MAS61015 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 9

Please hand in Exercises 2 and 9 by the Wednesday lecture of Week 3. I would prefer paper, but if that is not possible for some reason, then you can send me a scan by email.

Exercise 1. Write down all isomorphism classes of abelian groups of order 2, 4 and 8. Write down all isomorphism classes of abelian groups of order 6, 10, 15.

Exercise 2.

(a) If there is an exact sequence

$$0 \to \mathbb{Z}/4 \xrightarrow{\alpha} A \xrightarrow{\beta} \mathbb{Z}/2 \to 0,$$

what are the possible isomorphism types for A? If you think that A could be $\mathbb{Z}/10$, for example, you should give explicit maps $\mathbb{Z}/4 \xrightarrow{\alpha} \mathbb{Z}/10 \xrightarrow{\beta} \mathbb{Z}/2$ and check that they are well-defined and give a short exact sequence.

Optional extra: If there is an exact sequence

$$0 \to \mathbb{Z}/2 \xrightarrow{\alpha} \mathbb{Z}/4 \xrightarrow{\beta} B \xrightarrow{\gamma} \mathbb{Z}/4 \oplus \mathbb{Z}/2 \xrightarrow{\delta} C \xrightarrow{\epsilon} \mathbb{Z}/2 \to 0,$$

what are the possible isomorphism types for B and C? (There are many possibilities.)

(b) Show that if there is an exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} \mathbb{Z} \to 0$ then $B \cong A \oplus \mathbb{Z}$. You should start by showing that there is a homomorphism $\sigma\colon \mathbb{Z} \to B$ such that $\beta\sigma = 1$.

Exercise 3.

(a) Let $\phi\colon A \to B$ be a homomorphism between Abelian groups. Show that $\phi(na) = n\phi(a)$ for all $a \in A$ and $n \in \mathbb{Z}$. (Start with the case $n \geq 0$ and use induction.)

(b) Let B be an Abelian group, and let $\phi\colon \mathbb{Z}^2 \to B$ be a homomorphism. Show that there are elements $u, v \in B$ such that $\phi(n, m) = nu + mv$ for all $(n, m) \in \mathbb{Z}^2$.

(c) List all the homomorphisms from $\mathbb{Z}^2$ to $\mathbb{Z}/9$. How many of them are surjective?

(d) Prove that there is no homomorphism $\phi\colon \mathbb{Z}/4 \to \mathbb{Z}/12$ such that $\phi(1) = 1$.

(e) How much can you say about homomorphisms from $\mathbb{Z}/n$ to $\mathbb{Z}/m$ for arbitrary natural numbers n, m?

Exercise 4. What is the minimum number of generators for $\mathbb{Z}/2 \oplus \mathbb{Z}/2$? What is the minimum number of generators for $\mathbb{Z}/2 \oplus \mathbb{Z}/3$? Is $\mathbb{Q}$ a finitely generated abelian group?

Exercise 5. Let p and q be coprime integers, and let $\phi\colon \mathbb{Z} \to \mathbb{Z}/p$ be the homomorphism $\phi(n) = (nq \bmod p)$. Prove that ϕ is surjective. Prove also that the only homomorphism from $\mathbb{Z}/q$ to $\mathbb{Z}/p$ is the zero homomorphism

Exercise 6. Let A be a finite Abelian group, and let B be a free Abelian group. Prove that if $\phi\colon A \to B$ is a homomorphism, then $\phi = 0$.

Exercise 7. Suppose we have two sets D and E each with precisely two elements, say $D = \{p, q\}$ and $E = \{r, s\}$. Define a function $\psi\colon D \to \mathbb{Z}[E]$ by

$$\psi(p) = 3[r] + [s] \qquad \psi(q) = 5[r] + 2[s]$$

and let $\phi\colon \mathbb{Z}[D] \to \mathbb{Z}[E]$ be the linear extension of ψ. What is $\phi([p] - [q])$? What is $\phi(n[p] + m[q])$?

Now define a map $\zeta\colon E \to \mathbb{Z}[D]$ by

$$\zeta(r) = 2[p] - [q] \qquad \zeta(s) = -5[p] + 3[q]$$

and let ξ be the linear extension of ζ. What is $\xi\phi([p])$? Extend this calculation to show that $\xi = \phi^{-1}$.

Exercise 8. Let $\phi\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}/12$ be the homomorphism defined by

$$\phi(n, m) = (3n, (2n + 4m \bmod 12)).$$

Give an isomorphism $\psi\colon \mathbb{Z} \to \ker(\phi)$.

Exercise 9. Consider the following sequences of abelian groups and homomorphisms. The degrees are indicated by the top row, so for example $B_0 = \mathbb{Z}/6$ and $B_1 = B_2 = \mathbb{Z}/4$.

For each sequence, decide whether it is a chain complex. If it is a chain complex, find the homology, and decide whether the sequence is exact.

$$\begin{array}{lcccccccccccccc}
\text{degree} & & & & -2 & & -1 & & 0 & & 1 & & 2 & & 3 \\
A_* = & \cdots & \longleftarrow & & 0 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{6} & \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 \longleftarrow \cdots \\
B_* = & \cdots & \longleftarrow & & 0 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}/6 & \xleftarrow{3} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \longleftarrow & 0 \longleftarrow \cdots \\
C_* = & \cdots & \longleftarrow & & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} \longleftarrow \cdots \\
D_* = & \cdots & \longleftarrow & & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 \longleftarrow \cdots \\
E_* = & \cdots & \longleftarrow & & 0 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} \longleftarrow \cdots \\
F_* = & \cdots & \longleftarrow & & 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} \longleftarrow \cdots
\end{array}$$

(Here notation like $\mathbb{Z}/n \xrightarrow{p} \mathbb{Z}/m$ refers to the map $f\colon \mathbb{Z}/n \to \mathbb{Z}/m$ given by $f(i \pmod{n}) = pi \pmod{m}$. You should think about the conditions on n, m and p that are needed to make this well-defined.)