed set of f_4 is S^1.

Exercise 3. Suppose that $n > 0$. For each of the spaces $X = S^n, \mathbb{R}^n, OB^n$ define a continuous map $f\colon X \to X$ that has no fixed points.

Solution:

(a) We can define $f\colon S^n \to S^n$ by $f(x) = -x$. If $-x = x$ then $x = 0$ so $x \notin S^n$; this proves that f has no fixed points.

(b) We can define $g\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(x_1, \dots, x_n) = (x_1 + 1, \dots, x_n + 1).$$

This clearly has no fixed points.

(c) We can define a homeomorphism $h\colon OB^n \to \mathbb{R}^n$ by $h(x) = x/\sqrt{1 - \|x\|^2}$, with inverse $h^{-1}(y) = y/\sqrt{1 + \|y\|^2}$. We can then define $k\colon OB^n \to OB^n$ by $k = h^{-1} \circ g \circ h$, with g as in (b). Now if $k(x) = x$ then $h^{-1}(g(h(x))) = x$, so $g(h(x)) = h(x)$, so the point $y = h(x) \in \mathbb{R}^n$ is a fixed point of g, which is impossible. Thus, k has no fixed points.

For an alternative construction, pick any point $a \in OB^n$ with $a \neq 0$, and define $m(x) = x + (1 - \|x\|a)$. If $x \in OB^n$ then we can write $x = ru$ for some unit vector u and some $r \in [0, 1)$. We then have $m(x) = ru + (1-r)a$, which lies on the line segment joining u to a, but is not equal to u. This shows that $m(x) \in OB^n$ as required. We have $m(x) - x = (1 - \|x\|)a$, which is nonzero as $\|x\| < 1$ and $a \neq 0$. This shows that m has no fixed points.

Exercise 4. You can assume all homology calculations mentioned in the notes. Show that

(a) Neither of $\mathbb{R}P^1$ and $\mathbb{R}P^2$ is a homotopy retract of the other.
(b) The torus T^2 is a homotopy retract of T^3, but T^3 is not a homotopy retract of T^2.
(c) S^1 is a retract of $S^3 \setminus S^1$
(d) OB^2 is a homotopy retract of B^2, but not an actual retract.

Solution:

(a) Recall that $H_1(\mathbb{R}P^1) = \mathbb{Z}$ and $H_1(\mathbb{R}P^2) = \mathbb{Z}/2$. There is no injective homomorphism from $\mathbb{Z}$ to $\mathbb{Z}/2$ because $\mathbb{Z}$ is infinite and $\mathbb{Z}/2$ is finite. There is no injective homomorphism from $\mathbb{Z}/2$ to $\mathbb{Z}$, because $\mathbb{Z}/2$ has an element of order 2 and $\mathbb{Z}$ does not. It follows that neither of these spaces can be a homotopy retract of the other.

(b) Recall that $T^2 = S^1 \times S^1$ and $T^3 = S^1 \times S^1 \times S^1$. We can define $T^2 \xrightarrow{f} T^3 \xrightarrow{g} T^2$ by $f(u,v) = (u,v,1)$ and $g(u,v,w) = (u,v)$. Then $g \circ f = \mathrm{id}\colon T^2 \to T^2$, so T^2 is a retract of T^3. On the other hand, we have $H_1(T^k) = \mathbb{Z}^k$ and there is no injective homomorphism from $\mathbb{Z}^3$ to $\mathbb{Z}^2$ so T^3 is not a homotopy retract of T^2. (To prove the algebraic claim rigorously, let $\alpha\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ be a homomorphism. Let e_i be the basis vector in $\mathbb{Z}^3$, and put $u_i = \alpha(e_i) \in \mathbb{Z}^2 < \mathbb{Q}^2$. We have three vectors in the space $\mathbb{Q}^2$, so by standard linear algebra they must be linearly dependent, say $a_1u_1 + a_2u_2 + a_3u_3 = 0$ for some $a_i \in \mathbb{Q}$ with $(a_1, a_2, a_3) \neq (0,0,0)$. We can put these numbers a_i over a common denominator, say $(a_1, a_2, a_3) = (b_1/n, b_2/n, b_3/n)$ for some $b_1, b_2, b_3, n \in \mathbb{Z}$ with $n > 0$. We then have

$$\alpha(b) = b_1u_1 + b_2u_2 + b_3u_3 = n(a_1u_1 + a_2u_2 + a_3u_3) = 0,$$

so α is not injective.)

(c) Recall that

$$S^3 = \{(w,x,y,z) \in \mathbb{R}^4 \mid w^2 + x^2 + y^2 + z^2 = 1\}.$$

As usual, we identify S^1 with the subset

$$S^1 = \{(w,x,0,0) \in \mathbb{R}^4 \mid w^2 + x^2 = 1\} = \{(w,x,y,z) \in S^3 \mid (y,z) = (0,0)\},$$

so

$$S^3 \setminus S^1 = \{(w,x,y,z) \in S^3 \mid (y,z) \neq (0,0)\}.$$

Note that if $(w,x,y,z) \in S^3 \setminus S^1$ then $y^2 + z^2 > 0$ so we can legitimately define

$$g(w,x,y,z) = (y,z)/\sqrt{y^2+z^2} \in S^1.$$

This gives a continuous map $g\colon S^3 \setminus S^1 \to S^1$. In the opposite direction, we can define $f\colon S^1 \to S^3 \setminus S^1$ by $g(u,v) = (0,0,u,v)$. It is then clear that $f(g(u,v)) = (u,v)$ for all $(u,v) \in S^1$, so we have defined a retraction.

(d) We can define maps $OB^2 \xrightarrow{f} B^2 \xrightarrow{g} OB^2$ by $f(x) = 0$ and $g(x) = 0$. As OB^2 is convex, the composite $g \circ f$ is homotopic to the identity by a straight line homotopy. Thus, we have a homotopy retraction. We could make this closer to being an actual retraction by taking $f(x) = x$ and $g(x) = 0.99999x$. However, we cannot have an actual retraction. To see this, note that if $g \circ f = \mathrm{id}\colon OB^2 \to OB^2$ then any $x \in OB^2$ is equal to $g(f(x))$ and so lies in the image of g. This means that g is a surjective continuous map from the compact space B^2 to the non-compact space OB^2, contradicting Proposition 8.20.

Exercise 5. Let $p, q\colon \mathbb{C} \to \mathbb{C}$ be continuous maps such that p is a polynomial of degree $n > 0$ and q satisfies $|q(x)| < 1$ for all $x \in \mathbb{C}$. By adapting the proof of the Fundamental Theorem of Algebra, prove that there exists $x \in \mathbb{C}$ such that $p(x) = q(x)$.

Solution: We have

$$p(x) = a_0 + a_1x + \cdots + a_nx^n$$

for some coefficients a_i with $a_n \neq 0$. Put $f(x) = p(x) - q(x)$. Suppose, for a contradiction that $f(x)$ is never zero. Choose some very large radius R and define $h\colon [0,1]^2 \to \mathbb{C} \setminus \{0\}$ by

$$h(s,t) = f(Rse^{2\pi it})/f(Rs).$$

As we are assuming that f is never zero, the division is valid and $h(s,t)$ lies in $\mathbb{C} \setminus \{0\}$ as required. Now put $u(t) = h(1,t)$. We have $h(s,0) = h(s,1) = 1$ for all s, and $h(0,t) = 1$ for all t, so h is a pinned homotopy between the constant path and u. On the other hand, we have chosen R to be very large, so when $|x| = R$ the term a_nx^n will be much larger than all the other terms in $p(x)$, and also much larger than $q(x)$, because $|q(x)| < 1$ for all x. This gives

$$u(t) = \frac{f(Re^{2\pi it})}{f(R)} \simeq \frac{a_nR^ne^{2\pi int}}{a_nR^n} = e^{2\pi int}.$$

Thus, if we put $v(t) = e^{2\pi int}$ then $u(t)$ will be very close to $v(t)$ for all t, so the straight line path from $u(t)$ to $v(t)$ will not pass through the origin, so we have a pinned homotopy between u and v in $\mathbb{C} \setminus \{0\}$. We now conclude that the constant path is path homotopic to v. However, this is impossible, because in the group $H_1(\mathbb{C} \setminus \{0\}) \simeq H_1(S^1) \simeq \mathbb{Z}$ the constant path corresponds to 0 and v corresponds to n.