MAS61015 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 12

Please hand in Exercises 1 and 2 by the Wednesday lecture of Week 6. I would prefer paper, but if that is not possible for some reason, then you can send me a scan by email.

Exercise 1. We define groups U_k, V_k and W_k (for all $k \in \mathbb{Z}$) and maps between them as follows:

- U_k is a copy of $\mathbb{Z}/4$ with generator u_k, and V_k is a copy of $\mathbb{Z}/16$ with generator v_k, and W_k is a copy of $\mathbb{Z}/4$ with generator w_k.
- The maps $d\colon U_k \to U_{k-1}$ and $d\colon V_k \to V_{k-1}$ and $d\colon W_k \to W_{k-1}$ are given by $d(u_k) = 0$ and $d(w_k) = 0$ and $d(v_k) = 8v_{k-1}$.
- The map $i\colon U_k \to V_k$ is given by $i(u_k) = 4v_k$.
- The map $p\colon V_k \to W_k$ is given by $p(v_k) = w_k$.

(a) Prove that this makes U_*, V_* and W_* into chain complexes.
(b) Prove that i and p are chain maps.
(c) Prove that the sequence $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ is short exact.
(d) Find the homology groups of U_*, V_* and W_*.
(e) Describe the action of the maps i_* and p_* on these homology groups.
(f) By finding suitable snakes, describe the connecting map $\delta\colon H_k(W) \to H_{k-1}(U)$. Check that the resulting long sequence of homology groups is exact.

Note: In working through this problem you will need to refer to various homology classes $[z]$. You must remember that this notation is only meaningful when z is a cycle, i.e. it satisfies $d(z) = 0$. It is easy to violate this rule by accident; you should check your work carefully to ensure that you have not done so.

Exercise 2. Let U_* and W_* be chain complexes, and suppose we have maps $f_n\colon W_n \to U_{n-1}$ that satisfy $df = -fd\colon W_n \to U_{n-2}$. Put $V_n = U_n \oplus W_n$ and define $d\colon V_n \to V_{n-1}$ by

$$d(u, w) = (d(u) + f(w),\ d(w)).$$

Define maps $U_n \xrightarrow{i} V_n \xrightarrow{p} W_n$ by $i(u) = (u, 0)$ and $p(u, w) = w$.

(a) Prove that V_* is a chain complex.
(b) Prove that i and p are chain maps and that the sequence $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ is short exact.
(c) Prove that the resulting map $\delta\colon H_n(W) \to H_{n-1}(U)$ satisfies $\delta([w]) = [f(w)]$.

Exercise 3. Let $U_* \xrightarrow{i} V_* \xrightarrow{j} W_*$ be a short exact sequence of chain complexes and chain maps. Suppose that the groups $H_n(U)$ and $H_n(W)$ are finite for all n, and are zero when n is odd. Prove that $H_n(V)$ is finite for all n, with $|H_n(V)| = |H_n(U)||H_n(W)|$.

Exercise 4. Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain maps between chain complexes. Suppose that for every $w \in W_k$ with $dw = 0$ there exists $v \in V_k$ with $dv = 0$ and $pv = w$. Prove that the sequence $H_*(U) \xrightarrow{i_*} H_*(V) \xrightarrow{p_*} H_*(W)$ is short exact.