MAS61015 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 14

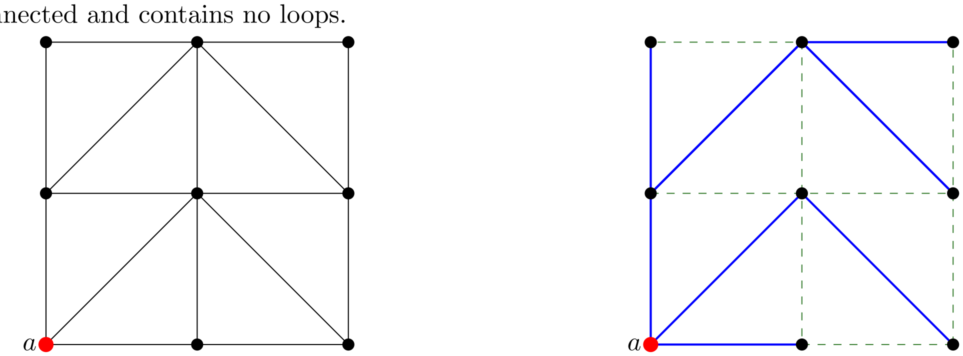
Please hand in exercise 1 by the end of Week 8.

Exercise 1. Let X be a graph, consisting of some points in $\mathbb{R}^2$ (called vertices) and straight edges between them. We assume that no two edges intersect except at the endpoints. In this exercise we will work through the standard calculation of $H_*(X)$.

Part (a) below shows an example. However, you should give answers that work for any X, except in cases where the question specifically tells you to use the example in (a).

By a *combinatorial path* in X we mean a sequence of vertices $u_0, \dots, u_r$ such that each pair (u_i, u_{i+1}) is an edge of X. The *combinatorial distance* between vertices a and b is the minimum possible length of a combinatorial path between them.

(a) A *spanning tree* is a subgraph $T \subseteq X$ that contains all of the vertices and some of the edges, with the property that it is connected and contains no loops.



X $\qquad$ T

Show that there always exists a spanning tree. (Just choose a connected loop-free subgraph containing a with as many edges as possible, and prove that it must be a spanning tree.) For the rest of this exercise, we choose a spanning tree T and a vertex $a \in T$.

(b) Show that if x is a vertex of X, then there is a unique combinatorial path u_x that goes from x to a without visiting any vertex twice. Draw some examples of paths u_x in the complex illustrated above.

(c) We also write u_x for the sum of the edges in u_x, considered as an element of $C_1(X)$. What is $\partial(u_x)$?

(d) Define $r(x)$ to be the first vertex on u_x after x (to be interpreted as $r(a) = a$ in the exceptional case where $x = a$). In other words, $r(x)$ is the vertex that we reach after taking one step towards a from x. Annotate the above diagram to show the effect of the map r.

(e) Let e be an edge of T. Show that there is a vertex x such that the endpoints of e are x and $r(x)$.

(f) Part (d) defined r as a map $\mathrm{vert}(T) \to \mathrm{vert}(T)$, where $\mathrm{vert}(T)$ is the set of vertices of T. Explain how to extend this to give a map $r\colon T \to T$. Show that r is homotopic to the identity (but not by a linear homotopy). Deduce that T is contractible.

(g) Now let the edges not in T be $e_1, \dots, e_m$, where $e_q = (x_q, y_q)$. Define $a_q, b_q, c_q \in e_q$ by

$$a_q = \tfrac{3}{4}x_q + \tfrac{1}{4}y_q \qquad b_q = \tfrac{1}{2}x_q + \tfrac{1}{2}y_q \qquad c_q = \tfrac{1}{4}x_q + \tfrac{3}{4}y_q$$

Put $z_q = \langle x_q, y_q \rangle - u_{x_q} + u_{y_q} \in C_1(X)$. Prove that $\partial(z_q) = 0$ (so we have a corresponding element $h_q = [z_q] \in H_1(X)$).

(h) Put $U = X \setminus T$, so U consists of the edges e_q with the endpoints removed. Put $V = X \setminus \{b_1, \dots, b_m\}$. Describe the homology of U, V and $U \cap V$ in terms of the points a_q, b_q, c_q and a.

(i) Use the Mayer-Vietoris sequence to show that $H_1(X) \simeq \mathbb{Z}^m$.

(j) In the construction of the Mayer-Vietoris sequence we use the subcomplex $C_*(U, V) = C_*(U) + C_*(V) \leq C_*(X)$. Show that $z_q \notin C_*(U, V)$. Find elements $z'_q \in C_1(U)$ and $z''_q \in C_1(V)$ such that $\mathrm{sd}^2(z_q) = z'_q + z''_q$, proving that $\mathrm{sd}^2(z_q) \in C_1(U, V)$. (For this you will need to think about $\mathrm{sd}^2(u_x)$ and $\mathrm{sd}^2(\langle x_q, y_q \rangle)$. You can just leave $\mathrm{sd}^2(u_x)$ as $\mathrm{sd}^2(u_x)$ but you will need to analyse $\mathrm{sd}^2(\langle x_q, y_q \rangle)$ in more detail.)

(k) Use z'_q and z''_q to find a snake involving $\mathrm{sd}^2(z_q)$ and thus compute $\delta(h_q)$ in the Mayer-Vietoris sequence. Conclude that the elements $h_1, \dots, h_m$ give a basis for $H_1(X)$.