Algebraic Topology Exam Questions

This is a collection of questions taken from Algebraic Topology exams over a number of years. To some extent, they have been modified to be compatible with the current version of the course, but some differences remain.

1 Compactness and the Hausdorff property

Some questions in this section use ideas that are specific to metric spaces rather than general topological spaces. These ideas are not developed in the current version of the course.

(1) Let X be a metric space.

(a) Let Y be a compact subspace of X. Prove that Y is closed in X.

(b) Let Y and Z be two compact subspaces of X. Prove that $Y \cup Z$ is compact.

(c) Deduce (or prove otherwise) that every finite space is compact.

(d) Let Y and Z be compact metric spaces. Prove that $Y \times Z$ is compact.

(e) Conversely, let Y and Z be metric spaces such that $Z \neq \emptyset$ and $Y \times Z$ is compact. Prove that Y is compact.

(f) Put $X = \{(x, y, z) \in \mathbb{R}^3 \; : \; x^4 + y^4 + z^4 = 1\}$. Prove that X is compact. You may use general theorems provided that you state them precisely.

Solution:

(a) Let (y_n) be a sequence in Y, converging to some point $x \in X$. Clearly any subsequence converges to x also. By compactness, some subsequence (y_{n_k}) converges to some $y \in Y$, and as limits are unique we must have $x = y$, so $x \in Y$. This means that Y is closed, as required.

(b) Let (x_n) be a sequence in $Y \cup Z$. Then either $x_n \in Y$ for infinitely many n, or $x_n \in Z$ for infinitely many n. In the first case, we can choose a subsequence (x'_n) of (x_n) such that $x'_n \in Y$ for all n in other words we have a sequence in Y. As Y is compact, some subsequence (x''_n) of (x'_n) converges in Y, and thus in $Y \cup Z$. The other case is similar, so in either case some subsequence of (x_n) converges in $Y \cup Z$. This implies that $Y \cup Z$ is compact.

(c) If X has only one point then every sequence converges so X is compact. If X has $n > 1$ points, we can write it in the form $X = Y \cup Z$ where $|Y| = n - 1$ and $|Z| = 1$, so Y and Z are compact by induction, so X is compact by (ii).

(d) Let (w_n) be a sequence in $Y \times Z$, with $w_n = (y_n, z_n)$ say. As Y is compact, some subsequence (y_{n_k}) converges to some $y \in Y$. Put $y'_k = y_{n_k}$ and $z'_k = z_{n_k}$ and $w'_k = (y'_k, z'_k) = w_{n_k}$. As Z is compact, some subsequence z'_{k_j} converges to some point $z \in Z$. Put $y''_j = y'_{k_j}$ and $z''_j = z'_{k_j}$ and $w''_j = (y''_j, z''_j) = w'_{k_j}$. As (y''_j) is a subsequence of the sequence (y'_k) which converges to y, we see that $y''_j \to y$. By assumption we have $z''_j \to z$, so $w''_j \to (y, z)$. Thus, some subsequence of (w_n) converges in $Y \times Z$, proving that $Y \times Z$ is compact as claimed.

(e) As $Z \neq \emptyset$ we can choose a point $a \in Z$. Let $p \colon Y \times Z \to Y$ be defined by $p(y, z) = y$. We have $p(y, a) = y$, which shows that p is surjective. In general, if $f \colon A \to B$ is a surjective continuous map of spaces and A is compact we know that B is compact. As $Y \times Z$ is assumed compact, we deduce that Y is compact.

(e) If $(x, y, z) \in X$ then $x^4 \leq x^4 + y^4 + z^4 = 1$ so $|x| \leq 1$. Similarly, we see that $|y| \leq 1$ and $|z| \leq 1$, which implies that X is bounded. I claim that it is also closed in $\mathbb{R}^3$. Indeed, suppose we have a sequence $a_n = (x_n, y_n, z_n)$ in X converging to some point $a = (x, y, z) \in \mathbb{R}^3$. then $x_n^4 + y_n^4 + z_n^4 = 1$ and $x_n \to x$, $y_n \to y$ and $z_n \to z$, so by the algebra of limits we have

$$x^4 + y^4 + z^4 = \lim(x_n^4 + y_n^4 + z_n^4) = 1,$$

so $a \in X$.

A bounded closed subset of $\mathbb{R}^n$ is compact, so we deduce that X is compact as claimed.

(2) Let X be a metric space.

(a) Let Y be a compact subspace of X. Prove that Y is closed in X.

(b) Let Y and Z be two compact subspaces of X. Prove that $Y \cup Z$ is compact.

(c) Deduce (or prove otherwise) that every finite space is compact.

(d) Let Y and Z be compact metric spaces. Prove that $Y \times Z$ is compact.

(e) Conversely, let Y and Z be metric spaces such that $Z \neq \emptyset$ and $Y \times Z$ is compact. Prove that Y is compact.

(f) Put $X = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^4 + y^4 + z^4 = 1\}$. Prove that X is compact. You may use general theorems provided that you state them precisely.

Solution:

(a) Let (y_n) be a sequence in Y, converging to some point $x \in X$. Clearly any subsequence converges to x also. By compactness, some subsequence (y_{n_k}) converges to some $y \in Y$, and as limits are unique we must have $x = y$, so $x \in Y$. This means that Y is closed, as required.

(b) Let (x_n) be a sequence in $Y \cup Z$. Then either $x_n \in Y$ for infinitely many n, or $x_n \in Z$ for infinitely many n. In the first case, we can choose a subsequence (x'_n) of (x_n) such that $x'_n \in Y$ for all n in other words we have a sequence in Y. As Y is compact, some subsequence (x''_n) of (x'_n) converges in Y, and thus in $Y \cup Z$. The other case is similar, so in either case some subsequence of (x_n) converges in $Y \cup Z$. This implies that $Y \cup Z$ is compact.

(c) If X has only one point then every sequence converges so X is compact. If X has $n > 1$ points, we can write it in the form $X = Y \cup Z$ where $|Y| = n - 1$ and $|Z| = 1$, so Y and Z are compact by induction, so X is compact by (ii).

(d) Let (w_n) be a sequence in $Y \times Z$, with $w_n = (y_n, z_n)$ say. As Y is compact, some subsequence (y_{n_k}) converges to some $y \in Y$. Put $y'_k = y_{n_k}$ and $z'_k = z_{n_k}$ and $w'_k = (y'_k, z'_k) = w_{n_k}$. As Z is compact, some subsequence z'_{k_j} converges to some point $z \in Z$. Put $y''_j = y'_{k_j}$ and $z''_j = z'_{k_j}$ and $w''_j = (y''_j, z''_j) = w'_{k_j}$. As (y''_j) is a subsequence of the sequence (y'_k) which converges to y, we see that $y''_j \to y$. By assumption we have $z''_j \to z$, so $w''_j \to (y, z)$. Thus, some subsequence of (w_n) converges in $Y \times Z$, proving that $Y \times Z$ is compact as claimed.

(e) As $Z \neq \emptyset$ we can choose a point $a \in Z$. Let $p\colon Y \times Z \to Y$ be defined by $p(y, z) = y$. We have $p(y, a) = y$, which shows that p is surjective. In general, if $f\colon A \to B$ is a surjective continuous map of spaces and A is compact we know that B is compact. As $Y \times Z$ is assumed compact, we deduce that Y is compact.

(f) If $(x, y, z) \in X$ then $x^4 \leq x^4 + y^4 + z^4 = 1$ so $|x| \leq 1$. Similarly, we see that $|y| \leq 1$ and $|z| \leq 1$, which implies that X is bounded. I claim that it is also closed in $\mathbb{R}^3$. Indeed, suppose we have a sequence $a_n = (x_n, y_n, z_n)$ in X converging to some point $a = (x, y, z) \in \mathbb{R}^3$. then $x_n^4 + y_n^4 + z_n^4 = 1$ and $x_n \to x$, $y_n \to y$ and $z_n \to z$, so by the algebra of limits we have
$$x^4 + y^4 + z^4 = \lim(x_n^4 + y_n^4 + z_n^4) = 1,$$
so $a \in X$.

A bounded closed subset of $\mathbb{R}^n$ is compact, so we deduce that X is compact as claimed.

(3)

(a) What does it mean to say that a metric space X is *compact*? **(3 marks)**

(b) Let $f\colon X \to Y$ be a continuous surjective map of metric spaces, where X is compact. Prove that Y is compact. **(6 marks)**

(c) Let Z be a closed subset of a compact space X. Prove that Z is compact. **(6 marks)**

(d) Put $U = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq 1\}$, and define $g\colon \mathbb{C} \to \mathbb{C}$ by $g(z) = e^z$.

  (i) Is U compact? **(2 marks)**

  (ii) Is $g(U)$ compact? **(4 marks)**

  (iii) Is $g(g(U))$ compact? **(4 marks)**

Justify your answers.

Solution:

(a) A metric space X is compact if for every sequence (x_n) in X there is a subsequence (x_{n_k}) and a point $x \in X$ such that $x_{n_k} \to x$. **[bookwork]**[3]

(b) Let $f\colon X \to Y$ be a continuous surjective map, and suppose that X is compact. Consider a sequence (y_n) in Y. As f is surjective, we can choose $x_n \in X$ for each n such that $f(x_n) = y_n$. As X is compact, there is a subsequence (x_{n_k}) of (x_n) and a point $x \in X$ such that $x_{n_k} \to x$. Put $y = f(x) \in Y$, and note that $y_{n_k} = f(x_{n_k})$. As f is continuous, it follows that $y_{n_k} \to y$. Thus our original sequence has a convergent subsequence, proving that Y is compact. **[bookwork]**[6]

(c) Let X be compact, and let Z be a closed subspace of X. Consider a sequence (z_n) in Z. We can regard this as a sequence in the compact space X, so some subsequence (z_{n_k}) converges to some point $x \in X$. However, Z is closed and z_{n_k} lies in Z for all k and $z_{n_k} \to x$, so x must actually lie in Z. Thus our original sequence has a subsequence that converges to a point in Z, proving that Z is compact. **[bookwork]**[6]

(d) Put $U = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq 1\}$, and define $g\colon \mathbb{C} \to \mathbb{C}$ by $g(z) = e^z$. Then U is clearly unbounded and thus not compact; the sequence $i, 2i, 3i, \ldots$ has no convergent subsequence. [2] On the other hand, we can use the fact that $g(x + iy) = e^x(\cos(y) + i\sin(y))$ to see that $g(U) = \{z \in \mathbb{C} \mid 1 \leq |z| \leq e\}$ [2]. This is bounded and closed and thus compact [2]. We can regard g as a continuous surjective map from $g(U)$ to $g(g(U))$ and it follows from (b) that $g(g(U))$ is compact [4] **[unseen]**. **The properties of the complex exponential map are reviewed in lectures and used in several examples.**

(4)

(a) What does it mean to say that a metric space X is *compact*? **(3 marks)**

(b) Let X and Y be compact metric spaces. Prove that $X \times Y$ is compact. **(8 marks)**

(c) Let $f\colon I \to Y$ be a continuous map (where $I = [0,1]$). Prove that $f(I)$ is closed in Y. **(7 marks)**

(d) Put $X = \mathbb{Z} \times \mathbb{Z}$ and $Y = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$, considered as subspaces of the plane $\mathbb{R}^2$.

  (i) Is X compact? **(2 marks)**

  (ii) Is Y compact? **(2 marks)**

  (iii) Is $X \cap Y$ compact? **(3 marks)**

Justify your answers.

Solution:

(a) A metric space X is compact if for every sequence (x_n) in X there is a subsequence (x_{n_k}) and a point $x \in X$ such that $x_{n_k} \to x$. **[bookwork]**[3]

(b) Consider a sequence $z_n = (x_n, y_n)$ in $X \times Y$ [1]. As X is compact, the sequence (x_n) has a convergent subsequence, say $(x_{n_1}, x_{n_2}, \ldots)$ converging to $x \in X$ [1]. We write $x'_m = x_{n_m}$ for convenience, and also put $y'_m = y_{n_m}$ and $z'_m = (x'_m, y'_m) = z_{n_m}$ [1]. Note that $x'_m \to x$ as $m \to \infty$ [1]. Next, observe that Y is compact, so the sequence (y'_m) has a convergent subsequence, say $(y'_{m_1}, y'_{m_2}, \ldots)$ converging to $y \in Y$ [1]. Now put $y''_k = y'_{m_k}$ and $x''_k = x'_{m_k}$ and $z''_k = (x''_k, y''_k)$ [1]. Then $y''_k \to y$ by assumption, and $x''_k \to x$ because (x''_k) is a subsequence of (x'_m), and $x'_m \to x$ [1]. This means that $z''_k = (x''_k, y''_k) \to (x, y)$, so (z''_k) is a convergent subsequence of our original sequence (z_n) [1]. This proves that $X \times Y$ is compact. **[bookwork]**

(c) We know that I is compact [2], and that the image of a compact set under any continuous map is again compact [2]. This means that $f(I)$ is a compact subspace of Y [1]. However, any compact subset of a metric space is automatically closed [2], so $f(I)$ is closed in Y as claimed. **[seen]**

(d) (i) X is unbounded and thus not compact. [2]

  (ii) Y is not closed, and thus is not compact. [2]

(iii) $X \cap Y$ is a finite set; explicitly,

$$X \cap Y = \{(-1,-1),(-1,0),(-1,1),(0,-1),(0,0),(0,1),(1,-1),(1,0),(1,1)\}.$$

It follows that $X \cap Y$ is compact. [3]

(5) 2021-22 Q2:

(a) Define what is meant by a *topology* on a set X. **(3 marks)**

(b) What does it mean to say that a topological space X is *Hausdorff*?
(If your definition relies on any other concepts, then you should define them.) **(3 marks)**

(c) What does it mean to say that a topological space X is *compact*?
(If your definition relies on any other concepts, then you should define them.) **(3 marks)**

(d) Let X and Y be topological spaces, and let $f\colon X \to Y$ be a continuous injective map. For each of the claims below, give a proof or a counterexample with justification.

(i) If X is Hausdorff, then Y must also be Hausdorff. **(4 marks)**

(ii) If X is compact, then Y must also be compact. **(4 marks)**

(iii) If Y is Hausdorff, then X must also be Hausdorff. **(4 marks)**

(iv) If Y is compact, then X must also be compact. **(4 marks)**

Solution:

(a) **Bookwork** A *topology* on X is a family τ of subsets of X (called open sets) such that

(1) The empty set and the whole set X are open [1]

(2) The union of any family of open sets is open [1]

(3) The intersection of any finite list of open sets is open. [1]

(b) **Bookwork** Let X be a topological space. Given $a, b \in X$ with $a \neq b$, a *Hausdorff separation* for (a, b) is a pair of open sets $U, V \subseteq X$ with $a \in U$ and $b \in V$ and $U \cap V = \emptyset$[2]. We say that X is *Hausdorff* if every pair of distinct points has a Hausdorff separation [1].

(c) **Bookwork** Let X be a topological space. An *open cover* of X is a family $(U_i)_{i \in I}$ of open sets whose union is all of X [1]. Given such a cover, a *finite subcover* is a subfamily $(U_i)_{i \in J}$ where $J \subseteq I$ is finite and the union is still all of X [1]. We say that X is *compact* if every open cover has a finite subcover [1].

(d) **Unseen**

(i) Let X be empty, take $Y = \{0, 1\}$ with the indiscrete topology, and let $f\colon X \to Y$ be the inclusion. Then f is continuous and injective and X is (vacuously) Hausdorff but Y is not Hausdorff (because there is no Hausdorff separation for the pair $(0, 1)$). [4]

(ii) Let X be empty, take $Y = \mathbb{Z}$ with the discrete topology, and let $f\colon X \to Y$ be the inclusion. Then f is continuous and injective and X is compact but Y is not compact (because the open cover by singletons has no finite subcover). [4]

(iii) Suppose that Y is Hausdorff; we will show that X is also Hausdorff. Suppose that $a, b \in X$ with $a \neq b$. Then $f(a), f(b) \in Y$, with $f(a) \neq f(b)$ because f is injective. As Y is Hausdorff, we can choose disjoint open sets $U, V \subseteq Y$ with $f(a) \in U$ and $f(b) \in V$. As f is continuous, the sets $f^{-1}(U), f^{-1}(V) \subseteq X$ are open. As $f(a) \in U$ and $f(b) \in V$ we have $a \in f^{-1}(U)$ and $b \in f^{-1}(V)$. As U and V are disjoint, we have $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$. Thus, the pair $(f^{-1}(U), f^{-1}(V))$ is a Hausdorff separation for (a, b). [4]

(iv) Take $X = (0, 1)$ and $Y = [0, 1]$ and let $f\colon X \to Y$ be the inclusion. Then f is continuous and injective. It is standard that subsets of $\mathbb{R}$ are compact iff they are bounded and closed, so Y is compact but X is not. [4]

(6) 2022-23 Q2:

(a) What does it mean to say that a topological space X is *compact*? If your explanation relies on any auxiliary terms, then you should define them. **(3 marks)**

(b) Let X be compact topological space, and let Y be a closed subset of X.

(i) Define the subspace topology on Y. **(2 marks)**

(ii) Prove that when equipped with the subspace topology, Y is again compact. **(5 marks)**

(iii) Give an example of a compact space X and a compact subpace Y such that Y is not closed in X. **(3 marks)**

(iv) Explain a commonly-satisfied condition on X that guarantees that compact subspaces are closed. If your explanation relies on any auxiliary terms, then you should define them. However, you need not prove anything. **(3 marks)**

(c) Put $X = \mathbb{Z} \times \mathbb{Z}$ and $Y = \{(x, y) \in \mathbb{R}^2 \mid 100 < x^2 + y^2 < 10000\}$, considered as subspaces of the plane $\mathbb{R}^2$.

(i) Is X compact? **(1 marks)**

(ii) Is Y compact? **(1 marks)**

(iii) Is $X \cap Y$ compact? **(2 marks)**

Justify your answers.

(d) Let X be a metric space such that $X \setminus \{x\}$ is compact for all $x \in X$. Prove that X is finite. **(5 marks)**

Solution:

(a) **Bookwork** Let X be a topological space. By an *open cover* of X we mean a family $(U_i)_{i \in I}$ of open subsets of X, such that each point $x \in X$ lies in U_i for at least one index i [1]. A *finite subcover* of such a cover is a finite subset $J = \{j_1, \dots, j_n\} \subseteq I$ such that $(U_j)_{j \in J}$ is still a cover, or equivalently $X = U_{j_1} \cup \dots \cup U_{j_n}$ [1]. We say that X is *compact* if every open cover has a finite subcover [1].

(b) (i) **Bookwork** For the subspace topology on Y, we declare that a subset $V \subseteq Y$ is open iff there exists an open subset U of X such that $V = U \cap Y$ [2].

(ii) **Bookwork** Suppose that X is compact, and that Y is closed in X, which means that the set $U^* = X \setminus Y$ is open in X.

Let $(V_i)_{i \in I}$ be a family of subsets of Y that are open with respect to the subspace topology; we must show that this has a finite subcover [1]. As each V_i is open in the subspace topology, we can choose an open subset U_i of X such that $V_i = U_i \cap Y$ [1]. We find that the sets U_i together with U^* cover all of the compact space X [1], so there must be a finite subcover [1]. This means that there exists a finite subset $J \subseteq I$ such that $X = U^* \cup \bigcup_{j \in J} U_j$. In particular, for $y \in Y$ we note that y cannot lie in U^* so it must lie in one of the sets U_j with $j \in J$, but that means that $y \in Y \cap U_j = V_j$. This shows that $Y = \bigcup_{j \in J} V_j$ as required [1].

(iii) **Unseen** Take $X = \{0, 1\}$ with the indiscrete topology, and $Y = \{0\}$. Then Y is compact (as it is finite) but not closed. [3]

(iv) **Bookwork** A space X is said to be *Hausdorff* if for all $x, y \in X$ with $x \neq y$, there exist open sets $U, V \subseteq X$ with $x \in U$ and $y \in V$ and $U \cap V = \emptyset$ [1]. If X is Hausdorff, then any compact subset of X is closed [2].

(c) **Similar problems seen** We use the standard fact that a subset of $\mathbb{R}^2$ is compact iff it is bounded and closed.

(i) The set X is unbounded and thus not compact. [1]

(ii) The set Y is not closed, and thus is not compact. [1]

(iii) For $(x, y) \in X \cap Y$ we have $x, y \in \mathbb{Z}$ with $x^2 + y^2 < 10000$ so $x, y \in \{-99, -98, \dots, 98, 99\}$. This shows that $X \cap Y$ is finite and so is compact. [2]

(d) **Unseen** Let X be a metric space, so X is Hausdorff [1]. Suppose that for each $x \in X$, the set $X \setminus \{x\}$ is compact. As in (b)(iv) this means that $X \setminus \{x\}$ is closed, so $\{x\}$ is open in X [2]. If X is empty then it is certainly finite. Otherwise we can choose $a \in X$. By hypothesis the set $X \setminus \{a\}$ is compact, so the open cover by sets $\{x\}$ with $x \neq a$ must have a finite subscover [1]. This forces the set $X \setminus \{a\}$ to be finite, and it follows that X is finite as well [1].

(7) 2023-24 Q1:

(a) What does it mean to say that a topological space X is *Hausdorff*?
(If your definition relies on any other concepts, then you should define them.) **(3 marks)**

(b) What does it mean to say that a topological space X is *compact*?
(If your definition relies on any other concepts, then you should define them.) **(3 marks)**

(c) Put $X = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^4 + y^4 + z^4 = 1\}$. Prove that X is compact. You may use general theorems provided that you state them precisely. **(5 marks)**

(d) Put $Y = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^4 + y^4 + z^4 < 1\}$. Prove that Y is not compact. Here you should argue directly from the definitions and not use any theorems. **(5 marks)**

(e) Let Y and Z be two compact subspaces of a topological space X. Prove that $Y \cup Z$ is also compact. **(4 marks)**

(f) Let Y and Z be topological spaces such that $Z \neq \emptyset$ and $Y \times Z$ is compact. Prove that Y is compact. You may use standard results so long as you state them clearly and verify carefully that they are applicable. **(5 marks)**

Solution:

(a) **Bookwork** Let X be a topological space. Given $a, b \in X$ with $a \neq b$, a *Hausdorff separation* for (a, b) is a pair of open sets $U, V \subseteq X$ with $a \in U$ and $b \in V$ and $U \cap V = \emptyset$[2]. We say that X is *Hausdorff* if every pair of distinct points has a Hausdorff separation [1].

(b) **Bookwork** Let X be a topological space. An *open cover* of X is a family $(U_i)_{i \in I}$ of open sets whose union is all of X [1]. Given such a cover, a *finite subcover* is a subfamily $(U_i)_{i \in J}$ where $J \subseteq I$ is finite and the union is still all of X [1]. We say that X is *compact* if every open cover has a finite subcover [1].

(c) **Similar examples seen** If $(x, y, z) \in X$ then $x^4 \leq x^4 + y^4 + z^4 = 1$ so $|x| \leq 1$. Similarly, we see that $|y| \leq 1$ and $|z| \leq 1$, which implies that X is bounded [2]. Also, we can define $f : \mathbb{R}^4 \to \mathbb{R}$ by $f(x, y, z) = x^4 + y^4 + z^4$. This is continuous (because it is polynomial) and $\{1\}$ is closed in $\mathbb{R}$ so the set $X = f^{-1}\{1\}$ is closed in $\mathbb{R}^4$ [2]. Any bounded closed subset of $\mathbb{R}^n$ is compact, so we deduce that X is compact as claimed [1].

(d) **Similar examples seen** For $n > 0$ put $U_n = \{(x, y, z) \in \mathbb{R}^3 \mid x^4 + y^4 + z^4 < 1 - n^{-1}\}$, so these sets are form an open cover of Y [2]. However, U_n is not all of Y, because the point $((1 - 1/(n + 1))^{1/4}, 0, 0)$ lies in $Y \setminus U_n$ [1]. If Y was compact then we would have a finite subcover, say $Y = U_{n_1} \cup \cdots \cup U_{n_p}$ and this would give $Y = U_n$ where $n = \max(n_1, \ldots, n_p)$, which is a contradiction; so Y is not compact. [2]

(e) Suppose that Y and Z are compact subsets of X; we claim that $Y \cup Z$ is also compact, To see this, let $(U_i)_{i \in I}$ be a family of open subsets of X that covers $Y \cup Z$; we must show that there is a finite subcover [1]. As the family covers $Y \cup Z$, it certainly covers Y, and Y is compact, so we can choose indices $i_1, \ldots, i_p$ with $Y \subseteq U_{i_1} \cup \cdots \cup U_{i_p}$ [1]. Similarly, we can choose indices $i_{p+1}, \ldots, i_{p+q}$ such that $Z \subseteq U_{i_{p+1}} \cup \cdots \cup U_{i_{p+q}}$. It follows that $Y \cup Z \subseteq U_{i_1} \cup \cdots \cup U_{i_{p+q}}$, so we have the required finite subcover [2].

(f) Let Y and Z be topological spaces such that $Z \neq \emptyset$ (so we can choose $z_0 \in Z$). Suppose that $Y \times Z$ is compact; we claim that Y is also compact. Because $\pi(y, z_0) = y$, we see that the projection $Y \times Z \to Y$ is surjective (and also continuous, by the definition of the product topology) [2]. A standard theorem says that if $f : A \to B$ is continuous and surjective and A is compact then B is also compact [2]. Using this, we see that Y is compact as claimed [1]. (It is also not hard to prove this directly by consideration of open covers.)

2 Path components

(8)

(a) What does it mean to say that a topological space X is *path-connected*?

(b) Prove that the space S^n is path-connected for all $n > 0$.

(c) Let X be a subset of $\mathbb{R}^n$, and let a be a point in X. What does it mean to say that X is *star-shaped* around a? Show that if X is star-shaped around a, then it is path-connected.

(d) Suppose that $f : X \to \mathbb{R}$ is continuous, $f(x)$ is nonzero for all x, and there exist $x_0, x_1 \in X$ with $f(x_0) < 0 < f(x_1)$. Prove that X is not path-connected.

(e) Recall that $GL_3(\mathbb{R})$ is the space of 3×3 invertible matrices over $\mathbb{R}$. Prove that this space is not path-connected.

Solution:

(a) A space X is *path-connected* if for each pair of points $x_0, x_1 \in X$ there exists a continuous map $u\colon I \to X$ such that $u(0) = x_0$ and $u(1) = x_1$.

(b) Suppose that $n > 1$ and that $x_0, x_1 \in S^n$. Suppose first that $x_1 \neq -x_0$, so that the line segment from x_0 to x_1 does not pass through the origin. Thus, if we put $f(t) = (1-t)x_0 + tx_1$ then $f(t) \neq 0$ for all $t \in I$. We can thus define a continuous map $u\colon I \to S^n$ by $u(t) = f(t)/\|f(t)\|$ and this satisfies $u(0) = x_0/\|x_0\| = x_0$ and $u(1) = x_1$ as required.

Now consider the exceptional case where $x_1 = -x_0$. As $n > 0$ the set S^n has more than two points so we can choose a point x_2 that is different from both $-x_0$ and $-x_1$. By the first part of the proof we can define a path u from x_0 to x_2 and a path v from x_1 to x_2 in S^n. This gives a path $w := u * \overline{v}$ from x_0 to x_2.

(c) A subset $X \subseteq \mathbb{R}^n$ is *star-shaped* around a point $a \in X$ if for all $x \in X$, the linear path from x to a (given by the formula $u(t) = (1-t)x + ta$, which is meaningful because x and a are vectors in $\mathbb{R}^n$) lies wholly in X.

Suppose that this holds. For any $x_0, x_1 \in X$ we can let u_0 be the linear path from x_0 to a and let u_1 be the linear path from x_1 to a. Then $u_0 * \overline{u}_1$ is a path from x_0 to x_1, showing that X is path-connected.

(d) Suppose that $f\colon X \to \mathbb{R}$ is continuous, $f(x)$ is nonzero for all x, and there exist $x_0, x_1 \in X$ with $f(x_0) < 0 < f(x_1)$. I claim that there is no continuous path in X from x_0 to x_1, so that X is not path-connected. Indeed, if u is such a path, put $g(t) = f(u(t))$, giving a continuous function $g\colon I \to \mathbb{R}$. We have $g(0) = f(x_0) < 0$ and $g(1) = f(x_1) > 0$. By the Intermediate Value Theorem, there must be some $t \in I$ with $g(t) = 0$, or in other words $f(u(t)) = 0$. However, $u(t) \in X$, and $f(x) \neq 0$ for all $x \in X$ by assumption. This contradiction shows that there can be no such map u.

(e) Consider the map $\det\colon GL_3(\mathbb{R}) \to \mathbb{R}$. As $\det(A)$ is a polynomial expression in the entries of the matrix A, we see that det is continuous. If $A \in GL_3(\mathbb{R})$ then A is invertible, so $\det(A) \neq 0$. The matrices

$$A_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

lie in $GL_3(\mathbb{R})$ and satisfy $\det(A_0) < 0 < \det(A_1)$. It follows from the previous part that $GL_3(\mathbb{R})$ is disconnected.

(9)

(a) Let X be a topological space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is an equivalence relation.

(b) Let $f\colon X \to Y$ be a continuous map. Define the induced map $f_*\colon \pi_0(X) \to \pi_0(Y)$, and prove that it is well-defined.

(c) Show that if $f, g\colon X \to Y$ are homotopic maps then $f_* = g_*\colon \pi_0(X) \to \pi_0(Y)$.

(d) Put $X = [-3,-2] \cup [-1,1] \cup [2,3]$ and $Y = [0,1] \cup [2,10]$, and define $f\colon X \to Y$ by $f(x) = x^2$. Describe the sets $\pi_0(X)$ and $\pi_0(Y)$ and the map $f_*\colon \pi_0(X) \to \pi_0(Y)$.

Solution:

(a) We write $x \sim y$ iff there is a path in X from x to y, in other words a continuous map $s\colon I \to X$ such that $s(0) = x$ and $s(1) = y$. For any $x \in X$ we can define $c_x\colon I \to X$ by $c_x(t) = x$ for all t; this is a path from x to x, proving that $x \sim x$. If $x \sim y$ then there is a path s from x to y and we can define a path $\overline{s}$ from y to x by $\overline{s}(t) = s(1-t)$; this shows that $y \sim x$. If there is also a path r from y to z then we can define a path $s * r$ from x to z by

$$(s * r)(t) = \begin{cases} s(2t) & \text{if } 0 \leq t \leq 1/2 \\ r(2t-1) & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

and this shows that $x \sim z$. Thus $\sim$ is reflexive, symmetric and transitive and thus is an equivalence relation.

(b) Let c be an element of $\pi_0(X)$, in other words a path component in X. For any $x \in c$ we have a point $f(x) \in Y$, and thus a path-component $[f(x)] \in \pi_0(Y)$. If x' is another point in c then $x \sim x'$ so we can choose a path s from x to x' in X. Thus $f \circ s\colon I \to Y$ is a path in Y from $f(x)$ to $f(x')$, so $f(x) \sim f(x')$, so $[f(x)] = [f(x')]$. We can thus define $f_*(c) = [f(x)]$; this is independent of the choice of x and thus is well-defined.

(c) If $f, g\colon X \to Y$ are homotopic then we can chooose a map $h\colon I \to X \to Y$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all x. If $c \in \pi_0(X)$ we can choose $x \in X$ and note that $f_*(c) = [f(x)]$ and $g_*(c) = [g(x)]$. We can also define a map $s\colon I \to Y$ by $s(t) = h(t, x)$. This gives a path from $s(0) = f(x)$ to $s(1) = g(x)$, so $[f(x)] = [g(x)]$, in other words $f_*(c) = g_*(c)$.

(d) Write

$$
\begin{aligned}
a &= [-3, -2] \\
b &= [-1, 1] \\
c &= [2, 3] \\
d &= [0, 1] \\
e &= [2, 11]
\end{aligned}
$$

Then $\pi_0(X) = \{a, b, c\}$ and $\pi_0(Y) = \{c, d\}$. The map $f_*\colon \pi_0(X) \to \pi_0(Y)$ is given by $f_*(a) = f_*(c) = e$ and $f_*(b) = d$.

(10)

(a) Let X be a topoological space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is indeed an equivalence relation. **(8 marks)**

(b) Let $f\colon X \to Y$ be a continuous map. Define the function $f_*\colon \pi_0(X) \to \pi_0(Y)$, and check that it is well-defined. **(5 marks)**

(c) Suppose that Y is path-connected and X is not. Show that there do not exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to the identity map id_X. **(6 marks)**

(d) Put $X = \{A \in M_2\mathbb{R} \mid A^2 = A\}$. What can you say about $\det(A)$ when $A \in X$? Show that X is not path-connected. **(6 marks)**

Solution:

(a) Write $x \sim y$ iff there is a path in X from x to y [1], or in other words a continuous map $u\colon I \to X$ such that $u(0) = x$ and $u(1) = y$ [1]. I claim that this is an equivalence relation. Indeed, given $x \in X$ we can define $c_x\colon I \to X$ by $c_x(t) = x$ for all t. This gives a path from x to itself, showing that $\sim$ is reflexive [1]. Next, suppose that $x \sim y$, so there exists a path u from x to y in X. We can then define $\overline{u}(t) = u(1-t)$ to get a path from y to x, showing that $y \sim x$, showing that $\sim$ is symmetric [2]. Finally, suppose we have a path u from x to y, and a path v from y to z. We then define a map $w\colon I \to X$ by

$$
w(t) = \begin{cases} u(2t) & \text{if } 0 \leq t \leq 1/2 \\ v(2t-1) & \text{if } 1/2 \leq t \leq 1.\text{[2]} \end{cases}
$$

This is well-defined and continuous because $u(1) = y = v(0)$. We have $w(0) = u(0) = x$ and $w(1) = v(1) = z$, so w gives a path from x to z; this proves that $\sim$ is transitive [1]. **[bookwork]**

(b) Let $f\colon X \to Y$ be a continuous map. We define $f_*\colon \pi_0(X) \to \pi_0(Y)$ by $f_*([x]) = [f(x)]$ [1](where $[x]$ is the equivalence class of x under the relation $\sim$). To see that this is well-defined, suppose that $[x_0] = [x_1]$ in $\pi_0(X)$ [1]. This means that $x_0 \sim x_1$, so there is a path $u\colon I \to X$ from x_0 to x_1 [1]. The function $f \circ u\colon I \to Y$ gives a path from $f(x_0)$ to $f(x_1)$ in Y [1], so $[f(x_0)] = [f(x_1)]$ as required [1]. **[bookwork]**

(c) Suppose that Y is path-connected, so $\pi_0(Y)$ has only a single element, which we will call b. Then $f_*\colon \pi_0(X) \to \pi_0(Y)$ must be the constant map with value b, so $g_*f_*\colon \pi_0(X) \to \pi_0(X)$ must be the constant map with value $g_*(b)$. On the other hand, if $gf \simeq 1$ then g_*f_* is the identity. Thus, the identity map of $\pi_0(X)$ is constant, so $\pi_0(X)$ can only have a single element. This means that X is path-connected, contrary to assumption. [6] **[similar examples seen]**

(d) Put $X = \{A \in M_2\mathbb{R} \mid A^2 = A\}$. For $A \in X$ we have $\det(A)^2 = \det(A)$ so $\det(A) \in \{0, 1\}$ [2]. We can thus regard det as a continuous map $X \to \mathbb{R}$ such that $\det(A) \neq 1/2$ for all A. The zero matrix and the identity matrix lie in X, with $\det(0) = 0 < 1/2$ and $\det(I) = 1 > 1/2$. It follows that 0 cannot be connected to I by a path in X, so X is not path-connected. [4] [**similar examples seen]A proposition proved in lectures says that if $f\colon X \to \mathbb{R}$ is nowhere zero and $f(x) < 0$ and $f(y) > 0$ then $x \not\sim y$. A number of examples were discussed, including some where the "missing value" is not zero. In particular, the trace was used to show that $\{A \in M_n\mathbb{R} \mid A^2 = A\}$ is disconnected for $n > 1$.**

(11) 2021-22 Mock Q2:

(a) Let X be a topological space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is an equivalence relation. **(6 marks)**

(b) Let $f\colon X \to Y$ be a continuous map. Define the induced map $f_*\colon \pi_0(X) \to \pi_0(Y)$, and prove that it is well-defined. **(4 marks)**

(c) Show that if $f, g\colon X \to Y$ are homotopic maps then $f_* = g_*\colon \pi_0(X) \to \pi_0(Y)$. **(4 marks)**

(d) Let Y and Z be topological spaces. Construct a bijection $\pi_0(Y \times Z) \to \pi_0(Y) \times \pi_0(Z)$, and prove that it is a bijection. **(5 marks)**

(e) Define $i\colon \mathbb{Z} \to \mathbb{R} \setminus \mathbb{Z}$ by $i(n) = n + \frac{1}{2}$. Prove that there do not exist continuous maps $\mathbb{Z} \xrightarrow{f} S^2 \times S^2 \xrightarrow{g} \mathbb{R} \setminus \mathbb{Z}$ such that i is homotopic to $g \circ f$. **(6 marks)**

Solution:

(a) We write $x \sim y$ iff there is a path in X from x to y, in other words a continuous map $s\colon I \to X$ such that $s(0) = x$ and $s(1) = y$ [2]. For any $x \in X$ we can define $c_x\colon I \to X$ by $c_x(t) = x$ for all t; this is a path from x to x, proving that $x \sim x$ [1]. If $x \sim y$ then there is a path s from x to y and we can define a path $\overline{s}$ from y to x by $\overline{s}(t) = s(1 - t)$; this shows that $y \sim x$ [1]. If there is also a path r from y to z then we can define a path $s * r$ from x to z by

$$(s * r)(t) = \begin{cases} s(2t) & \text{if } 0 \le t \le 1/2 \\ r(2t - 1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

This is well-defined because $s(1) = y = r(0)$, and it is continuous by closed patching [1]. This shows that $x \sim z$ [1]. Thus $\sim$ is reflexive, symmetric and transitive and thus is an equivalence relation.

(b) Let c be an element of $\pi_0(X)$, in other words a path component in X. For any $x \in c$ we have a point $f(x) \in Y$, and thus a path-component $[f(x)] \in \pi_0(Y)$. If x' is another point in c then $x \sim x'$ so we can choose a path s from x to x' in X [1]. Thus $f \circ s\colon I \to Y$ is a path in Y from $f(x)$ to $f(x')$ [1], so $f(x) \sim f(x')$, so $[f(x)] = [f(x')]$ [1]. We can thus define $f_*(c) = [f(x)]$; this is independent of the choice of x and thus is well-defined [1].

(c) If $f, g\colon X \to Y$ are homotopic then we can chooose a map $h\colon I \to X \to Y$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all x [1]. If $c \in \pi_0(X)$ we can choose $x \in X$ and note that $f_*(c) = [f(x)]$ and $g_*(c) = [g(x)]$. We can also define a map $s\colon I \to Y$ by $s(t) = h(t, x)$ [2]. This gives a path from $s(0) = f(x)$ to $s(1) = g(x)$, so $[f(x)] = [g(x)]$, in other words $f_*(c) = g_*(c)$ [1].

(d) Suppose we have topological spaces Y and Z. Let $p\colon Y \times Z \to Y$ and $q\colon Y \times Z \to Z$ be the projection maps, defined by $p(y, z) = y$ and $q(y, z) = z$ [1]. Define $\phi\colon \pi_0(Y \times Z) \to \pi_0(Y) \times \pi_0(Z)$ by $\phi(c) = (p_*(c), q_*(c))$, so $\phi([y, z]) = ([y], [z])$ [1]. Any element of $\pi_0(Y) \times \pi(0(Z)$ has the form (b, c), where $b \in \pi_0(Y)$ and $c \in \pi_0(Z)$. We can then choose $y \in Y$ and $z \in Z$ such that $b = [y]$ and $c = [z]$. This gives an element $(y, z) \in Y \times Z$ and a path component $[y, z] \in \pi_0(Y \times Z)$ with $\phi([y, z]) = ([y], [z]) = (b, c)$. This shows that ϕ is surjective [1]. Now suppose we have two path components $[y, z]$ and $[y', z']$ in $\pi_0(Y \times Z)$ which satisfy $\phi([y, z]) = \phi([y', z'])$. This means that $([y], [z]) = ([y'], [z'])$, so $[y] = [y']$ and $[z] = [z']$. As $[y] = [y']$ in $\pi_0(Y)$ we can choose a continuous map $v\colon [0, 1] \to Y$ with $v(0) = y$ and $v(1) = y'$. Similarly, we can choose a continuous map $w\colon [0, 1] \to Z$ with $w(0) = z$ and $w(1) = z'$. Now define $u\colon [0, 1] \to Y \times Z$ by $u(t) = (v(t), w(t))$, noting that this is continuous by the universal property of the product topology. We have $u(0) = (y, z)$ and $u(1) = (y', z')$ so $[y, z] = [y', z']$ in $\pi_0(Y \times Z)$[2]. This proves that ϕ is also injective, and so is a bijection.

(e) Suppose (for a contradiction) that i is homotopic to $g \circ f$ for some continuous maps $\mathbb{Z} \xrightarrow{f} S^2 \times S^2 \xrightarrow{g} \mathbb{R} \setminus \mathbb{Z}$ [1]. It then follows from (c) that $i_* = g_* \circ f_*$ [1]. However, it is standard that S^2 is path connected [1], or equivalently that $|\pi_0(S^2)| = 1$. It follows using (d) that $S^2 \times S^2$ is also path connected [1], so $f_*([-1]) = f_*([0])$ in $\pi_0(S^2 \times S^2)$, so $g_*(f_*([-1])) = g_*(f_*([0]))$, so $i_*([-1]) = i_*([0])$ in $\pi_0(\mathbb{R} \setminus \mathbb{Z})$ [1]. This means that there is a path from $-\frac{1}{2}$ to $\frac{1}{2}$ in $\mathbb{R} \setminus \mathbb{Z}$, which violates the Intermediate Value Theorem[1].

(12) 2023-24 Q2:

(a) Let X be a topological space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is indeed an equivalence relation. **(8 marks)**

(b) Let $f\colon X \to Y$ be a continuous map. Define the function $f_*\colon \pi_0(X) \to \pi_0(Y)$, and check that it is well-defined. **(5 marks)**

(c) Suppose that Y is path-connected and X is not. Show that there do not exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to the identity map id_X. **(6 marks)**

(d) Put $X = \{A \in M_2\mathbb{R} \mid A^2 = A\}$ (where $M_2\mathbb{R}$ is the space of 2×2 real matrices). What can you say about $\det(A)$ when $A \in X$? Show that X is not path-connected. **(6 marks)**

Solution:

(a) **Bookwork** Write $x \sim y$ iff there is a path in X from x to y [1], or in other words a continuous map $u\colon I \to X$ such that $u(0) = x$ and $u(1) = y$ [1]. I claim that this is an equivalence relation. Indeed, given $x \in X$ we can define $c_x\colon I \to X$ by $c_x(t) = x$ for all t. This gives a path from x to itself, showing that $\sim$ is reflexive [1]. Next, suppose that $x \sim y$, so there exists a path u from x to y in X. We can then define $\overline{u}(t) = u(1-t)$ to get a path from y to x, showing that $y \sim x$, showing that $\sim$ is symmetric [2]. Finally, suppose we have a path u from x to y, and a path v from y to z. We then define a map $w\colon I \to X$ by

$$w(t) = \begin{cases} u(2t) & \text{if } 0 \le t \le 1/2 \\ v(2t-1) & \text{if } 1/2 \le t \le 1. \text{[2]} \end{cases}$$

This is well-defined and continuous because $u(1) = y = v(0)$. We have $w(0) = u(0) = x$ and $w(1) = v(1) = z$, so w gives a path from x to z; this proves that $\sim$ is transitive [1].

(b) **Bookwork** Let $f\colon X \to Y$ be a continuous map. We define $f_*\colon \pi_0(X) \to \pi_0(Y)$ by $f_*([x]) = [f(x)]$ [1](where $[x]$ is the equivalence class of x under the relation $\sim$). To see that this is well-defined, suppose that $[x_0] = [x_1]$ in $\pi_0(X)$ [1]. This means that $x_0 \sim x_1$, so there is a path $u\colon I \to X$ from x_0 to x_1 [1]. The function $f \circ u\colon I \to Y$ gives a path from $f(x_0)$ to $f(x_1)$ in Y [1], so $[f(x_0)] = [f(x_1)]$ as required [1].

(c) **Slightly disguised bookwork** Suppose that Y is path-connected, so $\pi_0(Y)$ has only a single element, which we will call b. Then $f_*\colon \pi_0(X) \to \pi_0(Y)$ must be the constant map with value b, so $g_*f_*\colon \pi_0(X) \to \pi_0(X)$ must be the constant map with value $g_*(b)$. On the other hand, if $gf \simeq 1$ then g_*f_* is the identity. Thus, the identity map of $\pi_0(X)$ is constant, so $\pi_0(X)$ can only have a single element. This means that X is path-connected, contrary to assumption. [6]

(d) **Similar examples seen** Put $X = \{A \in M_2\mathbb{R} \mid A^2 = A\}$. For $A \in X$ we have $\det(A)^2 = \det(A)$ so $\det(A) \in \{0, 1\}$ [2]. We can thus regard det as a continuous map $X \to \mathbb{R}$ such that $\det(A) \neq 1/2$ for all A. The zero matrix and the identity matrix lie in X, with $\det(0) = 0 < 1/2$ and $\det(I) = 1 > 1/2$. It follows that 0 cannot be connected to I by a path in X, so X is not path-connected. [4]

3 The fundamental group

These questions involve material that is not covered in the current version of the course.

(13)

(a) Let X be a topoological space, and let x_0 and x_1 be points in X. What does it mean to say that two paths from x_0 to x_1 are *pinned homotopic*? Define the set $\pi_1(X; x_0, x_1)$.

(b) Let X be path-connected. Prove that the group $\pi_1(X; x_0)$ is isomorphic to the group $\pi_1(X; x_1)$.

(c) Put $X = \{(w,x,y,z) \in \mathbb{C}^4 \mid w \neq x \ , \ x \neq y \ , \ y \neq z\}$, and take $x_0 = (0,1,2,3)$ as the basepoint in X. Calculate $\pi_1(X)$. (You may wish to consider the expression $f(w,x,y,z) = (w, x-w, y-x, z-y)$.)

Solution:

(a) Let X be a topoological space, and let x_0 and x_1 be two points in X. Let $u, v \colon I \to X$ be two paths, both of which start at x_0 and end at x_1. We say that u and v are *pinned homotopic* if there exists a map $h \colon I \times I \to X$ such that

- $h(0,t) = u(t)$ for all $t \in I$
- $h(1,t) = v(t)$ for all $t \in I$
- $h(s,0) = x_0$ for all $s \in I$
- $h(s,1) = x_1$ for all $s \in I$.

This is an equivalence relation on the set of all paths from x_0 to x_1 in X; the set $\pi_1(X; x_0, x_1)$ is just the set of equivalence classes.

(b) Now suppose that X is path-connected, so we can choose a path u from x_0 to x_1 in X, and put $q = [u] \in \pi_1(X; x_0, x_1)$. If $a \in \pi_1(X; x_0) = \pi_1(X; x_0, x_0)$ then q^{-1} runs from x_1 to x_0, and a runs from x_0 to x_0, and q runs from x_0 to x_1, so $q^{-1}aq$ runs from x_1 to itself. We can thus define a function $f \colon \pi_1(X; x_0) \to \pi_1(X; x_1)$ by $f(a) = q^{-1}aq$. Similarly, we can define $g \colon \pi_1(X; x_1) \to \pi_1(X; x_0)$ by $g(b) = qbq^{-1}$. Clearly $g(f(a)) = qq^{-1}aqq^{-1} = a$ and similarly $f(g(b)) = b$, so f is a bijection with inverse g. Moreover, $f(a)f(a') = q^{-1}aqq^{-1}a'q = q^{-1}aa'q = f(aa')$, so f is a group homomorphism, and thus an isomorphism $\pi_1(X; x_0) \to \pi_1(X; x_1)$.

(c) Define $f \colon X \to \mathbb{C} \times \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times$ by $f(w,x,y,z) = (w, x-w, y-x, z-y)$ (where $\mathbb{C}^\times$ means $\mathbb{C} \setminus \{0\}$). This is a homeomorphism, with inverse $f^{-1}(a,b,c,d) = (a, a+b, a+b+c, a+b+c+d)$. On the other hand, $\mathbb{C}$ is homotopy equivalent to a point, and $\mathbb{C}^\times$ is homotopy equivalent to S^1; it follows that X is homotopy equivalent to $S^1 \times S^1 \times S^1$, and thus that $\pi_1(X)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

(14)

(a) Let X be a based topological space, and let Y be a subspace of X containing the basepoint. What does it mean to say that Y is a *retract* of X?

(b) Prove that if Y is a retract of X, then $|\pi_1(Y)| \leq |\pi_1(X)|$.

(c) Recall that $\mathbb{R}P^3$ is a subspace of the space $M_4(\mathbb{R})$ of all 4×4 matrices over $\mathbb{R}$, which is homeomorphic to $\mathbb{R}^{16}$. Prove that $\mathbb{R}P^3$ is not a retract of $M_4(\mathbb{R})$.

(d) Recall that $U(3)$ is the space of 3×3 matrices A over $\mathbb{C}$ such that $A^\dagger A = I$. You may assume that for such A we have $\det(A) \in S^1$. Define $j \colon S^1 \to U(3)$ by

$$j(z) = \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

What is $\det(j(z))$? Deduce that $\pi_1(U(3))$ is infinite.

Solution:

(a) Let X be a based topological space, let Y be a subspace of X containing the basepoint, and let $i \colon Y \to X$ be the inclusion map. We say that Y is a *retract* of X if there exists a continuous map $r \colon X \to Y$ such that $r \circ i = \mathrm{id}_Y$, or equivalently $r(y) = y$ for all $y \in Y$.

(b) Suppose that Y is a retract of X. We then have homomorphisms $i_* \colon \pi_1(Y) \to \pi_1(X)$ and $r_* \colon \pi_1(X) \to \pi_1(Y)$ such that $r_* i_* = 1 \colon \pi_1(Y) \to \pi_1(Y)$. Now let a and a' be two different elements of $\pi_1(Y)$. Then $r_*(i_*(a)) = a \neq a' = r_*(i_*(a'))$, so clearly $i_*(a)$ cannot be the same as $i_*(a')$. Thus, all the different elements of $\pi_1(Y)$ are mapped to different elements of $\pi_1(X)$, so there must be at least as many elements in $\pi_1(X)$ as there are in $\pi_1(Y)$. In other words, we have $|\pi_1(Y)| \leq |\pi_1(X)|$.

(c) We have $|\pi_1(\mathbb{R}P^3)| = 2$ and $M_4(\mathbb{R}) \simeq \mathbb{R}^{16}$ is contractible so $|\pi_1(M_4(\mathbb{R}))| = 1$. By the previous part, $\mathbb{R}P^3$ cannot be a retract of $M_4(\mathbb{R})$.

(d) It is easy to see that $\det(j(z)) = z$, so $\det \circ j = 1 \colon S^1 \to S^1$. It follows that the maps $j_* \colon \pi_1(S^1) \to \pi_1(U(3))$ and $\det_* \colon \pi_1(U(3)) \to \pi_1(S^1)$ satisfy $\det_* \circ j_* = 1 \colon \pi_1(S^1) \to \pi_1(S^1)$. By the logic of part (b) we see that $|\pi_1(U(3))| \geq |\pi_1(S^1)| = |\mathbb{Z}| = \infty$.

4 Homotopy equivalence

(15)

(a) Let $f, g\colon X \to Y$ be continuous maps between topological spaces. What does it mean to say that f is *homotopic* to g?

(b) Let X and Y be topological spaces. What does it mean to say that X and Y are *homotopy equivalent*?

(c) Show that if X and Y are homotopy equivalent then there is a bijection between the sets of path-components $\pi_0(X)$ and $\pi_0(Y)$.

(d) Consider the cross $X = \{(x,0) \mid -1 \leq x \leq 1\} \cup \{(0,y) \mid -1 \leq y \leq 1\}$, and let $C = \mathbb{R}^2 \setminus X$ be its complement. Prove that C is homotopy equivalent to S^1.

Solution:

(a) We say that f and g are homotopic if there exists a continuous map $h\colon I \times X \to Y$ such that $h(0,x) = f(x)$ and $h(1,x) = g(x)$ for all $x \in X$.

(b) We say that spaces X and Y are homotopy equivalent if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to 1_X and fg is homotopic to 1_Y.

(c) For any map $u\colon X \to Y$ we have an induced function $u_*\colon \pi_0(X) \to \pi_0(Y)$, given by $u_*\langle x\rangle = \langle u(x)\rangle$ for all $x \in X$. These maps satisfy $1_* = 1$ and $(vu)_* = v_*u_*$, and $u'_* = u_*$ if u' is homotopic to u. If f, g are as above we then have maps $f_*\colon \pi_0(X) \to \pi_0(Y)$ and $g_*\colon \pi_0(Y) \to \pi_0(X)$, satisfying

$$f_*g_* = (fg)_* = (1_Y)_* = 1_{\pi_0(Y)}$$

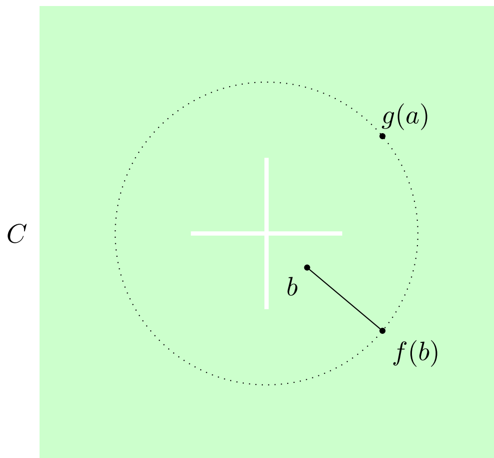
$$g_*f_* = (gf)_* = (1_X)_* = 1_{\pi_0(X)}.$$

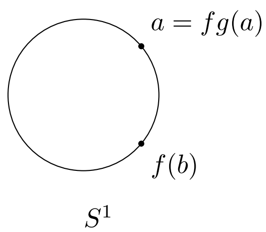
Thus g_* is an inverse for f_*, so f_* is a bijection.

(d) Define maps as follows:

$$\begin{array}{ll} f\colon C \to S^1 & f(x,y) = (x,y)/\sqrt{x^2+y^2} \\ g\colon S^1 \to C & g(x,y) = (2x,2y) \\ h\colon I \times C \to C & h(t,x,y) = (1-t)(x,y) + t(x,y)/\sqrt{x^2+y^2}. \end{array}$$

Then $fg = 1_{S^1}$, and h is a (linear) homotopy from 1_C to gf, so f is a homotopy equivalence.





(16) Consider a metric space X.

(a) (i) What does it mean to say that a subset U of X is open?

(ii) What does it mean to say that a subset F of X is closed?

(b) Show that a subset $F \subseteq X$ is closed iff for every sequence (x_n) in F that converges to a point $x \in X$, we actually have $x \in F$.

(c) Explain what it means for a subset $A \subseteq X$ to be compact. Show that if A is compact and $f\colon X \to Y$ is continuous then $f(A)$ is compact.

(d) Prove that the space $[0,1]$ is compact. Show that there is a continuous bijection $g\colon [-1,-1/2) \cup [1/2,1] \to [0,1]$; can it be chosen to be a homeomorphism?

Solution:

(a) We say that $U \subseteq X$ is *open* if for each point $x \in U$, there exists $\epsilon > 0$ such that the open ball $OB(x,\epsilon) = \{y \in X \mid d(x,y) < \epsilon\}$ is contained in U.

We say that $F \subseteq X$ is closed if the complement $X \setminus F$ is open.

(b) Suppose that F is closed, and that (x_n) is a sequence in F converging to a point $x \in X$. I claim that $x \in F$. If not, then x lies in the open set $X \setminus F$, so there exists $\epsilon > 0$ such that $\overset{\circ}{B}_\epsilon(x) \subseteq X \setminus F$, or equivalently $\overset{\circ}{B}_\epsilon(x) \cap F = \emptyset$. Because $x_n \to x$, there exists N such that $d(x_n, x) < \epsilon$ when $n \geq N$, or in other words $x_n \in \overset{\circ}{B}_\epsilon(x)$ when $n \geq N$. On the other hand, we have $x_n \in F$ for all n by assumption, so for $n \geq N$ we have $x_n \in \overset{\circ}{B}_\epsilon(x) \cap F = \emptyset$, which is impossible. Thus $x \in F$ after all.

Conversely, suppose that F satisfies the condition on sequences; we need to prove that F is closed, or equivalently that $X \setminus F$ is open. If not, then there exists $x \in X \setminus F$ such that $\overset{\circ}{B}_\epsilon(x)$ is not contained in $X \setminus F$ for any $\epsilon > 0$. In particular, $\overset{\circ}{B}_{1/n}(x)$ is not contained in $X \setminus F$, so we can choose a point $x_n \in \overset{\circ}{B}_{1/n}(x) \cap F$. As $x_n \in \overset{\circ}{B}_{1/n}(x)$ we have $d(x_n, x) < 1/n$ so $x_n \to x$. Thus (x_n) is a sequence in F converging to the point x outside F, contradicting the condition on sequences.

(c) Not written

(d) Not written

(17)

(a) What does it mean to say that a topological space X is *homotopy equivalent* to a topoological space Y? Show that the relation of homotopy equivalence is an equivalence relation.

(b) What does it mean for a space to be (a) *contractible* and (b) *path connected*? Show that any contractible space is path connected. Is the reverse implication true?

(c) Consider the rational comb space

$$X = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0 \text{ or } x \in \mathbb{Q}\}.$$

Show that X is homotopy equivalent to the upper half plane $Y = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$, and deduce that X is contractible.

Solution:

(a) We say that X is homotopy equivalent to Y if there exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$ (where $p \simeq q$ means that p is homotopic to q).

Clearly any space X is homotopy equivalent to itself, because we can take $f = g = \mathrm{id}_X$.

If X is homotopy equivalent to Y then by reversing the rôles of f and g we see that Y is homotopy equivalent to X.

Now suppose that X is homotopy equivalent to Y and that Y is homotopy equivalent to Z. We can then choose maps f and g as above, and also maps $u\colon Y \to Z$ and $v\colon Z \to Y$ such that $uv \simeq \mathrm{id}_Z$ and $vu \simeq \mathrm{id}_Y$. These give maps $uf\colon X \to Z$ and $gv\colon Z \to X$ such that

$$(uf)(gv) = u(fg)v \simeq u\,\mathrm{id}_Y\,v = uv \simeq \mathrm{id}_Z$$

$$(gv)(uf) = g(vu)f \simeq g\,\mathrm{id}_Y\,f = gf \simeq \mathrm{id}_X,$$

so X is homotopy equivalent to Z.

This shows that the relation of homotopy equivalence is an equivalence relation.

(b) A space X is contractible if it is equivalent to the one-point space $\{0\}$. It is path connected if for any two points $x, y \in X$ there is a path $s\colon I \to X$ with $s(0) = x$ and $s(1) = y$.

Suppose that X is contractible, so we have maps $f\colon X \to \{0\}$ and $g\colon \{0\} \to X$ and a homotopy $h\colon \mathrm{id}_X \simeq gf$. Write $a = g(0) \in X$. Note that we must have $f(x) = 0$ for all $x \in X$, because there are no other points in $\{0\}$ that $f(x)$ could be. Thus $gf(x) = a$ for all x. As h is a homotopy from 1 to gf, we have $h(0, x) = x$ and $h(1, x) = a$ for all x. Thus, for any point $x \in X$ we can define a path $s_x\colon I \to X$ by $s_x(t) = h(t, x)$. This starts at x and ands at a. If y is any other point in X we can take the join of s_x with the reverse of s_y to get a path from x to y. Thus X is path connected.

On the other hand, a path connected space need not be contractible. For example, the space S^1 is path-connected (we can define a path from $e^{i\theta}$ to $e^{i\phi}$ by $s(t) = e^{i((1-t)\theta + t\phi)}$) but not contractible (because $H_1(S^1) \neq 0$).

(c) Let $i\colon Y \to X$ be the inclusion, and define $r\colon X \to Y$ by $r(x, y) = (x, \max(0, y))$. We then have $rj = 1$. I claim that if $(x, y) \in X$ then the line segment joining (x, y) to $jr(x, y)$ is contained in X. If $y \geq 0$ then $rj(x, y) = (x, y)$ and the claim is clear. If $y < 0$ then (as $(x, y) \in X$) we must have $x \in \mathbb{Q}$. We also have $rj(x, y) = (x, 0)$ so the line segment in question is the set of points (x, w) with $y \leq w \leq 0$. As $x \in \mathbb{Q}$, all these points lie in X as required. Thus rj is linearly homotopic to id_X, which implies that j is a homotopy equivalence.

The set Y is convex and thus contractible, in other words homotopy equivalent to a point. As homotopy equivalence is an equivalence relation, we deduce that X is also homotopy equivalent to a point.

(18)

(a) What does it mean to say that a topoological space X is *homotopy equivalent* to a topoological space Y? Show that the relation of homotopy equivalence is an equivalence relation.

(b) What does it mean for a space to be (i) *contractible* and (ii) *path connected*? Show that any contractible space is path connected. Is the reverse implication true?

(c) Consider the rational comb space

$$X = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0 \text{ or } x \in \mathbb{Q}\}.$$

Show that X is homotopy equivalent to the upper half plane $Y = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, and deduce that X is contractible.

Solution:

(a) We say that X is homotopy equivalent to Y if there exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$ (where $p \simeq q$ means that p is homotopic to q).

Clearly any space X is homotopy equivalent to itself, because we can take $f = g = \mathrm{id}_X$.

If X is homotopy equivalent to Y then by reversing the rôles of f and g we see that Y is homotopy equivalent to X.

Now suppose that X is homotopy equivalent to Y and that Y is homotopy equivalent to Z. We can then choose maps f and g as above, and also maps $u\colon Y \to Z$ and $v\colon Z \to Y$ such that $uv \simeq \mathrm{id}_Z$ and $vu \simeq \mathrm{id}_Y$. These give maps $uf\colon X \to Z$ and $gv\colon Z \to X$ such that

$$(uf)(gv) = u(fg)v \simeq u\,\mathrm{id}_Y\, v = uv \simeq \mathrm{id}_Z$$
$$(gv)(uf) = g(vu)f \simeq g\,\mathrm{id}_Y\, f = gf \simeq \mathrm{id}_X,$$

so X is homotopy equivalent to Z.

This shows that the relation of homotopy equivalence is an equivalence relation.

(b) A space X is contractible if it is equivalent to the one-point space $\{0\}$. It is path connected if for any two points $x, y \in X$ there is a path $s\colon I \to X$ with $s(0) = x$ and $s(1) = y$.

Suppose that X is contractible, so we have maps $f\colon X \to \{0\}$ and $g\colon \{0\} \to X$ and a homotopy $h\colon \mathrm{id}_X \simeq gf$. Write $a = g(0) \in X$. Note that we must have $f(x) = 0$ for all $x \in X$, because there are no other points in $\{0\}$ that $f(x)$ could be. Thus $gf(x) = a$ for all x. As h is a homotopy from 1 to gf, we have $h(0, x) = x$ and $h(1, x) = a$ for all x. Thus, for any point $x \in X$ we can define a path $s_x\colon I \to X$ by $s_x(t) = h(t, x)$. This starts at x and ands at a. If y is any other point in X we can take the join of s_x with the reverse of s_y to get a path from x to y. Thus X is path connected.

On the other hand, a path connected space need not be contractible. For example, the space S^1 is path-connected (we can define a path from $e^{i\theta}$ to $e^{i\phi}$ by $s(t) = e^{i((1-t)\theta + t\phi)}$) but not contractible (because $\pi_1(S^1) \neq \{e\}$).

(c) Let $i\colon Y \to X$ be the inclusion, and define $r\colon X \to Y$ by $r(x,y) = (x, \max(0,y))$. We then have $rj = 1$. I claim that if $(x,y) \in X$ then the line segment joining (x,y) to $jr(x,y)$ is contained in X. If $y \geq 0$ then $rj(x,y) = (x,y)$ and the claim is clear. If $y < 0$ then (as $(x,y) \in X$) we must have $x \in \mathbb{Q}$. We also have $rj(x,y) = (x,0)$ so the line segment in question is the set of points (x,w) with $y \leq w \leq 0$. As $x \in \mathbb{Q}$, all these points lie in X as required. Thus rj is linearly homotopic to id_X, which implies that j is a homotopy equivalence.

The set Y is convex and thus contractible, in other words homotopy equivalent to a point. As homotopy equivalence is an equivalence relation, we deduce that X is also homotopy equivalent to a point.

(19)

(a) Let X be a subspace of $\mathbb{R}^n$, and let a be a point in X.

- (i) Explain what it means for X to be *star-shaped* around a. **(4 marks)**
- (ii) Prove that if X is star-shaped around a, then X is contractible. **(4 marks)**

(b)
- (i) Suppose that $\alpha, \beta > 0$ and that $0 \leq t \leq 1$. Show that $\alpha t + \beta(1-t)$ is strictly greater than zero. **(3 marks)**
- (ii) Suppose that $\gamma, \delta, \epsilon > 0$ and that $0 \leq t \leq 1$. Show that $\gamma t^2 + \delta t(1-t) + \epsilon(1-t)^2$ is strictly greater than zero. **(3 marks)**
- (iii) Consider a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2\mathbb{R}$. Put $\lambda = \mathrm{trace}(A)$ and $\mu = \det(A)$. Express $\mathrm{trace}((1-t)I + tA)$ and $\det((1-t)I + tA)$ in terms of λ, μ and t. **(6 marks)**
- (iv) Put $X = \{A \in M_2\mathbb{R} \mid \det(A) > 0 \text{ and } \mathrm{trace}(A) > 0\}$. Prove that X is contractible. **(5 marks)**

Solution:

(a)
- (i) We say that X is star-shaped around a if for each $t \in I$ and $x \in X$, the point $(1-t)x + ta$ lies in X. Equivalently, X is star-shaped around a if every linear path starting in X and ending at a lies wholly in X. [4]
- (ii) Suppose that X is star-shaped around a. We can then define a map $h\colon I \times X \to X$ by $h(t,x) = (1-t)x + ta$. We have $h(0,x) = x$ and $h(1,x) = a$ for all $x \in X$, so this gives a contraction of X. [4]

(b)
- (i) Suppose that $\alpha, \beta > 0$ and $0 \leq t \leq 1$. Then αt and $\beta(1-t)$ are both greater than or equal to 0. Moreover, αt is only zero when $t = 0$, and $\beta(1-t)$ is only zero when $t = 1$. Thus, for any t, at least one of the two terms is strictly positive, and thus $\alpha t + \beta(1-t) > 0$. [3]
- (ii) Suppose that $\gamma, \delta, \epsilon > 0$ and that $0 \leq t \leq 1$. Then γt^2, $\delta t(1-t)$ and $\epsilon(1-t)^2$ are all greater than or equal to zero. The first one is strictly greater than zero unless $t = 0$, and the last one is strictly greater than zero unless $t = 1$. Thus, for all t, at least one term is strictly positive, so their sum is strictly positive. [3] **[unseen]**
- (iii) We have $\lambda = a + d$ and $\mu = ad - bc$ and

$$(1-t)I + tA = \begin{bmatrix} 1 - t + ta & tb \\ tc & 1 - t + td \end{bmatrix},$$

so

$$\begin{aligned} \mathrm{trace}((1-t)I + tA)) &= (ta + 1 - t) + (td + 1 - t) \\ &= (a+d)t + 2(1-t) \\ &= \lambda t + 2(1-t) [3] \end{aligned}$$

and

$$\begin{aligned} \det((1-t)I + tA)) &= (ta + 1 - t)(td + 1 - t) - t^2 bc \\ &= (ad - bc)t^2 + (a+d)t(1-t) + (1-t)^2 \\ &= \mu t^2 + \lambda t(1-t) + (1-t)^2. [3] \end{aligned}$$

- (iv) Suppose that $A \in X$, so $\lambda, \mu > 0$, and suppose that $t \in I$. As $\lambda > 0$ and $2 > 0$, part (a) tells us that $\lambda t + 2(1-t) > 0$, so $\mathrm{trace}((1-t)I + tA) > 0$. As $\mu > 0$, $\lambda > 0$ and $1 > 0$, part (b) tells us that $\mu t^2 + \lambda t(1-t) + (1-t)^2 > 0$, so $\det((1-t)I + tA) > 0$. This shows that $(1-t)I + tA \in X$, so X is star-shaped around I, and thus contractible. [5]

(20)

(a) Given topoological spaces X, Y and continuous maps $f, g\colon X \to Y$, what does it mean for f and g to be homotopic? **(3 marks)**

(b) Show that if Y is contractible, then any two maps $f, g\colon X \to Y$ are homotopic. **(7 marks)**

(c) Show that if X is contractible and Y is path-connected, then any two maps $f, g\colon X \to Y$ are homotopic. **(10 marks)**

(d) Regard S^1 as $\{z \in \mathbb{C} \mid |z| = 1\}$, and put $T = S^1 \times S^1$. Define $f\colon T \to T$ by $f(z, w) = (iz, -iw)$. Prove that f is homotopic to the identity map. **(5 marks)**

Solution:

(a) Maps $f, g\colon X \to Y$ are homotopic iff there is a continuous map $h\colon I \times X \to Y$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in X$. [3] **[bookwork]**

(b) Suppose that Y is contractible, so we have a point $b \in Y$ and a map $m\colon I \times Y \to Y$ with $m(0, y) = y$ and $m(1, y) = b$ for all $y \in Y$ [2]. Let $c_b\colon X \to Y$ be the constant map with value b. Define $h\colon I \times X \to Y$ by $h(t, x) = m(t, f(x))$ [2]. This has $h(0, x) = f(x)$ and $h(1, x) = m(1, f(x)) = b = c_b(x)$, showing that f is homotopic to c_b [1]. Similarly, g is homotopic to c_b and thus to f [2]. **[seen]**

(c) Now suppose instead that X is contractible, so we have a point $a \in X$ and a map $n\colon I \times X \to X$ with $n(0, x) = x$ and $n(1, x) = a$ for all $x \in X$ [2]. Given $f\colon X \to Y$ we put $k(t, x) = f(n(t, x))$ [2]. This has $k(0, x) = f(x)$ and $k(1, x) = f(a) = c_{f(a)}(x)$, showing that $f \simeq c_{f(a)}$ [1]. Similarly $g \simeq c_{g(a)}$ [1]. Finally, as Y is path-connected, we can choose a path $u\colon I \to Y$ with $u(0) = f(a)$ and $u(1) = g(a)$ [2]. Put $l(t, x) = u(t)$; then $l(0, x) = u(0) = f(a) = c_{f(a)}(x)$ and $l(1, x) = u(1) = g(a) = c_{g(a)}(x)$, so $f \simeq c_{f(a)} \simeq c_{g(a)} \simeq g$ as required. [2] **The case $X = I$ is done in lectures, but otherwise this is unseen.**

(d) Define $h(t, z, w) = (e^{i\pi t/2} z, e^{-i\pi t/2} w)$. Then $h(0, z, w) = (z, w)$ and

$$h(1, z, w) = (e^{i\pi/2} z, e^{-i\pi/2} w) = (iz, -iw) = f(z, w),$$

showing that f is homotopic to the identity. [5] **[similar examples seen]**

(21) Let E be the figure eight space, so $E = E_- \cup E_+$ where $E_\pm$ is the circle of radius one centred at $(\pm 1, 0)$.

(a) Prove that E is not homotopy equivalent to the torus. **(4 marks)**

(b) Put $A = \{(1, 0), (-1, 0)\}$ and $X = \mathbb{R}^2 \setminus A$. Sketch a proof that X is homotopy equivalent to E. **(5 marks)**

(c) Put $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 = 1\ ,\ y = 0\}$ and $Y = \mathbb{R}^3 \setminus B$. Deduce that Y is homotopy equivalent to E. **(4 marks)**

(d) Put $C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 = 1\ ,\ y = xz\}$ and $Z = \mathbb{R}^3 \setminus C$. Deduce that Z is homotopy equivalent to E. You may wish to consider the expression

$$(x, \mathrm{rot}_{\pi x/4}(y, z)) = (x, \cos(\pi x/4) y - \sin(\pi x/4) z, \sin(\pi x/4) y + \cos(\pi x/4) z).$$

(12 marks)

Solution:

(a) The torus T has $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$ [1], which is abelian [1], but $\pi_1(E)$ is nonabelian [1], so $\pi_1(T) \not\simeq \pi_1(E)$, so T is not homotopy equivalent to E [1]. **[similar examples seen]**

(b) Let $g\colon X \to E$ be as illustrated in the following diagram:

C2def.eps[2]

Let $f\colon E \to X$ be the inclusion. Then $gf = \mathrm{id}_E$ [1], and the line joining $fg(a)$ to a lies wholly in X so fg is linearly homotopic to id_X [2]. This shows that X is homotopy equivalent to E. **[bookwork]**

(c) We observe that $B = A \times \mathbb{R}$ [2] and so $Y = X \times \mathbb{R}$ [1], and $\mathbb{R}$ is contractible so $Y \simeq X$ [1].

More explicitly, define $p\colon X \to Y$ by $p(x, y) = (x, y, 0)$ and $q\colon Y \to X$ by $q(x, y, z) = (x, y)$. Then $qp = \mathrm{id}_X$ and pq is linearly homotopic to id_Y, so $Y \simeq X \simeq E$.

(d) We will show that Z is homeomorphic to Y [2], and so homotopy equivalent to Y, X and E [1]. We define $r\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $r(x, y, z) = (x, \mathrm{rot}_{\pi x/4}(y, z))$ [1]. This is a homeomorphism, with inverse $s(x, y, z) = (x, \mathrm{rot}_{-\pi x/4}(y, z))$ [2]. The points in C have the form $(1, y, y)$ or $(-1, y, -y)$ [1]. We have $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, so $r(1, y, y) = (1, 0, \sqrt{2}y) \in B$ [1]. Similarly, we have $\cos(-\pi/4) = 1/\sqrt{2}$ and $\sin(-\pi/4) = -1/\sqrt{2}$, so $r(-1, y, -y) = (-1, 0, -\sqrt{2}y) \in B$ [1]. Using this, we see that $r(C) = B$ [1]and so r induces a homeomorphism

$$Z = \mathbb{R}^3 \setminus C \simeq \mathbb{R}^3 \setminus r(C) = \mathbb{R}^3 \setminus B = Y$$

as required [2]. **[unseen]**

5 Abelian groups and chain complexes

(22)

(a) In the context of Abelian groups, define the terms

- homomorphism **(2 marks)**
- subgroup **(2 marks)**
- kernel **(2 marks)**
- image. **(2 marks)**

(b) Let A and B be Abelian groups, and let $\phi\colon A \to B$ be a homomorphism. Prove that

(i) The kernel of ϕ is a subgroup of A **(3 marks)**

(ii) The kernel of ϕ is a subgroup of the kernel of the homomorphism 2ϕ. **(2 marks)**

(c) Let $\phi\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}/12$ be the homomorphism defined by

$$\phi(n, m) = (3n, (2n + 4m \bmod 12)).$$

Give an isomorphism $\psi\colon \mathbb{Z} \to \ker(\phi)$. **(6 marks)**

(d) Let A be a finite Abelian group, and let B be a free Abelian group. Prove that if $\phi\colon A \to B$ is a homomorphism, then $\phi = 0$. **(6 marks)**

Solution:

(a) (i) A homomorphism from an Abelian group A to an Abelian group B is a function $\phi\colon A \to B$ such that $\phi(a+a') = \phi(a)+\phi(a')$ for all $a, a' \in A$ [2] (from which it follows that $\phi(0) = 0$ and $\phi(a-a') = \phi(a)-\phi(a')$).

(ii) A subgroup of A is a subset $C \subseteq A$ with the property that $0 \in C$, and $-c \in C$ whenever $c \in C$, and $c + c' \in C$ whenever $c, c' \in C$. [2]

(iii) The kernel of a homomorphism $\phi\colon A \to B$ is $\{a \in A \mid \phi(a) = 0\}$. [2]

(iv) The image of a homomorphism $\phi\colon A \to B$ is $\{\phi(a) \mid a \in A\} = \{b \in B \mid b = \phi(a) \text{ for some } a \in A\}$. [2]

[bookwork]

(b) (i) First, we have $\phi(0_A) = 0_B$ so $0_A \in \ker(\phi)$ [1]. Next, suppose we have $c \in \ker(\phi)$, so $\phi(c) = 0$. We then have $\phi(-c) = -\phi(c) = -0 = 0$, so $-c \in \ker(\phi)$ [1]. Finally, suppose we have another element $c' \in \ker(\phi)$, so that $\phi(c') = 0$. Then $\phi(c + c') = \phi(c) + \phi(c') = 0 + 0 = 0$, so $c + c' \in \ker(\phi)$ [1]. This proves that $\ker(\phi)$ is a subgroup. **[bookwork]**

(ii) By the first part we know that $\ker(\phi)$ and $\ker(2\phi)$ are subgroups; we need only check that $\ker(\phi) \subseteq \ker(2\phi)$ [1]. If $a \in \ker(\phi)$ then $\phi(a) = 0$ so $(2\phi)(a) = 2\phi(a) = 2.0 = 0$, so $a \in \ker(2\phi)$ as required [1].

(c) We have $\phi(n, m) = 0$ iff $3n = 0$ and $2n + 4m = 0 \pmod{12}$ [2]. This is clearly equivalent to $n = 0$ and $4m = 0 \pmod{12}$, which means that $n = 0$ and m is divisible by 3 [2]. We can thus define an isomorphism $f\colon \mathbb{Z} \to \ker(\phi)$ by $f(k) = (0, 3k)$ [2]. **[similar examples seen]**

(d) Let A be a finite Abelian group, and let B be a free Abelian group, say $B = \mathbb{Z}[D]$ for some set D. Suppose that $a \in A$ and $\phi(a) = n_1[d_1] + \ldots + n_r[d_r]$ say, for some integers $n_1, \ldots, n_r$ and distinct elements $d_1, \ldots, d_r \in D$ [2]. As A is finite we know that $ma = 0$ for some $m > 0$ [1]. We thus have

$$mn_1[d_1] + \ldots + mn_r[d_r] = m\phi(a) = \phi(ma) = \phi(0) = 0. [1]$$

As the d_i are distinct, the only way this can happen is if $n_1 = \ldots = n_r = 0$, so $\phi(a) = 0$ [1]. This holds for all $a \in A$, so ϕ must be the zero homomorphism as claimed. [1]

(23) 2018-19 Q4: Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain complexes and chain maps.

(a) Define what is meant by saying that the above sequence is short exact. **(3 marks)**

Now recall that a *snake* for the above sequence is a system (c, w, v, u, a) such that

- $c \in H_n(W)$;
- $w \in Z_n(W)$ is a cycle such that $c = [w]$;
- $v \in V_n$ is an element with $p(v) = w$;
- $u \in Z_{n-1}(U)$ is a cycle with $i(u) = d(v) \in V_{n-1}$;
- $a = [u] \in H_{n-1}(U)$.

(b) Prove that for each $c \in H_n(W)$ there is a snake starting with c. **(8 marks)**

(c) Prove that if two snakes have the same starting point, then they also have the same endpoint. **(10 marks)**

(d) Suppose that the differential $d\colon V_{n+1} \to V_n$ is surjective. Show that any snake starting in $H_n(W)$ ends with zero. **(4 marks)**

Solution:

(a) The map i is injective, the map p is surjective, and the image of i is the same as the kernel of p. [3] **[Bookwork]**

(b) Consider an element $c \in H_n(W)$. As $H_n(W) = Z_n(W)/B_n(W)$ by definition, we can certainly choose $w \in Z_n(W)$ such that $c = [w]$ [1]. As the sequence $U \xrightarrow{i} V \xrightarrow{p} W$ is short exact, we know that $p\colon V_n \to W_n$ is surjective, so we can choose $v \in V_n$ with $p(v) = w$ [1]. As p is a chain map we have $p(d(v)) = d(p(v)) = d(w) = 0$ (the last equation because $w \in Z_n(W)$) [1]. This means that $d(v) \in \ker(p)$, but $\ker(p) = \operatorname{img}(i)$ because the sequence is exact, so we have $u \in U_{n-1}$ with $i(u) = d(v)$ [2]. Note also that $i(d(u)) = d(i(u)) = d(d(v)) = 0$ (because i is a chain map and $d^2 = 0$) [1]. On the other hand, exactness means that i is injective, so the relation $i(d(u)) = 0$ implies that $d(u) = 0$ [1]. This shows that $u \in Z_{n-1}(U)$, so we can put $a = [u] \in H_{n-1}(U)$ [1]. We now have a snake (c, w, v, u, a) starting with c as required. **[Bookwork]**

(c) Suppose we have two snakes that start with c. We can then subtract them to get a snake $(0, w, v, u, a)$ starting with 0 [1]. It will be enough to show that this ends with 0 as well, or equivalently that $a = 0$ [1]. The first snake condition says that $[w] = 0$, which means that $w = d(w')$ for some $w' \in W_{n+1}$ [1]. Because p is surjective we can also choose $v' \in V_{n+1}$ with $w' = p(v')$ [1], and this gives $w = d(w') = d(p(v')) = p(d(v'))$ [1]. The next snake condition says that $p(v) = w$. We can combine these facts to see that $p(v - d(v')) = 0$, so $v - d(v') \in \ker(p) = \operatorname{img}(i)$[1]. We can therefore find $u' \in U_n$ with $v - d(v') = i(u')$ [1]. We can apply d to this using $d^2 = 0$ and $di = id$ to get $d(v) = i(d(u'))$ [1]. On the other hand, the third snake condition tells us that $d(v) = i(u)$. Subtracting these gives $i(u - d(u')) = 0$, but i is injective, so $u = d(u')$, so $u \in B_{n-1}(U)$ [1]. The final snake condition now says that $a = [u] = u + B_{n-1}(U)$, but $u \in B_{n-1}(U)$ so $a = [u] = 0$ [1]. **[Bookwork]**

(d) Now suppose that $d\colon V_{n+1} \to V_n$ is surjective. As $d^2 = 0$ this means that $d\colon V_n \to V_{n-1}$ is zero. Now suppose we have a snake (c, w, v, u, a) with $c \in H_n(W)$ so $v \in V_n$. The condition $i(u) = d(v)$ now gives $i(u) = 0$, but i is injective so $u = 0$, so $a = [u] = 0$. [4] **[Unseen]**

(24) 2021-22 Mock Q3:

(a) Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain complexes and chain maps. Define what is meant by a *snake* for this sequence. **(5 marks)**

(b) Define the homomorphism $\delta\colon H_n(W) \to H_{n-1}(U)$. You should give a clear statement of the lemmas needed to ensure that your definition is meaningful, but you do not need to prove those lemmas. **(4 marks)**

(c) Suppose that $H_k(W)$ is finite for all k, and that $H_k(U) \simeq \mathbb{Z}$ for all k. Prove that $H_k(V)$ is infinite and that the map $p_*\colon H_k(V) \to H_k(W)$ is surjective. **(5 marks)**

(d) Consider the chain complex with $A_k = \mathbb{Z}^3$ for all $k \in \mathbb{Z}$ and $d(x,y,z) = (z,0,0)$.

  (i) Find the homology of A_*. **(2 marks)**

  (ii) Show that the formula $m(x,y,z) = (0,y,0)$ defines a chain map $m\colon A_* \to A_*$ **(2 marks)**

  (iii) Show that m is chain homotopic to the identity. **(3 marks)**

  (iv) Construct a chain complex A'_* where the differential is zero, and a chain homotopy equivalence from A'_* to A_*. **(4 marks)**

Solution:

(a) A snake is a list (c,w,v,u,a) where

- $c \in H_k(W)$ [1]
- $w \in Z_k(W)$ is a cycle with $c = [w]$ [1]
- $v \in V_k$ satisfies $p(v) = w$ [1]
- $u \in Z_{k-1}(U)$ satisfies $i(u) = d(v)$ [1]
- $a = [u] \in H_{k-1}(U)$. [1]

(b) It can be shown that

  (1) For any $c \in H_k(W)$, there exists a snake (c,w,v,u,a) starting with c. [1]

  (2) If we have snakes (c,w,v,u,a) and (c,w',v',u',a') both starting with c, then $a = a'$. [1]

  We can therefore define $\delta\colon H_k(W)$ by $\delta(c) = a$, for any snake (c,w,v,u,a) that starts with c. [2]

(c) The Snake Lemma gives exact sequences

$$H_{k+1}(W) \xrightarrow{\delta} H_k(U) \xrightarrow{i_*} H_k(V) \xrightarrow{p_*} H_k(W) \xrightarrow{\delta} H_{k-1}(U)[1]$$

For every element c in the finite group $H_k(W)$ we know that c has finite order, so the element $\delta(c) \in H_{k-1}(U)$ also has finite order. However, $H_{k-1}(U) \simeq \mathbb{Z}$ so the only element of finite order in this group is zero. It follows that all the maps δ are zero [1], and thus that the sequence

$$H_k(U) \xrightarrow{i_*} H_k(V) \xrightarrow{p_*} H_k(W)$$

is short exact [1]. This means that p_* is surjective [1], as required. It also means that i_* is injective and $H_k(U) \simeq \mathbb{Z}$ is infinite so $H_k(V)$ must also be infinite [1].

(d) (i) We have

$$\begin{aligned}
B_k(A) &= \operatorname{img}(d) = \mathbb{Z} \oplus 0 \oplus 0 \\
Z_k(A) &= \{(x,y,z) \mid (z,0,0) = (0,0,0)\} = \{(x,y,0) \mid x,y \in \mathbb{Z}\} \\
&= \mathbb{Z} \oplus \mathbb{Z} \oplus 0 \\
H_k(A) &= (\mathbb{Z} \oplus \mathbb{Z} \oplus 0)/(\mathbb{Z} \oplus 0 \oplus 0) \simeq \mathbb{Z}.
\end{aligned}$$

  Explicitly, we have $H_k(A) = \mathbb{Z}.h$, where $h = [(0,1,0)]$ [2].

  (ii) From the formulae $d(x,y,z) = (z,0,0)$ and $m(x,y,z) = (0,y,0)$ we get $d(m(x,y,z)) = d(0,y,0) = (0,0,0)$ and $m(d(x,y,z)) = m(z,0,0) = (0,0,0)$. This shows that $dm = md$, so m is a chain map [2].

  (iii) Now define $s(x,y,z) = (0,0,x)$ [1]. This has $d(s(x,y,z)) = d(0,0,x) = (x,0,0)$ and $s(d(x,y,z)) = s(z,0,0) = (0,0,z)$ so

$$(ds + sd)(x,y,z) = (x,0,z) = (\mathrm{id} - m)(x,y,z),$$

  so s gives a chain homotopy between id and m [2].

(iv) Now define $A'_k = \mathbb{Z}$, with $d' = 0 \colon A'_k \to A'_{k-1}$ [1]. Define $i \colon A'_k \to A_k$ by $i(y) = (0, y, 0)$ [1]and $r \colon A_k \to A'_k$ by $r(x, y, z) = y$ [1]. These are chain maps with r id and $ir = m$ so ir is chain homotopic to id [1]. This means that i is a chain homotopy equivalence from A'_* to A_*.

(25) 2021-22 Q3:

(a) Define the terms *chain complex*, *chain map* and *chain homotopy.* **(8 marks)**

(b) Prove that if two chain maps are chain homotopic, then they have the same effect on homology groups. **(5 marks)**

(c) Consider the chain complex T with $T_i = \mathbb{Z}/8$ for all i and $d(x) = 4x$ for all x. Find the homology groups of T. **(3 marks)**

(d) Suppose we have a short exact sequence $A_* \xrightarrow{i} B_* \xrightarrow{p} C_*$ of chain complexes and chain maps. Suppose that for all $i \in \mathbb{Z}$ we have $H_{2i+1}(A) = H_{2i+1}(C) = 0$ and $|H_{2i}(A)| = 3$ and $|H_{2i}(C)| = 5$. Prove that all homology groups of B are cyclic or trivial, and determine their orders. **(5 marks)**

(e) Let U_* be a chain complex in which all the differentials d_{2i} (for all $i \in \mathbb{Z}$) are surjective homomorphisms. What can we conclude about the homology groups of U_*? **(4 marks)**

Solution:

(a) **Bookwork**

(1) A *chain complex* is a sequence of abelian groups U_i (for $i \in \mathbb{Z}$) [1]equipped with homomorphisms $d_i \colon U_i \to U_{i-1}$ [1]satisfying $d_{i-1} \circ d_i = 0 \colon U_i \to U_{i-2}$ for all i (or more briefly, $d^2 = 0$). [1]

(2) Let U_* and V_* be chain complexes. A *chain map* from U_* to V_* is a sequence of homomorphisms $f_i \colon U_i \to V_i$ [1]such that $d_i \circ f_i = f_{i-1} \circ d_i \colon U_i \to V_{i-1}$ for all $i \in \mathbb{Z}$ (or more briefly, $df = fd$) [1].

(3) Let $f, g \colon U_* \to V_*$ be chain maps [1]. A *chain homotopy* between f and g is a sequence of homomorphisms $s_i \colon U_i \to V_{i+1}$ [1]with $ds + sd = g - f$ [1].

(b) **Bookwork** Suppose we have chain maps $f, g \colon U_* \to V_*$ and a chain homotopy s as above. Consider an element $u \in H_n(U)$, so $u = [z]$ for some cycle $z \in U_n$ with $d(z) = 0$ [1]. As s is a chain homotopy from f to g, we have $g(z) - f(z) = d(s(z)) + s(d(z))$ [1]. As $d(z) = 0$ this becomes $g(z) - f(z) = d(s(z)) \in \operatorname{img}(d)_n = B_n(V)$ [1], so the cosets $[g(z)] = g(z) + B_n(V)$ and $[f(z)] = f(z) + B_n(V)$ are the same [1], or in other words $g_*(u) = f_*(u)$ as required [1].

(c) **Similar examples have been seen** We can identify $\mathbb{Z}/8$ with $\{0, 1, 2, \dots, 7\}$. The map d sends $0, 2, 4$ and 6 to 0 and 1, 3, 5 and 7 to 4. It follows that $Z_i(T) = \{0, 2, 4, 6\}$ [1]and $B_i(T) = \{0, 4\}$ [1]so the quotient $H_i(T) = Z_i(T)/B_i(T)$ has order $4/2 = 2$ and is isomorphic to $\mathbb{Z}/2$ [1].

(d) **Unseen** Let $A_* \xrightarrow{i} B_* \xrightarrow{p} C_*$ be as described. The Snake Lemma then gives exact sequences

$$0 = H_{2i+1}(A) \to H_{2i+1}(B) \to H_{2i+1}(C) = 0.$$

As the two outer groups are zero the middle one is also zero, so $H_{2i+1}(B) = 0$ [2]. We also have exact sequences

$$H_{2i+1}(C) \to H_{2i}(A) \to H_{2i}(B) \to H_{2i}(C) \to H_{2i-1}(A)$$

The two outer groups are zero, so the middle three groups form a short exact sequence. As $|H_{2i}(A)| = 3$ and $|H_{2i}(C)| = 5$ it follows that $|H_{2i}(B)| = 15$. Up to isomorphism, the only abelian group of order 15 is $\mathbb{Z}/3 \oplus \mathbb{Z}/5$, and this is isomorphic to $\mathbb{Z}/15$ by the Chinese Remainder Theorem (as 3 and 5 are coprime). It follows that $H_{2i}(B) \simeq \mathbb{Z}/15$ [3].

(e) **Unseen** Let U_* be a chain complex in which $d_{2i} \colon U_{2i} \to U_{2i-1}$ is always surjective, so $B_{2i-1}(U) = U_{2i-1}$. This means that every element $u \in U_{2i-1}$ can be expressed as $u = d(u')$ for some u', so $d(u) = d^2(u') = 0$ [1]. Thus, the homomorphism $d_{2i-1} \colon U_{2i-1} \to U_{2i-2}$ is zero. We now have $Z_{2i-1}(U) = B_{2i-1}(U) = U_{2i-1}$, so $H_{2i-1}(U) = U_{2i-1}/U_{2i-1} = 0$ [1]. We also have $B_{2i}(U) = 0$ [1]and so $H_{2i}(U) \simeq Z_{2i}(U) = \ker(d_{2i} \colon U_{2i} \to U_{2i-1})$ [1].

(26) 2022-23 Q3: Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain complexes and chain maps.

(a) Define what is meant by saying that the above sequence is short exact. **(3 marks)**

Now recall that a *snake* for the above sequence is a system (c, w, v, u, a) such that

- $c \in H_n(W)$;
- $w \in Z_n(W)$ is a cycle such that $c = [w]$;
- $v \in V_n$ is an element with $p(v) = w$;
- $u \in Z_{n-1}(U)$ is a cycle with $i(u) = d(v) \in V_{n-1}$;
- $a = [u] \in H_{n-1}(U)$.

(b) Prove that for each $c \in H_n(W)$ there is a snake starting with c. **(7 marks)**

(c) Explain how the connecting homomorphism $\delta\colon H_n(W) \to H_{n-1}(U)$ is defined in terms of snakes. If any further lemmas are needed to ensure that your definition is meaningful, then you should state those lemmas carefully, but you need not prove them. **(4 marks)**

(d) Consider the following example. For each $k \in \mathbb{Z}$ we have

$$\begin{aligned} U_k &= \mathbb{Z}/24 = \mathbb{Z}/(2^3 \times 3) & d^U(x) &= 12x = 2^2 \times 3 \times x \\ V_k &= \mathbb{Z}/1296 = \mathbb{Z}/(2^4 \times 3^4) & d^V(x) &= 36x = 2^2 \times 3^2 \times x \\ W_k &= \mathbb{Z}/54 = \mathbb{Z}/(2 \times 3^3) & d^W(x) &= -18x = -2 \times 3^2 \times x. \end{aligned}$$

The maps
$$U_k \xrightarrow{i} V_k \xrightarrow{p} W_k$$
are $i(a \pmod{24}) = 54a \pmod{1296}$ and $p(b \pmod{1296}) = b \pmod{54}$.

(i) Check that i and p are chain maps. (You may assume that they give a short exact sequence.) **(3 marks)**

(ii) Calculate the groups $H_k(U)$, $H_k(V)$ and $H_k(W)$. **(5 marks)**

(iii) By finding an appropriate snake, calculate the homomorphism $\delta\colon H_k(W) \to H_{k-1}(U)$. **(3 marks)**

Solution:

(a) **Bookwork** For each n, the map $i_n\colon U_n \to V_n$ is injective, the map $p_n\colon V_n \to W_n$ is surjective, and the image of i_n is the same as the kernel of p_n. [3]

(b) **Bookwork** Consider an element $c \in H_n(W)$. As $H_n(W) = Z_n(W)/B_n(W)$ by definition, we can certainly choose $w \in Z_n(W)$ such that $c = [w]$ [1]. As the sequence $U \xrightarrow{i} V \xrightarrow{p} W$ is short exact, we know that $p\colon V_n \to W_n$ is surjective, so we can choose $v \in V_n$ with $p(v) = w$ [1]. As p is a chain map we have $p(d(v)) = d(p(v)) = d(w) = 0$ (the last equation because $w \in Z_n(W)$) [1]. This means that $d(v) \in \ker(p)$, but $\ker(p) = \operatorname{img}(i)$ because the sequence is exact, so we have $u \in U_{n-1}$ with $i(u) = d(v)$ [1]. Note also that $i(d(u)) = d(i(u)) = d(d(v)) = 0$ (because i is a chain map and $d^2 = 0$) [1]. On the other hand, exactness means that i is injective, so the relation $i(d(u)) = 0$ implies that $d(u) = 0$ [1]. This shows that $u \in Z_{n-1}(U)$, so we can put $a = [u] \in H_{n-1}(U)$ [1]. We now have a snake (c, w, v, u, a) starting with c as required.

(c) **Bookwork** In addition to (b), we need the following lemma: given any two snakes (c, w, v, u, a) and (c, w', v', u', a') that both start with c, the endpoints a and a' are also the same [2]. This makes it possible to define $\delta\colon H_n(W) \to H_{n-1}(U)$ by the following rule: for any element $c \in H_n(W)$, we define $\delta(c)$ to be the endpoint of any snake that starts with c [2].

(d) **Similar examples seen**

(i) To show that i is a chain map, we must show that $d^V(i(x)) = i(d^U(x))$ in $\mathbb{Z}/1296$ for all $x \in \mathbb{Z}/24$, or equivalently that $54 \times 12 \times k = 36 \times 54 \times k \pmod{1296}$ for all $k \in \mathbb{Z}$. This holds because $(36 \times 54) - (54 \times 12) = 54 \times 24 = 2 \times 3^3 \times 2^3 \times 3 = 1296$ [2]. Similarly, to show that p is a chain map we just need to check that $36 = -18 \pmod{54}$, which is clear [1].

(ii) For $H_n(U)$ we note that $12k$ is divisible by 24 iff k is divisible by 2, so $Z_n(U) = \{0, 2, 4, \ldots, 22\} \simeq \mathbb{Z}/12$, but $B_n(U) = \{0, 12\}$ so $H_n(U) \simeq \mathbb{Z}/6$, with generator $a = [2]$. [2]
For $H_n(V)$ we note that $36k$ is divisible by $1296 = 36^2$ iff k is divisible by 36, so $Z_n(V) = B_n(V) = 36V_n$ and $H_n(V) = 0$. [1]
For $H_n(W)$ we note that $-18k$ is divisible by $54 = 3 \times 18$ iff k is divisible by 3, so $Z_n(W) = \{0, 3, 6, \ldots, 51\} \simeq \mathbb{Z}/18$. On the other hand, $B_n(W) = \{0, 18, 36\} \simeq \mathbb{Z}/3$, so $H_n(W) \simeq \mathbb{Z}/6$ with generator $c = [3]$. [2]

(iii) The sequence

$$(c, 3 \pmod{54}, 3 \pmod{1296}, 108 \pmod{1296}, 2 \pmod{24}, a)$$

is a snake, proving that $\delta(c) = a$. Thus, the homomorphism $\delta\colon (\mathbb{Z}/6).c \to (\mathbb{Z}/6).a$ is just given by $\delta(kc) = ka$. [3]

(27)

(a) Define the terms *chain map*, *chain homotopy*, *chain homotopic* and *chain homotopy equivalence*. **(8 marks)**

(b) Show that if $f, g\colon U_* \to V_*$ are chain maps that are chain homotopic to each other, then $f_* = g_*\colon H_*(U) \to H_*(V)$. **(5 marks)**

(c) Consider the chain complex T_* with $T_i = \mathbb{Z}^2$ for all i and $d_i(x, y) = (y, 0)$ for all $(x, y) \in T_i$. Show that T_* is chain homotopy equivalent to the zero complex. **(4 marks)**

(d) Suppose we have a short exact sequence $A_* \xrightarrow{i} B_* \xrightarrow{p} C_*$ of chain complexes and chain maps. Suppose that for all $k \in \mathbb{Z}$ we have $H_k(B) = 0$. Suppose also that $H_k(A) = \mathbb{Z}/2^k$ for $k \geq 0$ and $H_k(A) = 0$ for $k < 0$. Determine the homology groups of C_*. **(3 marks)**

(e) Let U_* be a chain complex in which $U_k = 0$ for $k < 0$ and $|U_k| = 2^k$ for $k \geq 0$ and $d_{2i}\colon U_{2i} \to U_{2i-1}$ is surjective for all i. Find the homology groups of U_*. **(5 marks)**

Solution:

(a) **Bookwork**

(1) Let U_* and V_* be chain complexes. A *chain map* from U_* to V_* is a sequence of homomorphisms $f_i\colon U_i \to V_i$ [1]such that $d_i \circ f_i = f_{i-1} \circ d_i\colon U_i \to V_{i-1}$ for all $i \in \mathbb{Z}$ (or more briefly, $df = fd$) [1].

(2) Let $f, g\colon U_* \to V_*$ be chain maps [1]. A *chain homotopy* between f and g is a sequence of homomorphisms $s_i\colon U_i \to V_{i+1}$ [1]with $ds + sd = g - f$ [1].

(3) We say that chain maps $f, g\colon U_* \to V_*$ are *chain homotopic* if there exists a chain homotopy as in (2). [1]

(4) A chain map $f\colon U_* \to V_*$ is a *chain homotopy equivalence* if there is a chain map $g\colon V_* \to U_*$ [1]such that $g \circ f\colon U_* \to U_*$ and $f \circ g\colon V_* \to V_*$ are chain homotopic to the corresponding identity maps [1].

(b) **Bookwork** Suppose we have chain maps $f, g\colon U_* \to V_*$ and a chain homotopy s as above. Consider an element $u \in H_n(U)$, so $u = [z]$ for some cycle $z \in U_n$ with $d(z) = 0$ [1]. As s is a chain homotopy from f to g, we have $g(z) - f(z) = d(s(z)) + s(d(z))$ [1]. As $d(z) = 0$ this becomes $g(z) - f(z) = d(s(z)) \in \operatorname{img}(d)_n = B_n(V)$ [1], so the cosets $[g(z)] = g(z) + B_n(V)$ and $[f(z)] = f(z) + B_n(V)$ are the same [1], or in other words $g_*(u) = f_*(u)$ as required [1].

(c) **Unseen** Let $i\colon 0 \to T_*$ and $r\colon T_* \to 0$ be the zero maps, so $r \circ i\colon 0 \to 0$ is the identity, and $i \circ r = 0\colon T_* \to T_*$. Define $s_k\colon T_k \to T_{k+1}$ by $s_k(x, y) = (0, x)$ [2]. Then

$$(ds + sd)(x, y) = d(0, x) + s(y, 0) = (x, 0) + (0, y) = (x, y),$$

so $ds + ds = 1 = 1 - 0 = 1 - i \circ r$, so $i \circ r$ is chain homotopic to the identity. This means that i and r are chain homotopy equivalences [2].

(d) **Unseen** Let $A_* \xrightarrow{i} B_* \xrightarrow{p} C_*$ be as described. The Snake Lemma then gives exact sequences

$$0 = H_k(B) \to H_k(C) \xrightarrow{\delta} H_{k-1}(A) \to H_{k-1}(B) = 0,$$

which means that the map δ is an isomorphism [2]. It follows that when $k > 0$ we have $H_k(C) \simeq H_{k-1}(A) \simeq \mathbb{Z}/2^{k-1}$ and when $k \leq 0$ we have $H_k(C) = 0$ [1].

(e) **Unseen** Let U_* be a chain complex as described. As $d_{2i}\colon U_{2i} \to U_{2i-1}$ is surjective, we see that $B_{2i-1}(U) = U_{2i-1}$. This means that every element $u \in U_{2i-1}$ can be expressed as $u = d(u')$ for some u', so $d(u) = d^2(u') = 0$. Thus, the homomorphism $d_{2i-1}\colon U_{2i-1} \to U_{2i-2}$ is zero [2]. We now have $Z_{2i-1}(U) = B_{2i-1}(U) = U_{2i-1}$, so $H_{2i-1}(U) = U_{2i-1}/U_{2i-1} = 0$. We also have $B_{2i}(U) = 0$ and so $H_{2i}(U) \simeq Z_{2i}(U) = \ker(d_{2i}\colon U_{2i} \to U_{2i-1})$ [1]. As d_{2i} is surjective with $|U_{2i}| = 2^{2i}$ and $|U_{2i-1}| = 2^{2i-1}$ we see that $|\ker(d_{2i})| = 2$ and so $\ker(d_{2i}) \simeq \mathbb{Z}/2$. In summary, we have

$$H_k(U) = \begin{cases} \mathbb{Z}/2 & \text{if } k \text{ is even and } k > 0 \\ 0 & \text{otherwise. [2]} \end{cases}$$

(28) 2023-24 Q3:

(a) Define the terms *chain map*, *chain homotopy*, *chain homotopic* and *chain homotopy equivalence*. **(8 marks)**

(b) Show that if $f, g\colon U_* \to V_*$ are chain maps that are chain homotopic to each other, then $f_* = g_*\colon H_*(U) \to H_*(V)$. **(5 marks)**

(c) Consider the chain complex T_* with $T_i = \mathbb{Z}^2$ for all i and $d_i(x, y) = (y, 0)$ for all $(x, y) \in T_i$. Show that T_* is chain homotopy equivalent to the zero complex. **(4 marks)**

(d) Suppose we have a short exact sequence $A_* \xrightarrow{i} B_* \xrightarrow{p} C_*$ of chain complexes and chain maps. Suppose that for all $k \in \mathbb{Z}$ we have $H_k(B) = 0$. Suppose also that $H_k(A) = \mathbb{Z}/2^k$ for $k \geq 0$ and $H_k(A) = 0$ for $k < 0$. Determine the homology groups of C_*. **(3 marks)**

(e) Let U_* be a chain complex in which $U_k = 0$ for $k < 0$ and $|U_k| = 2^k$ for $k \geq 0$ and $d_{2i}\colon U_{2i} \to U_{2i-1}$ is surjective for all i. Find the homology groups of U_*. **(5 marks)**

Solution:

(a) **Bookwork**

(1) Let U_* and V_* be chain complexes. A *chain map* from U_* to V_* is a sequence of homomorphisms $f_i\colon U_i \to V_i$ [1]such that $d_i \circ f_i = f_{i-1} \circ d_i\colon U_i \to V_{i-1}$ for all $i \in \mathbb{Z}$ (or more briefly, $df = fd$) [1].

(2) Let $f, g\colon U_* \to V_*$ be chain maps [1]. A *chain homotopy* between f and g is a sequence of homomorphisms $s_i\colon U_i \to V_{i+1}$ [1]with $ds + sd = g - f$ [1].

(3) We say that chain maps $f, g\colon U_* \to V_*$ are *chain homotopic* if there exists a chain homotopy as in (2). [1]

(4) A chain map $f\colon U_* \to V_*$ is a *chain homotopy equivalence* if there is a chain map $g\colon V_* \to U_*$ [1]such that $g \circ f\colon U_* \to U_*$ and $f \circ g\colon V_* \to V_*$ are chain homotopic to the corresponding identity maps [1].

(b) **Bookwork** Suppose we have chain maps $f, g\colon U_* \to V_*$ and a chain homotopy s as above. Consider an element $u \in H_n(U)$, so $u = [z]$ for some cycle $z \in U_n$ with $d(z) = 0$ [1]. As s is a chain homotopy from f to g, we have $g(z) - f(z) = d(s(z)) + s(d(z))$ [1]. As $d(z) = 0$ this becomes $g(z) - f(z) = d(s(z)) \in \operatorname{img}(d)_n = B_n(V)$ [1], so the cosets $[g(z)] = g(z) + B_n(V)$ and $[f(z)] = f(z) + B_n(V)$ are the same [1], or in other words $g_*(u) = f_*(u)$ as required [1].

(c) **Unseen** Let $i\colon 0 \to T_*$ and $r\colon T_* \to 0$ be the zero maps, so $r \circ i\colon 0 \to 0$ is the identity, and $i \circ r = 0\colon T_* \to T_*$. Define $s_k\colon T_k \to T_{k+1}$ by $s_k(x, y) = (0, x)$ [2]. Then

$$(ds + sd)(x, y) = d(0, x) + s(y, 0) = (x, 0) + (0, y) = (x, y),$$

so $ds + ds = 1 = 1 - 0 = 1 - i \circ r$, so $i \circ r$ is chain homotopic to the identity. This means that i and r are chain homotopy equivalences [2].

(d) **Unseen** Let $A_* \xrightarrow{i} B_* \xrightarrow{p} C_*$ be as described. The Snake Lemma then gives exact sequences

$$0 = H_k(B) \to H_k(C) \xrightarrow{\delta} H_{k-1}(A) \to H_{k-1}(B) = 0,$$

which means that the map δ is an isomorphism [2]. It follows that when $k > 0$ we have $H_k(C) \simeq H_{k-1}(A) \simeq \mathbb{Z}/2^{k-1}$ and when $k \leq 0$ we have $H_k(C) = 0$ [1].

(e) **Unseen** Let U_* be a chain complex as described. As $d_{2i}: U_{2i} \to U_{2i-1}$ is surjective, we see that $B_{2i-1}(U) = U_{2i-1}$. This means that every element $u \in U_{2i-1}$ can be expressed as $u = d(u')$ for some u', so $d(u) = d^2(u') = 0$. Thus, the homomorphism $d_{2i-1}: U_{2i-1} \to U_{2i-2}$ is zero [2]. We now have $Z_{2i-1}(U) = B_{2i-1}(U) = U_{2i-1}$, so $H_{2i-1}(U) = U_{2i-1}/U_{2i-1} = 0$. We also have $B_{2i}(U) = 0$ and so $H_{2i}(U) \simeq Z_{2i}(U) = \ker(d_{2i}: U_{2i} \to U_{2i-1})$ [1]. As d_{2i} is surjective with $|U_{2i}| = 2^{2i}$ and $|U_{2i-1}| = 2^{2i-1}$ we see that $|\ker(d_{2i})| = 2$ and so $\ker(d_{2i}) \simeq \mathbb{Z}/2$. In summary, we have

$$H_k(U) = \begin{cases} \mathbb{Z}/2 & \text{if } k \text{ is even and } k > 0 \\ 0 & \text{otherwise. [2]} \end{cases}$$

6 Singular chains

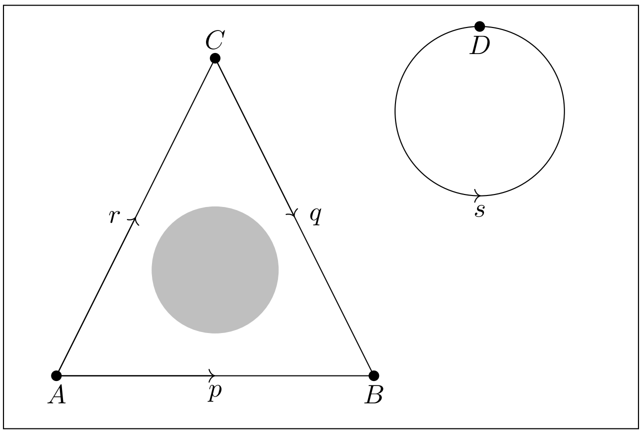
(29) Let X be a topological space.

(a) Let $c: \Delta_1 \to X$ be a constant path. Prove that c is homologous to 0.

(b) Let $s: \Delta_1 \to X$ be a path. Define the reversed path $\overline{s}$, and prove that $\overline{s}$ is homologous to $-s$.

(c) Let $r, s: \Delta_1 \to X$ be paths such that $r(e_1) = s(e_0)$. Write down a path $u: \Delta_1 \to X$ and prove that u is homologous to $r + s$.

(d) Let X be the complement of the shaded disc in the diagram below. Write down a path $u: \Delta_1 \to X$ such that u is homologous to $2p - 2q - 2r + s$.



Solution:

(a) As c is constant, there is a point $x \in X$ with $c(t) = x$ for all $t \in \Delta_1$. Define $d: \Delta_2 \to X$ by $d(t) = x$ for all $t \in \Delta_2$. Then $d\delta_0 = d\delta_1 = d\delta_2 = c: \Delta_1 \to X$, so $\partial(d) = c - c + c = c$, so $[c] = [0]$ in $H_1(X)$.

(b) First, we put $\overline{s}(t_0, t_1) = s(t_1, t_0)$. Next, we put $a = s(1, 0)$, and we define $r: \Delta_2 \to X$ by $r(t_0, t_1, t_2) = s(t_0+t_2, t_1)$. Then $r\delta_0(t_0, t_1) = r(0, t_0, t_1) = s(t_1, t_0) = \overline{s}(t_0, t_1)$ and $r\delta_1(t_0, t_1) = r(t_0, 0, t_1) = s(t_0 + t_1, 0) = s(1, 0) = a$ and $r\delta_2(t_0, t_1) = r(t_0, t_1, 0) = s(t_0, t_1)$, so $\partial(r) = \overline{s} - c_a + s$ so $\overline{s} + s - c_a \in B_1(X)$. From (i) we also know that $c_a \in B_1(X)$ so $\overline{s} + s \in B_1(X)$.

(c) We define a path $u = r * s: \Delta_1 \to X$ by

$$u(t_0, t_1) = \begin{cases} r(t_0 - t_1, 2t_1) & \text{if } t_0 \geq t_1 \\ s(2t_0, t_1 - t_0) & \text{if } t_0 \leq t_1. \end{cases}$$

This is well-defined and continuous (by closed patching) because $r(e_1) = s(e_0)$. We also define $w: \Delta_2 \to X$ by

$$w(t_0, t_1, t_2) = \begin{cases} r(t_0 - t_2, t_1 + 2t_2) & \text{if } t_0 \geq t_2 \\ s(2t_0 + t_1, t_2 - t_0) & \text{if } t_0 \leq t_2. \end{cases}$$

When $t_0 = t_2$ the first clause gives $r(e_1)$ and the second gives $s(e_0)$, and these are the same by assumption, so w is well-defined and continuous (by closed patching again). Now $w\delta_0(t_0, t_1) = w(0, t_0, t_1) = s(t_0, t_1)$ and $w\delta_1(t) = w(t_0, 0, t_1) = u(t_0, t_1)$ and $w\delta_2(t) = w(t_0, t_1, 0) = r(t_0, t_1)$, so $\partial(w) = s - u + r$, so $u + B_1(X) = (r + B_1(X)) + (s + B_1(X))$ as required.

(d) The path s in the diagram is linearly homotopic in X to the constant map with value D. As $s(e_0) = s(e_1) = D$ we see that this is a pinned homotopy, so s is homologous to a constant map and thus to 0. As p ends where $\overline{q}$ starts, and $\overline{q}$ ends where $\overline{r}$ starts, we can join these together to get a path $v = (p * \overline{q}) * \overline{r}$. This has $v = p - q - r \pmod{B_1(X)}$. It starts and ends at the same place, so we can form $u = v * v$, and this has $u = 2v = 2p - 2q - 2r + s \pmod{B_1(X)}$ as required.

(30)

(a) Let X be a topological space.

(i) Define the groups $C_n(X)$ for all nonnegative integers n. **(2 marks)**

(ii) Define the homomorphisms ∂_n. **(3 marks)**

(iii) Prove that $\partial_1 \circ \partial_2 = 0$. **(3 marks)**

(iv) Define the groups $H_n(X)$. **(4 marks)**

(b) Describe (without proof, but with careful attention to any special cases) the groups $H_n(\mathbb{R}^k \setminus \{0\})$ for all $n \geq 0$ and all $k \geq 1$. **(5 marks)**

(c) Let $u = n_1 s_1 + \ldots + n_k s_k$ be an m-cycle in S^n (where $m > 0$), and suppose that there is a point $a \in S^n$ that is not contained in any of the sets $s_1(\Delta_m), \ldots, s_k(\Delta_m)$. Prove that u is a boundary. **(8 marks)**

Solution:

(a) (i) The group $C_n(X)$ is the free Abelian group [1]generated by the set of continuous maps $s \colon \Delta_n \to X$ [1], where $\Delta_n = \{t \in \mathbb{R}^{n+1} \mid t_i \geq 0, \ \sum_i t_i = 1\}$. **[bookwork]**

(ii) We define continuous maps $\delta_0, \ldots, \delta_n \colon \Delta_{n-1} \to \Delta_n$ by

$$\delta_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

For any continuous map $s \colon \Delta_n \to X$ we define

$$\partial_n(s) = \sum_{k=0}^{n} (-1)^k (s \circ \delta_k) \in C_{n-1}(X)[1].$$

This can be extended in a unique way to give a homomorphism $\partial_n \colon C_n(X) \to C_{n-1}(X)$. [1]**[bookwork]**

(iii) From the definitions, we have

$$\begin{aligned}
\partial_1 \partial_2 [s] &= \partial_1([s\delta_0] - [s\delta_1] + [s\delta_2]) \\
&= [s\delta_0\delta_0] - [s\delta_0\delta_1] - [s\delta_1\delta_0] + [s\delta_1\delta_1] + [s\delta_2\delta_0] - [s\delta_2\delta_1] \\
&= ([s\delta_0\delta_0] - [s\delta_1\delta_0]) - ([s\delta_0\delta_1] - [s\delta_2\delta_0]) + ([s\delta_1\delta_1] - [s\delta_2\delta_1]).[1]
\end{aligned}$$

Whenever $k \leq l$ we have $\delta_k \delta_l = \delta_{l+1} \delta_k$; this shows that each of the bracketed terms is zero [1]. Thus $\partial_2 \partial_1$ vanishes on all singular 2-simplices, so it vanishes on all singular 2-chains [1]. **[bookwork]**

(iv) We define $Z_n(X) = \ker(\partial_n \colon C_n(X) \to C_{n-1}(X))$ [1]and $B_n(X) = \operatorname{img}(\partial_{n+1} \colon C_{n+1}(X) \to C_n(X))$ [1]. We have $\partial_n \partial_{n+1} = 0$, which implies that $B_n(X) \leq Z_n(X)$ [1], so we can define a quotient group $H_n(X) = Z_n(X)/B_n(X)$ [1]. **[bookwork]**

(b) As $\mathbb{R}^k \setminus \{0\}$ is homotopy equivalent to S^{k-1}, we have

$$H_n(\mathbb{R}^k \setminus \{0\}) = \begin{cases} \mathbb{Z}^2 & \text{if } n = 0, k = 1[2] \\ \mathbb{Z} & \text{if } n = 0, k > 1[1] \text{ or } n = k - 1 > 0[1] \\ 0 & \text{otherwise } [1]. \end{cases}$$

[bookwork]

(c) The space $S^n \setminus \{a\}$ [2] is homeomorphic to $\mathbb{R}^n$ [1]by stereographic projection, and thus is contractible [1]. This implies that $H_m(S^n \setminus \{a\}) = 0$ for $m > 0$ [1], so every m-cycle in $S^n \setminus \{a\}$ is a boundary [1]. We can regard u as an m-cycle in $S^m \setminus \{a\}$, so it is a boundary in $S^n \setminus \{a\}$ [1]and thus in S^n [1], as required. **[unseen]**

(31)

(a) Let X be a topological space.

(i) Define the groups $C_0(X)$ and $C_1(X)$, and the homomorphism $\partial_1\colon C_1(X) \to C_0(X)$.

(ii) Define the subdivision homomorphism $\mathrm{sd}\colon C_1(X) \to C_1(X)$.

(iii) Prove that $\partial_1 \mathrm{sd}^n(u) = \partial_1(u)$ for all $n \geq 1$.

(iv) Prove that if $u \in B_1(X)$ then $\mathrm{sd}(u) \in B_1(X)$.

(v) Let A and B be points in a vector space V. Give an expression for $\mathrm{sd}\langle A, B\rangle$ in terms of paths of the form $\langle C, D\rangle$.

(b) Describe without proof the groups $H_1(S^1)$, $H_1(S^1 \times S^1)$, $H_1(\mathbb{R}P^2)$ and $H_1(\mathbb{R}^3 \setminus \{0\})$.

(c) For each element $u \in H_1(\mathbb{R}P^2)$, give a path s in $\mathbb{R}P^2$ such that $u = [s]$.

Solution:

(a) (i) $C_0(X) = \mathbb{Z}[X]$ is the free Abelian group on the set X, or in other words the group of all $\mathbb{Z}$-combinations of points of X. We also write $S_1(X)$ for the set of paths in X (in other words, continuous maps $s\colon \Delta_1 \to X$) and $C_1(X) = \mathbb{Z}[S_1X]$ for the group of 1-chains (in other words, $\mathbb{Z}$-combinations of paths). The homomorphism $\partial_1\colon C_1(X) \to C_0(X)$ is defined by

$$\partial_1(s) = s(e_1) - s(e_0),$$

extended linearly as usual.

(ii) Define maps $l, r\colon \Delta_1 \to \Delta_1$ by $l(1-t, t) = ((1+t)/2, (1-t)/2)$ and $r(1-t, t) = ((1-t)/2, (1+t)/2)$. The subdivision homomorphism $\mathrm{sd}\colon C_1(X) \to C_1(X)$ is defined by $\mathrm{sd}(s) = s \circ r - s \circ l$.

(iii) Given any path $s\colon \Delta_1 \to X$ we can define $q\colon \Delta_2 \to X$ by $q(t_0, t_1, t_2) = s(t_0/2 + t_1, t_0/2 + t_2)$. This satisfies

$$\begin{aligned}
(q\delta_0)(1-t, t) &= q(0, 1-t, t) = s(1-t, t) \\
(q\delta_1)(1-t, t) &= q(1-t, 0, t) = s((1-t)/2, (1+t)/2) = sr(1-t, t) \\
(q\delta_2)(1-t, t) &= q(1-t, t, 0) = s((1+t)/2, (1-t)/2) = sl(1-t, t).
\end{aligned}$$

Thus $\partial_2(q) = s - (s \circ r) + (s \circ (l) = s - \mathrm{sd}(s)$, showing that $s = \mathrm{sd}(s) \pmod{B_1(X)}$. By linear extension, we see that $u = \mathrm{sd}(u) \pmod{B_1(X)}$ for all $u \in C_1(X)$.

I claim that in fact $u \sim \mathrm{sd}^n(u)$ for all 1-chains u and all $n \geq 0$. The case $n = 0$ is clear because $\mathrm{sd}^0(u) = u$, and we have just done the case $n = 1$. Assume that the case $n = k-1$ holds. For any chain v we can apply the case $n = k-1$ to v to see that $v = \mathrm{sd}^{k-1}(v) \pmod{B_1(X)}$, and we can apply the case $n = 1$ to the chain $u = \mathrm{sd}^{k-1}(v)$ to see that $\mathrm{sd}^{k-1}(v) = \mathrm{sd}^k(v) \pmod{B_1(X)}$, and by putting these together we see that $v = \mathrm{sd}^k(v) \pmod{B_1(X)}$. By induction, this holds for all k.

(iv) Suppose that $u \in B_1(X)$, so $u = \partial(a)$ for some $a \in C_2(X)$. By part (iii) we have $u = \mathrm{sd}(u) \pmod{B_1(X)}$, in other words $\mathrm{sd}(u) - u = \partial(b)$ for some $b \in C_2(X)$. Thus $\mathrm{sd}(u) = \partial(a + b) \in B_1(X)$, as required.

(v) Put $C = (A + B)/2$ (the midpoint of the path $\langle A, B\rangle$). We have

$$\begin{aligned}
\langle A, B\rangle(l(t)) &= \langle A, B\rangle((1+t)/2, (1-t)/2) = ((1+t)/2)A + ((1-t)/2)B = (1-t)(A+B)/2 + tA \\
\langle A, B\rangle(r(t)) &= \langle A, B\rangle((1-t)/2, (1+t)/2) = ((1-t)/2)A + ((1+t)/2)B = (1-t)(A+B)/2 + tB,
\end{aligned}$$

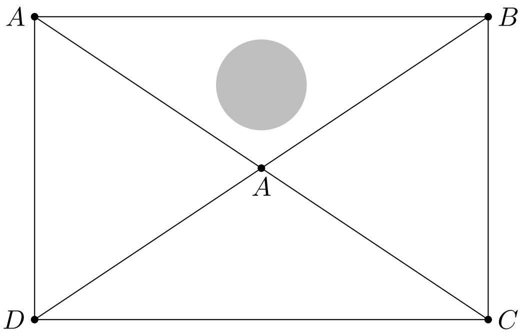
so $\langle A, B\rangle \circ l = \langle C, A\rangle)$ and $\langle A, B\rangle \circ r = \langle C, B\rangle$. Thus

$$\mathrm{sd}\langle A, B\rangle = \langle C, B\rangle - \langle C, A\rangle.$$

(b) $H_1(S^1) \simeq \mathbb{Z}$; $H_1(S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z}$; $H_1(\mathbb{R}P^2) \simeq \mathbb{Z}/2$; $H_1(\mathbb{R}^3 \setminus \{0\}) \simeq 0$.

(c) There are only two elements in $H_1(\mathbb{R}P^2)$, say 0 and v. For $u = 0$ we take s to be the constant path $s(1-t, t) = q(1, 0, 0)$. For $u = v$ we take $s(1-t, t) = q(\cos(\pi t), \sin(\pi t), 0)$.

(32) Consider the following diagram.



Let X be the complement in $\mathbb{R}^2$ of the shaded disc. Define $u, v, w \in C_1(X)$ by

$$\begin{aligned} u =& \langle A, B\rangle + \langle B, E\rangle + \langle E, A\rangle \\ v =& \langle A, B\rangle + \langle B, C\rangle + \langle C, D\rangle + \langle D, A\rangle \\ w =& \langle A, E\rangle + \langle E, D\rangle + \langle D, A\rangle. \end{aligned}$$

(a) Prove that u is a cycle. **(2 marks)**

(b) Prove that $\langle B, B\rangle$ is homologous to 0 in X. **(3 marks)**

(c) Prove that $\langle E, B\rangle$ is homologous to $-\langle B, E\rangle$ in X. **(4 marks)**

(d) Prove in detail that u is homologous to v in X, justifying each step. **(8 marks)**

(e) Write down a basic 1-chain s that is homologous in X to $\langle A, B\rangle + \langle B, C\rangle$ **(5 marks)**

(f) Is u homologous to w? Give a brief reason for your answer. **(3 marks)**

Solution:

(a) We have $\partial\langle U, V\rangle = V - U$ [1], so

$$\partial(u) = B - A + E - B + A - E = 0, [1]$$

so u is a cycle. **[similar examples seen]**

(b) We have $\partial\langle U, V, W\rangle = \langle V, W\rangle - \langle U, W\rangle + \langle U, V\rangle$ [1], so $\partial\langle B, B, B\rangle = \langle B, B\rangle - \langle B, B\rangle + \langle B, B\rangle = \langle B, B\rangle$ [1], which means that $\langle B, B\rangle$ is homologous to 0 [1]. **[bookwork]**

(c) We have $\partial\langle B, E, B\rangle = \langle E, B\rangle - \langle B, B\rangle + \langle B, E\rangle$ [2], so (using the previous part) we have

$$\langle E, B\rangle \sim \langle B, B\rangle - \langle B, E\rangle \sim -\langle B, E\rangle [2].$$

[similar examples seen]

(d) If we define $a = \langle A, D, E\rangle + \langle D, C, E\rangle + \langle C, B, E\rangle$ [3] we find that

$$\begin{aligned} \partial(a) =& \langle D, E\rangle - \langle A, E\rangle + \langle A, D\rangle + \\ & \langle C, E\rangle - \langle D, E\rangle + \langle D, C\rangle + \\ & \langle B, E\rangle - \langle C, E\rangle + \langle C, B\rangle \\ =& (\langle A, D\rangle + \langle D, C\rangle + \langle C, B\rangle) - \\ & (\langle A, E\rangle - \langle B, E\rangle). [3] \end{aligned}$$

Using this and part (iii), we see that

$$v \sim (\langle A, E\rangle - \langle B, E\rangle) \sim (\langle A, E\rangle + \langle E, B\rangle) = u. [2]$$

[similar examples seen]

(e) Define $s = \sigma(A,B) * \sigma(B,C)$, so

$$s(1-t,t) = \begin{cases} (1-2t)A + 2tB & \text{if } t \le \frac{1}{2} \\ (2-2t)B + (2t-1)C & \text{if } t \ge \frac{1}{2}. \end{cases}$$

Then $[s]$ is homologous to $\langle A,B\rangle + \langle B,C\rangle$ in X. [5]

(f) The chain u winds once around the hole, and w does not wind around the hole at all, so u is not homologous to w. [3]

(33) 2023-24 Q4:

(a) Let X be a topological space.

  (i) Define the groups $C_n(X)$ for all nonnegative integers n. **(2 marks)**

  (ii) Define the homomorphisms $\partial_n \colon C_n(X) \to C_{n-1}(X)$. **(3 marks)**

  (iii) Prove that $\partial_1 \circ \partial_2 = 0$. **(3 marks)**

  (iv) Define the groups $H_n(X)$. **(4 marks)**

(b) Describe (without proof, but with careful attention to any special cases) the groups $H_n(\mathbb{R}^k \setminus \{0\})$ for all $n \ge 0$ and all $k \ge 1$. **(5 marks)**

(c) Let $u = n_1 s_1 + \ldots + n_k s_k$ be an element of $Z_m(S^n)$ (where $m > 0$), and suppose that there is a point $a \in S^n$ that is not contained in any of the sets $s_1(\Delta_m), \ldots, s_k(\Delta_m)$. Prove that u is a boundary. (You may assume standard results and calculations from the course so long as you state them carefully.) **(8 marks)**

Solution:

(a) (i) **Bookwork** The group $C_n(X)$ is the free Abelian group [1]generated by the set of continuous maps $s \colon \Delta_n \to X$ [1], where $\Delta_n = \{t \in \mathbb{R}^{n+1} \mid t_i \ge 0,\ \sum_i t_i = 1\}$.

  (ii) **Bookwork** We define continuous maps $\delta_0, \ldots, \delta_n \colon \Delta_{n-1} \to \Delta_n$ by

$$\delta_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})[1].$$

  For any continuous map $s \colon \Delta_n \to X$ we define

$$\partial_n(s) = \sum_{k=0}^{n} (-1)^k (s \circ \delta_k) \in C_{n-1}(X)[1].$$

  This can be extended in a unique way to give a homomorphism $\partial_n \colon C_n(X) \to C_{n-1}(X)$. [1]

  (iii) **Bookwork** From the definitions, we have

$$\begin{aligned} \partial_1\partial_2[s] &= \partial_1([s\delta_0] - [s\delta_1] + [s\delta_2]) \\ &= [s\delta_0\delta_0] - [s\delta_0\delta_1] - [s\delta_1\delta_0] + [s\delta_1\delta_1] + [s\delta_2\delta_0] - [s\delta_2\delta_1] \\ &= ([s\delta_0\delta_0] - [s\delta_1\delta_0]) - ([s\delta_0\delta_1] - [s\delta_2\delta_0]) + ([s\delta_1\delta_1] - [s\delta_2\delta_1]).[1] \end{aligned}$$

  Whenever $k \le l$ we have $\delta_k\delta_l = \delta_{l+1}\delta_k$; this shows that each of the bracketed terms is zero [1]. Thus $\partial_2\partial_1$ vanishes on all singular 2-simplices, so it vanishes on all singular 2-chains [1].

  (iv) **Bookwork** We define $Z_n(X) = \ker(\partial_n \colon C_n(X) \to C_{n-1}(X))$ [1]and $B_n(X) = \operatorname{img}(\partial_{n+1} \colon C_{n+1}(X) \to C_n(X))$ [1]. We have $\partial_n\partial_{n+1} = 0$, which implies that $B_n(X) \le Z_n(X)$ [1], so we can define a quotient group $H_n(X) = Z_n(X)/B_n(X)$ [1].

(b) **Bookwork** As $\mathbb{R}^k \setminus \{0\}$ is homotopy equivalent to S^{k-1}, we have

$$H_n(\mathbb{R}^k \setminus \{0\}) = \begin{cases} \mathbb{Z}^2 & \text{if } n = 0, k = 1[2] \\ \mathbb{Z} & \text{if } n = 0, k > 1[1] \text{ or } n = k-1 > 0[1] \\ 0 & \text{otherwise } [1]. \end{cases}$$

(c) **Unseen** The space $S^n \setminus \{a\}$ [2] is homeomorphic to $\mathbb{R}^n$ [1]by stereographic projection, and thus is contractible [1]. This implies that $H_m(S^n \setminus \{a\}) = 0$ for $m > 0$ [1], so every m-cycle in $S^n \setminus \{a\}$ is a boundary [1]. We can regard u as an m-cycle in $S^m \setminus \{a\}$, so it is a boundary in $S^n \setminus \{a\}$ [1]and thus in S^n [1], as required.

7 True or false

(34) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(a) The punctured disc $X = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$ is compact.

(b) The circle S^1 is homeomorphic to $S^1 \times I$.

(c) The circle S^1 is homotopy equivalent to $S^1 \times I$.

(d) $\mathbb{C} \setminus S^1$ is homotopy equivalent to $Y = \{z \in \mathbb{C} \mid z = 0 \text{ or } |z| = 1\}$.

(e) Every continuous bijection from $[0, 1] \cup (2, 3]$ to $[0, 1]$ is a homeomorphism.

Solution:

(a) False. The space X is not closed in $\mathbb{R}^2$, because the sequence $(0, 1/n)$ in X converges in $\mathbb{R}^2$ to the point $(0, 0)$, which does not lie in X. A subspace of $\mathbb{R}^n$ is compact iff it is bounded and closed, so X is not compact.

(b) False. Removing any two points disconnects S^1, but $S^1 \times I$ cannot be disconnected by removing any finite set of points.

(c) True. Define maps as follows:

$$\begin{aligned} &f\colon S^1 \to S^1 \times I && f(z) = (z, 0) \\ &g\colon S^1 \times I \to S^1 && g(z, r) = z \\ &h\colon I \times (S^1 \times I) \to S^1 \times I && h(t, (z, r)) = (z, tr). \end{aligned}$$

Then $gf = 1\colon S^1 \to S^1$ and h is a (linear) homotopy from fg to $1_{S^1 \times I}$, so f is a homotopy equivalence with homotopy inverse g.

(d) True. Define maps as follows:

$$\begin{aligned} &f\colon \mathbb{C} \setminus S^1 \to Y && f(z) = \begin{cases} z/|z| & \text{if } |z| > 1 \\ 0 & \text{if } |z| < 1 \end{cases} \\ &g\colon Y \to \mathbb{C} \setminus S^1 && g(z) = 2z. \end{aligned}$$

Then $fg = 1_Y$ and gf is linearly homotopic to $1_{\mathbb{C} \setminus S^1}$, so f is a homotopy equivalence with homotopy inverse g.

(e) False. Define $f\colon [0, 1] \cup (2, 3] \to [0, 1]$ by

$$f(t) = \begin{cases} t/2 & \text{if } t \in [0, 1] \\ (t - 1)/2 & \text{if } t \in (2, 3]. \end{cases}$$

Then f is a continuous bijection, but f^{-1} is not continuous (because $1/2 + 1/2n \to 1/2$ but $f^{-1}(1/2 + 1/2n) = 2 + 1/n$ does not converge to $f^{-1}(1/2) = 1$), so f is not a homeomorphism.

(35) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(a) S^3 is contractible.

(b) If a space X is the union of two closed, path-connected subspaces A and B, then X is path-connected.

(c) $(\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1.

(d) $(\mathbb{R} \times \mathbb{R}^2) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1.

(e) The space $\mathbb{C} \setminus \{0, 1\}$ is homeomorphic to $\mathbb{C} \setminus \{i, -i\}$.

(f) The space $\mathbb{C} \setminus \{0, 1\}$ is homotopy equivalent to $\mathbb{C} \setminus \{0, 1, 2\}$.

Solution:

(a) False. We have $H_3(S^3) = \mathbb{Z}$ but H_3 of a point is zero, so S^3 is not homotopy equivalent to a point.

(b) False. Put $X = \{0, 1\}$ and $A = \{0\}$ and $B = \{1\}$. Then A and B are closed path connected subsets of X with $X = A \cup B$, but X is not path connected. (You would not be required to say this, but I remark that if $X = A \cup B$ where A and B are path connected (not necessarily closed) and $A \cap B \neq \emptyset$ then X is path connected.)

(c) False. Write

$$X = (\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{R} \times \{0\}) = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}.$$

We can then define a map $f \colon X \to \mathbb{R}$ by $f(x, y) = y$. This is never zero and it is positive at $(0, 1)$ and negative at $(0, -1)$, so $(0, 1)$ cannot be joined to $(0, -1)$ by a path in X, so X is not path connected. However, S^1 is path connected and anything homotopy equivalent to a path connected space is again path connected so X is not homotopy equivalent to S^1.

(d) True. Write $Y = (\mathbb{R} \times \mathbb{R}^2) \setminus (\mathbb{R} \times \{0\})$, and define maps as follows

$$\begin{aligned} f &\colon Y \to S^1 & f(x, y, z) &= (y, z)/\sqrt{y^2 + z^2} \\ g &\colon S^1 \to Y & g(y, z) &= (0, y, z). \end{aligned}$$

Then $gf = 1_{S^1}$, and fg is linearly homotopic to 1_Y.

(e) True. We can define a homeomorphism $f \colon \mathbb{C} \setminus \{0, 1\} \to \mathbb{C} \setminus \{i, -i\}$ by $f(z) = 2iz - i$, with inverse $f^{-1}(w) = (w + i)/2i$.

(f) False. We have $H_1(\mathbb{C} \setminus \{0, 1\}) \simeq \mathbb{Z}^2$, and this is not isomorphic to $H_1(\mathbb{C} \setminus \{0, 1, 2\}) \simeq \mathbb{Z}^3$, so $\mathbb{C} \setminus \{0, 1\}$ is not homotopy equivalent to $\mathbb{C} \setminus \{0, 1, 2\}$.

(36) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(a) The identity map of the unit circle is homotopic to the constant map $c \colon S^1 \to S^1$ defined by $c(z) = 1$ for all z.

(b) Let $f_n \colon S^1 \to S^1$ be defined by $f_n(z) = z^n$. Then f_n is not homotopic to f_m when $n \neq m$.

(c) $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^3$.

(d) If $f \colon X \to X$ is a homotopy equivalence, then $f_* \colon H_1(X) \to H_1(X)$ is the identity map.

Solution:

(a) False. Define $s_1 \colon \Delta_1 \to S^1$ by $s_1(t) = e(t)$, so that $u_1 = ([s_1] \bmod B_1(S^1))$ is the usual generator of $H_1(S^1)$. Then $c \circ s_1 \colon \Delta_1 \to S^1$ is a constant path, so $c_*[s_1] \sim 0$, so $c_*(u_1) = 0$ in $H_1(S^1)$. Thus c_* is not the identity map on $H_1(S^1)$, so c is not homotopic to the identity map on S^1.

(b) True. Define $s_n = f_n \circ s_1 \colon \Delta_1 \to S^1$, so $s_n(t) = e(t)^n = e(nt)$, so s_n can be unwound to the path $\tilde{s}_n(t) = nt$ in $\mathbb{R}$. It follows that the usual isomorphism $\overline{\phi} \colon H_1(S^1) \to \mathbb{Z}$ satisfies

$$\overline{\phi}([s_n] \bmod B_1(S^1)) = \tilde{s}_n(1) - \tilde{s}_n(0) = n = \overline{\phi}(nu_1),$$

so $f_{n*}(u_1) = nu_1$. It follows that $f_{n*} \neq f_{m*}$ when $n \neq m$, so f_n is not homotopic to f_m when $n \neq m$.

(c) False. If $f \colon \mathbb{R}^2 \to \mathbb{R}^3$ were a homeomorphism, then it would give a homeomorphism $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^3 \setminus \{f(0)\}$. However, $\mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to S^1 and $\mathbb{R}^3 \setminus \{f(0)\}$ is homeomorphic to $\mathbb{R}^3 \setminus \{0\}$ and thus homotopy equivalent to S^2. We know that $H_1(S^1) \simeq \mathbb{Z}$ and $H_1(S^2) \simeq 0$ so S^1 is not homotopy equivalent to S^2. It follows that $\mathbb{R}^2 \setminus \{0\}$ is not homotopy equivalent (and thus certainly not homeomorphic) to $\mathbb{R}^3 \setminus \{f(0)\}$, so no such map f can exist.

(d) False. The map $f_{-1} \colon S^1 \to S^1$ is a homeomorphism and thus a homotopy equivalence, and $(f_{-1})_*(u_1) = -u_1$ so $(f_{-1})_*$ is not the identity map.

(37) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(a) The torus $T = S^1 \times S^1$ is homotopy equivalent to S^2.

(b) There is a map $r\colon B^4 \to S^3$ such that rj is homotopic to id_{S^3}, where $j\colon S^3 \to B^4$ is the inclusion map.

(c) $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^3$.

(d) Every continuous function $f\colon S^2 \to \mathbb{R}^3$ is homotopic to a constant function.

(e) Let $K \subset S^3$ be a trefoil knot. Then $S^3 \setminus K$ is homotopy equivalent to $\mathbb{R}P^2$.

Solution:

(a) False. We have $H_1(T) \simeq \mathbb{Z}^2$, but $H_1(S^2) = 0$, so T is not homotopy equivalent to S^2.

(b) False. Let u_3 be the usual generator of H_3S^3. If there were such a map r, we would have $r_*j_* = (rj)_* = 1_* = 1\colon H_3S^3 \to H_3S^3$, so $u_3 = r_*(j_*(u_3))$. This is impossible, because B^4 is contractible so $H_3B^4 = 0$ and $j_*(u_3) \in H_3B^4$ so $j_*(u_3) = 0$ so $r_*(j_*(u_3)) = 0$.

(c) False. If $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ were a homeomorphism, then it would give a homeomorphism $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^3 \setminus \{f(0)\}$. However, $\mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to S^1 and $\mathbb{R}^3 \setminus \{f(0)\}$ is homeomorphic to $\mathbb{R}^3 \setminus \{0\}$ and thus homotopy equivalent to S^2. We know that $H_1(S^1) \simeq \mathbb{Z}$ and $H_1(S^2) \simeq 0$ so S^1 is not homotopy equivalent to S^2. It follows that $\mathbb{R}^2 \setminus \{0\}$ is not homotopy equivalent (and thus certainly not homeomorphic) to $\mathbb{R}^3 \setminus \{f(0)\}$, so no such map f can exist.

(d) True. We can just define $h(t,x) = tf(x)$; this is a homotopy from the constant map with value 0 to f.

(e) False. We have $H_1(S^3 \setminus K) \simeq \mathbb{Z}$ by the generalised Jordan Curve Theorem, but $H_1(\mathbb{R}P^2) \simeq \mathbb{Z}/2$.

(38) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems or calculations of homology groups without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) $\mathbb{R}P^1$ is homeomorphic to S^1.

(b) The Möbius strip is homotopy equivalent to S^2.

(c) $S^2 \setminus S^1$ is homotopy equivalent to $\mathbb{R} \setminus \{0\}$.

(d) The letter A is homeomorphic to the letter D.

(e) Any compact convex subset of $\mathbb{R}^2$ is homeomorphic to B^2.

Solution:

(a) True. There is a homeomorphism $f\colon S^1 \to \mathbb{R}P^1$ given by

$$f(x,y) = \frac{1}{2}\begin{pmatrix} 1+x & y \\ y & 1-x \end{pmatrix}.$$

(b) This is false, because the Möbius strip M is homotopy equivalent to S^1, so $\pi_1(M) \simeq \pi_1(S^1) \simeq \mathbb{Z}$, whereas $\pi_1(S^2) = 0$.

(c) This is true because both spaces are homotopy equivalent to the space with two points. Indeed, $\mathbb{R} \setminus \{0\}$ is the disjoint union of two contractible spaces $(-\infty, 0)$ and $(0, \infty)$, each of which is homotopy equivalent to a point, so $\mathbb{R} \setminus \{0\}$ is homotopy equivalent to two points. Similarly, $S^2 \setminus S^1$ is the disjoint union of the sets $U_+ = \{(x,y,z) \in S^2 \mid z > 0\}$ and $U_- = \{(x,y,z) \in S^2 \mid z < 0\}$. If we put $V = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ then V is contractible and there is a homeomorphism $f_+\colon V \to U_+$ given by $f(x,y) = (x, y, \sqrt{1 - x^2 - y^2})$, so U_+ is contractible. Similarly U_- is contractible, so $S^2 \setminus S^1$ is again homotopy equivalent to two points.

(d) This is false, because A can be disconnected by removing a point in the middle of one of the legs, but D cannot be disconnected by removing a single point.

(e) This is false: the closed line segment from $(-1,0)$ to $(1,0)$ is compact and convex but not homeomorphic to B^2. (The theorem states that if $X \subseteq \mathbb{R}^2$ is compact and convex *and contains an open ball* then X is homeomorphic to B^2.)

(39) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems or calculations of homology groups without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) $\mathbb{R}P^1$ is homeomorphic to S^1.

(b) The Möbius strip is homotopy equivalent to S^2.

(c) $SO(3)$ is homeomorphic to $\mathbb{R}P^3$.

(d) $S^2 \setminus S^1$ is homotopy equivalent to $\mathbb{R} \setminus \{0\}$.

(e) The letter A is homeomorphic to the letter D.

(f) Any compact convex subset of $\mathbb{R}^2$ is homeomorphic to B^2.

Solution:

(a) True. If we regard S^1 as the unit circle in the complex plane then we have $z \sim w$ iff $z^2 = w^2$, so there is a well-defined function $f : \mathbb{R}P^1 \to S^1$ given by $f(q(z)) = z^2$, and this is a homeomorphism.

(b) This is false, because the Möbius strip M is homotopy equivalent to S^1, so $H_1(M) \simeq H_1(S^1) \simeq \mathbb{Z}$, whereas $H_1(S^2) = 0$.

(c) This is true by a formula given in the notes, but it turns out that I won't have time to explain this properly.

(d) This is true because both spaces are homotopy equivalent to the space with two points. Indeed, $\mathbb{R} \setminus \{0\}$ is the disjoint union of two contractible spaces $(-\infty, 0)$ and $(0, \infty)$, each of which is homotopy equivalent to a point, so $\mathbb{R} \setminus \{0\}$ is homotopy equivalent to two points. Similarly, $S^2 \setminus S^1$ is the disjoint union of the sets $U_+ = \{(x, y, z) \in S^2 \mid z > 0\}$ and $U_- = \{(x, y, z) \in S^2 \mid z < 0\}$. If we put $V = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ then V is contractible and there is a homeomorphism $f_+ : V \to U_+$ given by $f(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$, so U_+ is contractible. Similarly U_- is contractible, so $S^2 \setminus S^1$ is again homotopy equivalent to two points.

(e) This is false, because A can be disconnected by removing a point in the middle of one of the legs, but D cannot be disconnected by removing a single point.

(f) This is false: the closed line segment from $(-1, 0)$ to $(1, 0)$ is compact and convex but not homeomorphic to B^2. (The theorem states that if $X \subseteq \mathbb{R}^2$ is compact and convex *and contains an open ball* then X is homeomorphic to B^2.)

(40) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems or calculations of homology groups without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) S^1 is homotopy equivalent to S^2. **(3 marks)**

(b) S^1 is homotopy equivalent to the Möbius strip. **(4 marks)**

(c) S^1 is homeomorphic to the Möbius strip. **(4 marks)**

(d) $\mathbb{R}P^2$ is homeomorphic to $S^1 \times S^1$. **(4 marks)**

(e) $SU(2) \setminus \{I\}$ is homeomorphic to $\mathbb{R}^3$. **(5 marks)**

(f) $\Delta_n \times \Delta_m$ is homeomorphic to Δ_{n+m}. **(5 marks)**

Solution:

(a) False. We have $H_1(S^1) \simeq \mathbb{Z}$ [1]but $H_1(S^2) = 0$ [1]. If X is homotopy equivalent to Y, then $H_n(X) \simeq H_nY$ for all n [1], so we conclude that $S^1 \not\simeq S^2$. **[seen]**

(b) True. The Möbius strip M is the quotient of $\mathbb{R} \times [-1, 1]$ by the equivalence relation

$$(x, y) \sim (x', y') \text{ iff } (x - x' \in \mathbb{Z} \text{ and } y = (-1)^{x-x'}y').$$

The circle can be thought of as the quotient of $\mathbb{R}$ by the equivalence relation

$$x \sim x' \text{ iff } x - x' \in \mathbb{Z}.$$

We thus have a map $j\colon S^1 \to M$ defined by $j\langle x\rangle = \langle x, 0\rangle$ [1], and a map $q\colon M \to S^1$ defined by $q\langle x, y\rangle = \langle x\rangle$ [1]; these clearly satisfy $qj = 1$. We also have a map $h\colon I \times M \to M$ defined by $h(t, \langle x, y\rangle) = \langle x, ty\rangle$ [1]. This has $h(1, \langle x, y\rangle) = \langle x, y\rangle$, and $h(0, \langle x, y\rangle) = \langle x, 0\rangle = jq\langle x, y\rangle$, so that $jq \simeq 1$. [1]**[seen]**

(c) False. If we remove any two distinct points from S^1 it becomes disconnected, but this is clearly not true for M. [4] **[similar examples seen]**

(d) We calculated in lectures that $H_1(\mathbb{R}P^2) \simeq \mathbb{Z}/2$ [1], whereas $H_1(S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z}$ [2]. It follows as in (a) that $\mathbb{R}P^2$ is not homotopy equivalent (and so not homeomorphic [1]) to $S^1 \times S^1$. **[similar examples seen]**

(e) True. There is a homeomorphism $f\colon S^3 \to SU(2)$ given by

$$f(a, b, c, d) = \begin{pmatrix} a + ib & c + id \\ c - id & a - ib \end{pmatrix}.\text{[2]}$$

If we define $P = (1, 0, 0, 0)$ then $f(P) = I$, so f induces a homeomorphism $S^3 \setminus \{P\} \to SU(2) \setminus \{I\}$. On the other hand, stereographic projection gives a homeomorphism $g\colon S^3 \setminus \{P\} \to \mathbb{R}^3$. Explicitly, we have

$$g(a, b, c, d) = (b/(1 - a), c/(1 - a), d/(1 - a)).\text{[3]}$$

(f) True. We proved in lectures that if $X \subseteq \mathbb{R}^k$ is bounded, closed and convex and contains an open ball, then X is homeomorphic to B^k [3]. This applies to both the sets $\Delta_n \times \Delta_m$ and Δ_{n+m}, so $\Delta_n \times \Delta_m \simeq B^{n+m} \simeq \Delta_{n+m}$ [2]. **[similar examples seen]**

(41) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) There is a continuous surjective map from $S^1 \times S^1$ to $\mathbb{R}$

(b) $\mathbb{C} \setminus \{2\}$ is homotopy equivalent to S^1

(c) $\mathbb{C} \setminus \{-1, 1\}$ is homotopy equivalent to S^1

(d) $S^2 \setminus \{\text{the north pole}\}$ is homeomorphic to $\mathbb{C}$.

(e) The letter X (considered as a subspace of $\mathbb{R}^2$) is homeomorphic to the letter Y.

(f) The letter X (considered as a subspace of $\mathbb{R}^2$) is homotopy equivalent to the letter Y.

Solution:

(a) **False.** The space $S^1 \times S^1$ is compact, and $\mathbb{R}$ is not compact, so there can be no continuous surjection from $S^1 \times S^1$ to $\mathbb{R}$.

(b) **True.** The map $f\colon \mathbb{C} \setminus \{2\} \to S^1$ given by $f(z) = (z - 2)/|z - 2|$ is a homotopy equivalence.

(c) **False.** The space $\mathbb{C} \setminus \{-1, 1\}$ is homotopy equivalent to the figure eight, so its fundamental group is nonabelian, whereas $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$ and thus is abelian. This shows that the two spaces have non-isomorphic fundamental groups, so they cannot be homotopy equivalent.

(d) **True.** The space $S^2 \setminus \{\text{the north pole}\}$ is homeomorphic to $\mathbb{R}^2$ by stereographic projection, and of course $\mathbb{R}^2$ is homeomorphic to $\mathbb{C}$ by the correspondence $(x, y) \leftrightarrow x + iy$.

(e) **False.** We can remove the central point from the letter X and the resulting space has four path components; but if we remove a point from the letter Y, the remaining space has at most three path components. This shows that X is not homeomorphic to Y.

(f) **True.** The letter X is star-shaped around its central point, so it is contractible, and the same applies to Y. Thus, they are both homotopy equivalent to a point and hence to each other.

(42) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) There is a continuous surjective map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R} \setminus \{0\}$ **(5 marks)**

(b) $S^2 \setminus S^1$ is homeomorphic to $\mathbb{R}^2$ **(5 marks)**

(c) $SO(2)$ is homotopy equivalent to the Möbius strip **(5 marks)**

(d) $SO(3)$ is homotopy equivalent to the torus **(5 marks)**

(e) The space $X = S^1 \cup \{(x, 0) \mid x \in \mathbb{R}\}$ is homeomorphic to $Y = S^1 \cup \{(x, 1) \mid x \in \mathbb{R}\}$. **(5 marks)**

Solution: In each part, two marks will be awarded for a correct true/false answer with no justification, and up to three (or exceptionally four) marks may be awarded for a reasonable line of argument leading to the wrong answer.

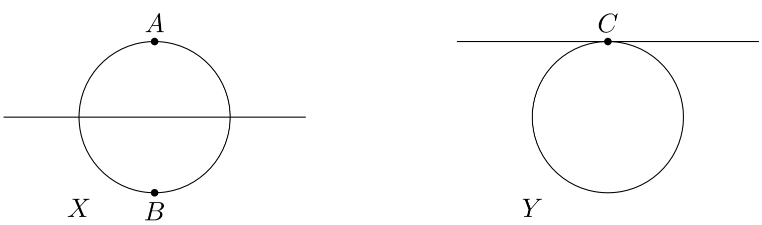
(a) False. The space $\mathbb{R}^2$ is path-connected and $\mathbb{R} \setminus \{0\}$ is not, so there cannot be a continuous surjective map from $\mathbb{R}^2$ to $\mathbb{R} \setminus \{0\}$. [5] **[similar examples seen]**

(b) False. The space $S^2 \setminus S^1$ is homotopy equivalent to S^0, and thus is not path-connected; but $\mathbb{R}^2$ is evidently path-connected, by linear paths. [5] **[seen]The homotopy equivalence $S^n \setminus S^m \simeq S^{n-m-1}$ is in the summary, so I expect that the students will use it. There are of course more direct proofs that $S^2 \setminus S^1$ is disconnected; they are also acceptable.**

(c) True. $SO(2)$ is homeomorphic to S^1 **[seen]**, and the Möbius strip M is homotopy equivalent to the circle running along the middle of the strip **[seen]**, so $SO(2)$ is homotopy equivalent to M. [5] **The summary contains various lists of spaces that are all homotopy equivalent to each other; one such list contains S^1, $SO(2)$ and the Möbius strip.**

(d) False. We know that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$, so $H_1(SO(3))$ has order two. On the other hand, $H_1(T) \simeq \mathbb{Z} \times \mathbb{Z}$ is infinite, and thus not isomorphic to $H_1(SO(3))$, so T is not homotopy equivalent to $SO(3)$. [5] **[similar examples seen]The facts that $SO(3) \simeq \mathbb{R}P^3$, that $H_1(\mathbb{R}P^3) = \mathbb{Z}/2$, and that $H_1(T) = \mathbb{Z} \times \mathbb{Z}$ are all in the summary. It is mentioned explicitly in the notes that $H_1(SO(3)) = \mathbb{Z}/2$. Many examples are given where we use H_1 to show that two spaces are not homotopy equivalent.**

(e) False. The picture is as follows:



The points A and B can be removed from X without disconnecting it; but removing any two points from Y disconnects it. (Note in particular that removing C already disconnects Y). [5] **[similar examples seen]**

(43) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) If X and Y are both path-connected subsets of $\mathbb{R}^2$, then $X \cap Y$ is also path-connected. **(5 marks)**

(b) The torus is homotopy equivalent to S^2. **(5 marks)**

(c) If $f\colon X \to Y$ and $g\colon Y \to X$ are continuous, based maps and $gf = \mathrm{id}_X$ then $\pi_1(X) \simeq \pi_1(Y)$. **(5 marks)**

(d) If two letters of the alphabet, considered as subspaces of $\mathbb{R}^2$, both have infinite H_1, then they are homotopy equivalent. **(5 marks)**

(e) The space $GL_3(\mathbb{R})$ is path-connected. **(5 marks)**

Solution: In each part, two marks will be awarded for a correct true/false answer with no justification, and up to three (or exceptionally four) marks may be awarded for a reasonable line of argument leading to the wrong answer.

(a) False. If $X = \{(x,y) \in S^1 \mid y \geq 0\}$ and $Y = \{(x,y) \in S^1 \mid y \leq 0\}$ then X and Y are path-connected but $X \cap Y$ is not.

(b) False. We have $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(S^2) = 0$ so T cannot be homotopy equivalent to S^2.

(c) False. Take $X = \{0\}$, $Y = S^1$, $f(0) = 1$ and $g(x,y) = 0$. Then $gf = \mathrm{id}_X$ but $\pi_1(X) = 0$ whereas $\pi_1(Y) = \mathbb{Z}$.

(d) False. The letter O has $H_1(O) = \mathbb{Z}$, whereas B has $H_1(B) = \mathbb{Z}^2$. These two homology groups are both infinite but are not isomorphic, so the spaces are not homotopy equivalent.

(e) False. The determinant gives a continuous map $\det\colon GL_3(\mathbb{R}) \to \mathbb{R} \setminus \{0\}$, which takes a positive value at I and a negative value at $-I$, so there can be no path from I to $-I$ in $GL_3(\mathbb{R})$.

(44) 2018-19 Q2: Are the following true or false? Justify your answers.

(a) S^5 is a Hausdorff space. **(4 marks)**

(b) The Klein bottle is a retract of $S^1 \times S^1 \times S^1$. **(4 marks)**

(c) There is a connected space X with $\pi_1(X) \simeq \mathbb{Z}/2$ and $H_1(X) \simeq \mathbb{Z}$. **(4 marks)**

(d) There is a short exact sequence $\mathbb{Z}/9 \to \mathbb{Z}/99 \to \mathbb{Z}/11$. **(4 marks)**

(e) If K is a simplicial complex and L is a subcomplex and $H_3(K) = 0$ then $H_3(L) = 0$. **(4 marks)**

(f) If K and L are simplicial complexes and $f\colon |K| \to |L|$ is a continuous map then there is a simplicial map $s\colon K \to L$ such that f is homotopic to $|s|$. **(5 marks)**

Solution:

(a) This is true [1], because the standard topology on S^5 comes from the Euclidean metric on $\mathbb{R}^6$, and metric spaces are always Hausdorff. [3] **[It was proved in lectures that metric spaces are Hausdorff.]**

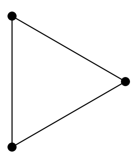
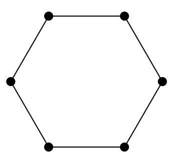
(b) This is false [1]. Let X be the Klein bottle. If this was a retract of $(S^1)^3$, then $\pi_1(X)$ would be a retract of the group $\pi_1((S^1)^3) = \mathbb{Z}^3$, so in particular it would be a subgroup of $\mathbb{Z}^3$ and so would be abelian. However, it is standard that $\pi_1(X)$ is nonabelian, so this is a contradiction. [3] **[Similar examples have been seen.]**

(c) This is false [1]. For a connected space X, the group $H_1(X)$ is always the abelianisation of $\pi_1(X)$. Thus, if $\pi_1(X)$ is $\mathbb{Z}/2$ then $H_1(X)$ must also be $\mathbb{Z}/2$. [3] **[Unseen]**

(d) This is true [1]: there is a short exact sequence $\mathbb{Z}/9 \xrightarrow{i} \mathbb{Z}/99 \xrightarrow{p} \mathbb{Z}/11$ given by $i(a \pmod 9) = 11a \pmod{99}$ and $p(b \pmod{99}) = b \pmod{11}$. [3] Alternatively, as 9 and 11 are coprime we can use the Chinese Remainder Theorem to identify $\mathbb{Z}/99$ with $\mathbb{Z}/9 \times \mathbb{Z}/11$. We then have a short exact sequence $\mathbb{Z}/9 \xrightarrow{j} \mathbb{Z}/9 \times \mathbb{Z}/11 \xrightarrow{q} \mathbb{Z}/11$ given by $j(x) = (x,0)$ and $q(x,y) = y$. **[Similar examples have been seen.]**

(e) This is false [1]. For example, if $K = \Delta^4$ and $L = \partial\Delta^4 \subset K$ then $H_3(K) = 0$ but $H_3(L) = \mathbb{Z}$. [3] **[Seen]**

(f) This is false. [1]For example, K and L could be as follows:

$K =$  $L =$ 

If $s\colon K \to L$ is a simplicial map, it is easy to see that the image can only be a single point or a single edge of L, and thus that $|s|$ is homotopic to a constant map. However, it is easy to produce a homeomorphism $f\colon |K| \to |L|$ and then f is not homotopic to a constant, so it cannot be homotopic to $|s|$ for any s. [4] (By the Simplicial Approximation Theorem, for any $f\colon |K| \to |L|$ we can find a corresponding map $s\colon K^{(r)} \to L$ for sufficiently large r; but that is not relevant here, because the question specifies that s should be defined on K itself.) **[Similar examples have been seen.]**

(45) 2023-24 Q5: Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems and calculations, provided that you state them clearly.

(a) S^3 is contractible. **(3 marks)**

(b) $\mathbb{R}P^3$ is a homotopy retract of S^3. **(3 marks)**

(c) If a space X is the union of two closed, path-connected subspaces A and B, then X is path-connected. **(3 marks)**

(d) $(\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1. **(4 marks)**

(e) $(\mathbb{R} \times \mathbb{R}^2) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1. **(4 marks)**

(f) The space $\mathbb{C} \setminus \{0, 1\}$ is homeomorphic to $\mathbb{C} \setminus \{i, -i\}$. **(4 marks)**

(g) The space $\mathbb{C} \setminus \{0, 1\}$ is homotopy equivalent to $\mathbb{C} \setminus \{0, 1, 2\}$. **(4 marks)**

Solution:

(a) False [1]. We have $H_3(S^3) = \mathbb{Z}$ but H_3 of a point is zero, so S^3 is not homotopy equivalent to a point [2].

(b) False [1]. If $\mathbb{R}P^3$ was a homotopy retract of S^3 then the group $H_1(\mathbb{R}P^3) = \mathbb{Z}/2$ would be isomorphic to a subgroup of the group $H_1(S^3) = 0$, which is clearly not true [2].

(c) False [1]. Put $X = \{0, 1\}$ and $A = \{0\}$ and $B = \{1\}$. Then A and B are closed path connected subsets of X with $X = A \cup B$, but X is not path connected [2]. (You would not be required to say this, but I remark that if $X = A \cup B$ where A and B are path connected (not necessarily closed) and $A \cap B \neq \emptyset$ then X is path connected.)

(d) False [1]. Write

$$X = (\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{R} \times \{0\}) = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}.$$

We can then define a map $f\colon X \to \mathbb{R}$ by $f(x, y) = y$. This is never zero and it is positive at $(0, 1)$ and negative at $(0, -1)$, so $(0, 1)$ cannot be joined to $(0, -1)$ by a path in X, so X is not path connected [1]. However, S^1 is path connected [1]and anything homotopy equivalent to a path connected space is again path connected so X is not homotopy equivalent to S^1 [1].

(e) True [1]. Write $Y = (\mathbb{R} \times \mathbb{R}^2) \setminus (\mathbb{R} \times \{0\})$, and define maps as follows

$$\begin{aligned} f&\colon Y \to S^1 & f(x, y, z) &= (y, z)/\sqrt{y^2 + z^2}\text{[1]} \\ g&\colon S^1 \to Y & g(y, z) &= (0, y, z)\text{[1]}. \end{aligned}$$

Then $fg = 1_{S^1}$, and gf is linearly homotopic to 1_Y [1].

(f) True [1]. We can define a homeomorphism $f\colon \mathbb{C} \setminus \{0, 1\} \to \mathbb{C} \setminus \{i, -i\}$ by $f(z) = 2iz - i$, with inverse $f^{-1}(w) = (w + i)/2i$ [3].

(g) False [1]. We have $H_1(\mathbb{C} \setminus \{0, 1\}) \simeq \mathbb{Z}^2$, and this is not isomorphic to $H_1(\mathbb{C} \setminus \{0, 1, 2\}) \simeq \mathbb{Z}^3$, so $\mathbb{C} \setminus \{0, 1\}$ is not homotopy equivalent to $\mathbb{C} \setminus \{0, 1, 2\}$ [3].

8 Examples

(46) Give examples of the following things, with careful justification.

(a) A noncompact topoological space X with a sequence of compact subspaces $Y_1 \subset Y_2 \subset \dots$ such that the union of all the sets Y_n is equal to X.

(b) A topoological space X with two noncompact subsets Y, Z such that $Y \cup Z$ is compact.

(c) A sequence in $\mathbb{R}$ with no convergent subsequence.

(d) A non-surjective map $f\colon X \to Y$ such that $f_*\colon \pi_0(X) \to \pi_0(Y)$ is surjective.

(e) An injective map $f\colon X \to Y$ such that $f_*\colon \pi_0(X) \to \pi_0(Y)$ is not injective.

Solution:

(a) Take $X = \mathbb{R}$, $Y_n = [-n, n]$. The sequence $(1, 2, 3, \dots)$ in $\mathbb{R}$ has no convergent subsequence, so $\mathbb{R}$ is noncompact. Moreover, Y_n is a bounded closed subspace of $\mathbb{R}$ and thus is compact. For any $x \in \mathbb{R}$ we can choose an integer $n > |x|$ and then $x \in Y_n$, which shows that $X = Y_1 \cup Y_2 \cup \dots$.

(b) Put $X = [0, 1]$ and $Y = (0, 1]$ and $Z = [0, 1)$. Then $Y \cup Z = [0, 1]$ which is compact. The sequence $(1/n)$ in Y has n subsequence that converges in Y, so Y is noncompact. Similarly, the sequence $(1 - 1/n)$ in Z has no subsequence that converges in Z, so Z is noncompact.

(c) The sequence $1, 2, 3, \dots$ in $\mathbb{R}$ has no convergent subsequence, because any two distict terms have distance at least one apart so no subsequence can be Cauchy.

(d) Let $X = \{0\}$ and $Y = [0, 1]$ and define $f\colon X \to Y$ by $f(0) = 0$. Then X and Y are both path-connected, so $\pi_0(X)$ and $\pi_0(Y)$ have only one point each. The map $f_*\colon \pi_0(X) \to \pi_0(Y)$ sends the only point in $\pi_0(X)$ to the only point in $\pi_0(Y)$, so f_* is a bijection and in particular is surjective. However f is obviously not surjective, as 1 does not lie in the image of f for example.

(e) Put $X = \{0, 1\}$ and $Y = [0, 1]$ and let $f\colon X \to Y$ be the inclusion map, which is clearly injective. If we write a for the component of 0 in X and b for the component of 1 in X and c for the component of 0 in Y then $\pi_0(X) = \{a, b\}$ and $\pi_0(Y) = \{c\}$ and $f_*(a) = f_*(b) = c$, so f_* is not injective.

(47) Give examples of the following things, with careful justification.

(a) A continuous bijection that is not a homeomorphism. **(3 marks)**

(b) An infinite sequence of open sets whose intersection is not open. **(3 marks)**

(c) Two metric spaces X, Y such that X is bounded, Y is unbounded, and X is homeomorphic to Y. **(4 marks)**

(d) A sequence in $(0, 1)$ such that no subsequence converges in $(0, 1)$. **(5 marks)**

(e) Two contractible subsets of $\mathbb{R}^2$ whose intersection is not contractible. **(5 marks)**

(f) Two topoological spaces X, Y and points $x \in X$, $y \in Y$ such that X is homotopy equivalent to Y but $X \setminus \{x\}$ is not homotopy equivalent to $Y \setminus \{y\}$. **(5 marks)**

Solution:

(a) Put $X = ([-1, 0] \times \{0\}) \cup ((0, 1] \times \{1\}) \subset \mathbb{R}^2$, and $Y = [-1, 1] \subset \mathbb{R}$ [1]. The map $q\colon X \to Y$ defined by $q(x, y) = x$ [1]is a continuous bijection, but not a homeomorphism (because Y is sequentially compact and X is not, for example) [1]. **[seen]**

(b) Put $U_n = (-1/n, 1/n)$ [2], which is open in $\mathbb{R}$. The intersection of all the sets U_n is the one-point set $\{0\}$, [1]which is not open in $\mathbb{R}$. **[seen]**

(c) Put $X = (0, 1)$ and $Y = (1, \infty)$, so clearly X is bounded and Y is not. Define $f\colon X \to Y$ by $f(x) = 1/x$. This is a homeomorphism, with $f^{-1}(y) = 1/y$. [4] **[seen]**

(d) Put $x_n = 1/n$ [2]. This converges in $\mathbb{R}$ to 0, so any subsequence converges in $\mathbb{R}$ to 0 [2], so it has no limit in $(0,1)$. [1][**seen**]

(e) Put

$$X_+ = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1 \text{ and } y \geq 0\}$$
$$X_- = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1 \text{ and } y \leq 0\}.\text{[2]}$$

Then X_+ and X_- are homeomorphic to I and thus are contractible [1], but $X_+ \cap X_- = \{(-1,0),(1,0)\}$ is not path-connected [1]and thus certainly not contractible [1]. [**unseen**]

(f) Take $X = Y = I$ and $x = 0$ and $y = 1/2$ [3]. Then X and Y are contractible, and thus certainly homotopy equivalent to each other [1]. However, $X \setminus \{x\}$ is path-connected and $Y \setminus \{y\}$ is not, so $X \setminus \{x\}$ is not homotopy equivalent to $Y \setminus \{y\}$ [1]. [**unseen**]

(48) Give examples of the following things, with justification.

(a) Connected sets $X, Y \subseteq \mathbb{R}^2$ such that $X \cap Y$ is not connected.

(b) A sequence of open sets $U_n \subseteq \mathbb{R}$ such that the set $X = U_1 \cap U_2 \cap \ldots = \bigcap_n U_n$ is not open.

(c) A surjective map $f \colon X \to Y$ of topological spaces such that the homomorphism $f_* \colon H_1(X) \to H_1(Y)$ is not surjective.

(d) A path connected space X that is homotopy equivalent to $X \times X$.

(e) A path connected space X that is not homotopy equivalent to $X \times X$.

Solution:

(a) Put

$$\begin{aligned} X &= \text{the upper half of the unit circle} \\ &= \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\ ,\ y \geq 0\} \\ Y &= \text{the lower half of the unit circle} \\ &= \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\ ,\ y \leq 0\} \end{aligned}$$

Then X and Y are both connected, but $X \cap Y = \{(-1,0),(1,0)\}$, which is disconnected.

(b) Put $U_n = \{x \in \mathbb{R} \mid |x| < 1/n\} = (-1/n, 1/n)$. Then U_n is open in $\mathbb{R}$, but

$$\begin{aligned} \bigcap_n U_n &= \{x \in \mathbb{R} \mid |x| < 1/n \text{ for all } n\} \\ &= \{x \in \mathbb{R} \mid |x| = 0\} \\ &= \{0\}, \end{aligned}$$

which is not open.

(c) Define $\eta \colon \mathbb{R} \to S^1$ by $\eta(t) = \exp(2\pi i t)$, which gives a surjective, continuous map. As $H_1(\mathbb{R}) = \{e\}$ and $H_1(S^1)$ is infinite, it is clear that $\eta_* \colon H_1(\mathbb{R}) \to H_1(S^1)$ cannot be surjective.

(d) The spaces I and $I \times I$ are both homotopy equivalent to a point, and thus to each other. (For a more degenerate example, one could just take X to be a point.)

(e) The space S^1 is not homotopy equivalent to $S^1 \times S^1$ (because $H_1(S^1) = \mathbb{Z}$ is not isomorphic to $H_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$).

(49) Give examples of the following things.

(a) A space X and a point $x \in X$ such that X is not contractible but $X \setminus \{x\}$ is contractible. **(3 marks)**

(b) A subspace $X \subseteq \mathbb{R}^2$ that is homotopy equivalent to $S^4 \setminus S^2$. (You need not give a proof.) **(4 marks)**

(c) Spaces X and Y, a discontinuous map $f\colon X \to Y$, and an open subset $V \subseteq Y$ such that $f^{-1}V$ is not open in X. (You should justify your answer carefully.) **(6 marks)**

(d) A space X and a point $x \in X$ such that $\pi_1(X)$ is abelian and $\pi_1(X \setminus \{x\})$ is nonabelian. (You should state what $\pi_1(X)$ and $\pi_1(X \setminus \{x\})$ are, but no further justification is required.) **(6 marks)**

(e) A space X such that $a(X) = 2$ and $b(X) = 2$, where as usual

$$\begin{aligned} a(X) &= \max\{|S| \mid S \text{ is a finite subset of } X \text{ and } X \setminus S \text{ is path-connected }\} \\ &= \text{the largest number of points that can be} \\ &\quad\ \text{removed from } X \text{ without disconnecting it} \\ b(X) &= \min\{|S| \mid S \text{ is a finite subset of } X \text{ and } X \setminus S \text{ is not path-connected }\} \\ &= \text{the smallest number of points that have to be} \\ &\quad\ \text{removed from } X \text{ to disconnect it} \end{aligned}$$

(You should justify your answer, but complete rigour is not required.) **(6 marks)**

Solution:

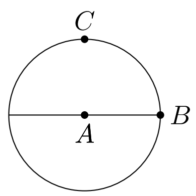
(a) S^1 is not contractible (because $H_1(S^1) = \mathbb{Z}$ is nontrivial) but $S^1 \setminus \{1\}$ is homeomorphic to $\mathbb{R}$ and thus is contractible. [3] **[seen]These facts are in the summary.**

(b) In general, $S^n \setminus S^m$ is homotopy equivalent to S^{n-m-1}. In particular, the space $S^4 \setminus S^2$ is homotopy equivalent to S^1, which is a subset of $\mathbb{R}^2$ [4]. **[seen]The homotopy equivalence $S^n \setminus S^m \simeq S^{n-m-1}$ is in the summary.**

(c) Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x > 0$. [1]This is discontinuous at $x = 0$, [1]because $1/n \to 0$ but $f(1/n) = 1 \not\to 0 = f(0)$ [1]. If we put $V = (-1, 1) \subset \mathbb{R}$ [1]then $f^{-1}V = (-\infty, 0]$ [1]. Thus V is open but $f^{-1}V$ is not [1]. **[seen]**

(d) Put $X = T = S^1 \times S^1$ and $x = (1, 1)$ [2]. Then $\pi_1(X) = \mathbb{Z} \times \mathbb{Z}$ [1], which is abelian. However, $X \setminus \{x\}$ is homotopy equivalent to a figure eight [1], so $\pi_1(X \setminus \{x\})$ is the free group on two generators [1], which is not abelian [1]. **[seen]These facts are in the summary. The only spaces ever mentioned with nonabelian fundamental group are the figure eight, the torus with one puncture, and the plane with two punctures. The fact that these three spaces are homotopy equivalent is also in the summary.**

(e) We can take X to be the letter B, or the following space, which is homeomorphic to the letter B: [3]



It is clear that $X \setminus \{A, C\}$ is connected (so $a(X) \geq 2$) and $X \setminus \{A, B\}$ is not (so $b(X) \leq 2$). By inspection, if we remove any one point, then X remains connected, so $b(X) = 2$. Also, if we remove any three points, then X becomes disconnected, so $a(X) = 2$. [3] **[seen]The calculation of a(letter B) and b(letter B) has been seen (and similarly for various other letters, and some other spaces). However, the students have not previously been asked to find a space with prescribed values of $a(X)$ and $b(X)$.**

(50) Give one example of each of the following things, with justification.

(a) A path connected space X with $H_1(X) = \mathbb{Z} \oplus (\mathbb{Z}/2)$. **(4 marks)**

(b) A path-connected space X and points $a, b, c \in X$ such that $X \setminus \{a, b, c\}$ is still path-connected. **(3 marks)**

(c) A path-connected space X and a point $a \in X$ such that $H_1(X)$ and $H_1(X \setminus \{a\})$ are both trivial. **(5 marks)**

(d) A continuous, surjective map $f\colon X \to Y$, where Y is compact but X is not. **(3 marks)**

(e) A space X and points $a, b \in X$ such that $\pi_1(X)$ is nonabelian but the space $Y = X \setminus \{a, b\}$ is simply connected. **(5 marks)**

(f) A continuous bijection that is not a homeomorphism. **(5 marks)**

Solution:

(a) For path connected spaces Y and Z, the product $Y \times Z$ is also path connected and has $H_1(Y \times Z) = H_1(Y) \oplus H_1(Z)$. The spaces S^1 and $\mathbb{R}P^2$ are path connected with $H_1(S^1) = \mathbb{Z}$ [1]and $H_1(\mathbb{R}P^2) = \mathbb{Z}/2$ [2] so $H_1(S^1 \times \mathbb{R}P^2) = \mathbb{Z} \oplus (\mathbb{Z}/2)$ [1].

(b) The simplest example is $X = \mathbb{R}^2$, $a = (-1, 0)$, $b = (0, 0)$, $c = (1, 0)$. It is also easy to exhibit one-dimensional examples (eg the wedge of three circles), and this may well be the most popular type of answer. [3] **[similar examples seen]**

(c) Take $X = S^2$, and let a be any point in X. Then $\pi_1(X) = 0$. Moreover, $X \setminus \{a\}$ is homeomorphic to $\mathbb{R}^2$, which is contractible, so $\pi_1(X \setminus \{a\})$ is again trivial. [5] **The individual facts mentioned are in the summary.**

(d) Take $X = \mathbb{R}$, $Y = \{0\}$, $f(x) = 0$. I expect that students will generally give more complicated examples. [3] **[unseen]**

(e) Let X be the figure eight [2], or in other words the union of the circles of radius one centred at $(1, 0)$ and $(-1, 0)$, so $\pi_1(X)$ is nonabelian [1]. Put $a = (-2, 0)$ and $b = (2, 0)$, so $X \setminus \{a, b\}$ is homeomorphic to the union of two lines meeting at a point. This means that $X \setminus \{a, b\}$ is contractible, and thus simply connected [2]. **The space X is mentioned repeatedly as an example with nonabelian fundamental group, and no other examples are given.**

(f) Define $e\colon [0, 2\pi) \to S^1$ by $e(t) = \exp(i\theta)$ [1]. Every point $z \in S^1$ can be written as $z = \exp(i\theta)$ for a unique angle θ in the range $0 \leq \theta < 2\pi$, so e is a bijection [1]. It is well-known to be continuous [1], but e^{-1} is not continuous [1] because $\exp(-i/n) \to 1$ in S^1 but $e^{-1}(\exp(-i/n)) = 2\pi - 1/n \not\to 0$ [1]. **[bookwork]**

(51) 2020-21 Q1: Give examples as follows, justifying your answers.

(a) Topological spaces X and Y, together with injective functions $f\colon X \to Y$ and $g\colon Y \to X$ such that f, $f \circ g$ and $g \circ f$ are all continuous, but g is not continuous. **(4 marks)**

(b) A compact, path-connected space X together with a continuous map $f\colon X \to X$ with no fixed points. **(4 marks)**

(c) A space X such that $H_1(X)$ is not a free abelian group. (Note here that the zero group is free abelian with no generators, so in particular $H_1(X)$ must be nonzero.) **(4 marks)**

(d) A space X together with points $a, b, c \in X$ such that $|\Pi(X; a, b)| \neq |\Pi(X; b, c)|$. **(4 marks)**

(e) A space X such that $\pi_1(X)$ is a free group with 3 generators, and $H_2(X) = \mathbb{Z}$. **(4 marks)**

Solution: In each case, two marks will be awarded for a correct example, and two further marks for justifying it. Up to two marks may also be awarded for intelligent discussion of an incorrect example. Note that in addition to the main lecture notes, students have access to a two-page summary of examples.

(a) We can use the standard example of a continuous bijection that is not a homeomorphism (Example 4.8):

$$X = (-\infty, 0] \cup (1, \infty) \qquad\qquad Y = \mathbb{R}$$

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x - 1 & \text{if } x > 1 \end{cases} \qquad\qquad g(y) = \begin{cases} y & \text{if } y \leq 0 \\ y + 1 & \text{if } y > 0. \end{cases}$$

Here f is continuous because the domains of the two clauses are both open in X, and $f \circ g$ and $g \circ f$ are identity maps so they are certainly continuous, but g is discontinuous at $y = 0$. [4]

(b) We can take $X = S^n$ for any $n > 0$, and $f(x) = -x$. (Example 9.15 mentions that S^n is compact, as an easy application of Proposition 9.14. It is path-connected by Proposition 5.11. This example of a fixed-point-free endomorphism is mentioned in the solution to Exercise 3 of Problem Sheet 9.) [4]

(c) We can take $X = \mathbb{R}P^2$, then $H_1(X) = \mathbb{Z}/2$, which is not free abelian. (Example 12.15 shows that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2$, and Theorem 18.18 shows that $H_1(\mathbb{R}P^2)$ is the abelianisation of this, which is $\mathbb{Z}/2$ again.) [4]

(d) We can take $X = \{0\} \amalg \mathbb{R}P^2$, with $a = 0$ and $b = c =$ basepoint of $\mathbb{R}P^2$. Then $\Pi(X; a, b) = \emptyset$ and $\Pi(X; b, c) = \pi_1(\mathbb{R}P^2, b) = C_2$ so $|\Pi(X; a, b)| = 0$ but $|\Pi(X; b, c)| = 2$. [4]

(e) We can take $X = S^1 \vee S^1 \vee S^1 \vee S^2$. Using Corollary 15.20 (a special case of the van Kampen Theorem) we see that $\pi_1(X)$ is the free product of three copies of $\pi_1(S^1) = \mathbb{Z}$ together with one copy of $\pi_1(S^2) = 1$, so it is free on three generators. Similarly, we can use Lemma 21.4 (a special case of the Mayer-Vietoris Theorem) to show that $H_2(X) = 0 \oplus 0 \oplus 0 \oplus \mathbb{Z} = \mathbb{Z}$ as required. [4]

Feedback: For part (a), another good answer (given by several students) is to define $f : [0, 2\pi) \to S^1$ by $f(x) = e^{ix}$, note that this is bijective, and take $g = f^{-1}$. Most people answered (b) correctly, using the same example as in the solution above. Some people gave answers for (c) where they claimed that $H_1(X)$ was not abelian, but homology groups are always abelian. Most people answered (d) correctly (but sometimes with inadequate justification); correct answers for (e) were rare.

(52) 2021-22 Mock Q4: For each of the following, either give an example (with justification) or prove that no example can exist.

(a) A continuous map $f : X \to Y$ such that $f_* : H_1(X) \to H_1(Y)$ is injective but not surjective, and $f_* : H_{10}(X) \to H_{10}(Y)$ is surjective but not injective. **(5 marks)**

(b) A path connected space X that is homotopy equivalent to $X \times X$. **(5 marks)**

(c) A path connected space X that is not homotopy equivalent to $X \times X$. **(5 marks)**

(d) A space X and a point $x \in X$ such that X is not contractible but $X \setminus \{x\}$ is contractible. **(5 marks)**

(e) A subspace $X \subseteq \mathbb{R}^2$ that is homotopy equivalent to $S^4 \setminus S^2$. **(5 marks)**

Solution:

(a) Let $f : S^{10} \to S^1$ be the constant map sending all of S^{10} to the point $e_0 \in S^1$ [3]. Then $f_* : H_1(S^{10}) \to H_1(S^1)$ is the inclusion $0 \to \mathbb{Z}$, which is injective but not surjective [1]. Moreover, $f_* : H_{10}(S^{10}) \to H_{10}(S^1)$ is the zero homomorphism $\mathbb{Z} \to 0$, which is surjective but not injective [1].

(b) The spaces $I = [0, 1]$ and $I \times I$ are both homotopy equivalent to a point, and thus to each other [5]. (For a more degenerate example, one could just take X to be a point.)

(c) The space S^1 is not homotopy equivalent to $S^1 \times S^1$ [3] (because $H_1(S^1) = \mathbb{Z}$ is not isomorphic to $H_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$) [2].

(d) S^1 [3] is not contractible (because $H_1(S^1) = \mathbb{Z}$ is nontrivial [1]) but $S^1 \setminus \{1\}$ is homeomorphic to $\mathbb{R}$ and thus is contractible [1].

(e) In general, $S^n \setminus S^m$ is homotopy equivalent to S^{n-m-1} [2]. In particular, the space $S^4 \setminus S^2$ is homotopy equivalent to S^1, which is a subset of $\mathbb{R}^2$ [3].

(53) 2021-22 Q4: For each of the following, either give an example (with justification) or prove that no example can exist.

(a) A continuous injective map $i : X \to Y$ such that the map $i_* : H_2(X) \to H_2(Y)$ is not injective. **(5 marks)**

(b) A continuous surjective map $p : X \to Y$ such that the map $p_* : H_2(X) \to H_2(Y)$ is not surjective. **(5 marks)**

(c) A contractible space X and a homeomorphism $f : X \to X$ with no fixed points. **(5 marks)**

(d) A continuous injective map $f : S^1 \to S^3$ such that $S^3 \setminus f(S^1)$ is homotopy equivalent to S^1. **(5 marks)**

(e) A continuous injective map $f : S^1 \to S^3$ such that $S^3 \setminus f(S^1)$ is contractible. **(5 marks)**

Solution:

(a) **Similar examples have been seen** Let i be the inclusion $S^2 \to B^3$ [3]. This is continuous and injective, but $H_2(S^2) \simeq \mathbb{Z}$ and $H_2(B^3) = 0$ so the map $i_* : H_2(S^2) \to H_2(B^3)$ is zero and is not injective [2].

(b) **Unseen** Let $p : [0, 1]^2 \to T = S^1 \times S^1$ be the usual gluing map, given by $p(s, t) = (e^{2\pi is}, e^{2\pi it})$ [3]. This is continuous and surjective, but $H_2([0, 1]^2) = 0$ and $H_2(T) \simeq \mathbb{Z}$ so the map $p_* : H_2([0, 1]^2) \to H_2(T)$ is zero and is not surjective [2].

(c) **Similar examples have been seen** Take $X = \mathbb{R}$ and define $f\colon X \to X$ by $f(x) = x + 1$ [3]. Then X is contractible and f is a homeomorphism (with $f^{-1}(x) = x - 1$) and f has no fixed points [2].

(d) **Bookwork** Let $f\colon S^1 \to S^3$ be the standard inclusion given by $f(u, v) = (u, v, 0, 0)$, and put $X = S^3 \setminus f(S^1)$ [2]. We then have

$$X = \{(u, v, w, x) \in \mathbb{R}^4 \mid u^2 + v^2 + w^2 + x^2 = 1,\ (w, x) \neq (0, 0)\}.$$

We can thus define $i\colon S^1 \to X$ and $r\colon X \to S^1$ and $h\colon [0, 1] \times X \to X$ by $i(w, x) = (0, 0, w, x)$ and $r(u, v, w, x) = (w^2 + x^2)^{-1/2}(w, x)$ and

$$h(t, (u, v, w, x)) = (t^2u^2 + t^2v^2 + w^2 + x^2)^{-1/2})(tu, tv, w, x).$$

We then find that $r \circ i = \mathrm{id}$ and h gives a homotopy between $i \circ r$ and the identity so i is a homotopy equivalence [3].

(e) **Immediate consequence of bookwork** The generalised Jordan Curve Theorem says that for any continuous injective map $f\colon S^1 \to S^3$, the complement $S^3 \setminus f(S^1)$ has the same homology as S^1 [3] and so cannot be contractible [2].

(54) 2022-23 Q4: For each of the following, either give an example (with justification) or prove that no example can exist.

(a) A topological space X with two noncompact subsets Y, Z such that $Y \cup Z$ is compact. **(5 marks)**

(b) Subsets $A, B, C \subseteq \mathbb{R}^2$ such that A, B, C, $A \cup B$, $A \cup C$ and $B \cup C$ are all contractible, but $A \cup B \cup C$ is not contractible. **(5 marks)**

(c) A topological space X with two open subsets U and V such that U, V and $U \cap V$ are all homotopy equivalent to S^1, and $X = U \cup V$, and X is homotopy equivalent to S^4. **(5 marks)**

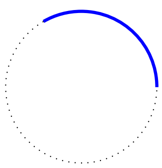
(d) A path connected space X such that $H_*(X)$ is not isomorphic to $H_*(X \times X)$. **(5 marks)**

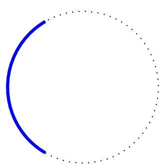
(e) Spaces X and Y such that X is path connected, Y is not path connected, and $H_k(X) \simeq H_k(Y)$ for all k. **(5 marks)**

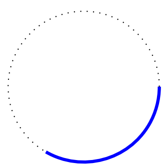
Solution:

(a) Take $X = S^1 \subset \mathbb{C}$ and $Y = X \setminus \{-1\}$ and $Z = X \setminus \{1\}$. Then neither Y nor Z is closed in $\mathbb{C}$, so they are both noncompact. However, $Y \cup Z = X$, and this is bounded and closed in $\mathbb{C}$ and is therefore compact. [5]

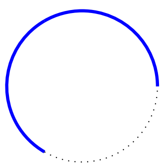
(b) Take A, B and C as follows:

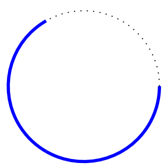

$A = \{e^{2\pi i t/3} \mid 0 \leq t \leq 1\}$

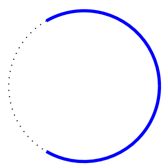

$B = \{e^{2\pi i t/3} \mid 1 \leq t \leq 2\}$


$C = \{e^{2\pi i t/3} \mid 2 \leq t \leq 3\}$

These are clearly contractible, as are the unions $A \cup B$, $B \cup C$ and $C \cup A$:


$A \cup B$


$B \cup C$


$C \cup A$

However, $A \cup B \cup C$ is the full circle S^1, which is not contractible. [5]

(c) This is not possible. If X, U and V were as specified, we would have $H_4(U) \simeq H_4(V) \simeq H_4(S^1) = 0$ and $H_3(U \cap V) \simeq H_3(S^1) \simeq 0$, whereas $H_4(X) \simeq H_4(S^4) \simeq \mathbb{Z}$. Thus, the Mayer-Vietoris sequence $H_4(U) \oplus H_4(V) \to H_4(X) \to H_3(U \cap V)$ would have the form $0 \to \mathbb{Z} \to 0$, which is not exact. [5]

(d) Take $X = S^1$, so $X \times X$ is a torus. It is clear that X is path connected, and standard calculations give $H_1(X) \simeq \mathbb{Z}$ and $H_1(X \times X) \simeq \mathbb{Z}^2$, so $H_*(X)$ is not isomorphic to $H_*(X \times X)$. [5]

(e) This is not possible. For any space Z we know that $H_0(Z)$ is the free abelian group generated by $\pi_0(Z)$, so $H_0(Z) \simeq \mathbb{Z}$ iff Z is path connected. Thus if X is path connected and Y is not, we cannot have $H_0(X) \simeq H_0(Y)$. [5]

9 Real projective space

In the current version of the course, in the Introduction we define $\mathbb{R}P^n = S^n/(x \sim -x)$ and

$$P_n = \{A \in M_{n+1}(\mathbb{R}) \mid A^2 = A^T = A,\ \mathrm{trace}(A) = 1\},$$

and we mention that $\mathbb{R}P^n$ is homeomorphic to P_n. A proof is given in Problem Sheet 5. In some earlier versions of the course, $\mathbb{R}P^n$ was just defined to be the same as P_n. Problems in this section should be approached from that point of view.

(55)

(a) Define the set $\mathbb{R}P^2$ and the map $q\colon S^2 \to \mathbb{R}P^2$.

(b) Define the usual metric on $\mathbb{R}P^2$, and prove that it is a metric.

(c) Define the space Δ_2, and prove carefully that there is a surjective continuous map $f\colon \mathbb{R}P^2 \to \Delta_2$ satisfying $fq(u, v, w) = (u^2, u^2 + v^2)$ for all $(u, v, w) \in \Delta_2$. You may use general theorems provided that you state them precisely.

Solution:

(a) We can define an equivalence relation $\sim$ on S^2 by $x \sim y$ iff ($x = y$ or $x = -y$). The set $\mathbb{R}P^2$ is the set of equivalence classes for this relation. The map $q\colon S^2 \to \mathbb{R}P^2$ is defined by $q(x) = \langle x \rangle$, the equivalence class of x.

(b) We define $e\colon \mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}$ by $e(x, y) = \min(\|x - y\|, \|x + y\|)$. This is clearly nonegative and symmetric, and we have $e(x, y) = 0$ iff one of $\|x - y\|$ and $\|x + y\|$ is zero, iff either $x = -y$ or $x = y$, or in other words iff $x \sim y$. Clearly also

$$e(x, y) = e(-x, y) = e(x, -y) = e(-x, -y),$$

and it follows that there is a well-defined function $d\colon \mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}$ such that $d(q(x), q(y)) = e(x, y)$. It is clear that $d(u, v) \geq 0$, with equality iff $u = v$, and that $d(u, v) = d(v, u)$. All that is left is to check the triangle inequality. Suppose we have $u, v, w \in \mathbb{R}P^2$. Choose $x \in S^2$ such that $u = q(x)$. Next, choose $y \in S^2$ such that $q(y) = v$. After replacing y by $-y$ if necessary, we may assume that $\|x - y\| \leq \|x + y\|$, so that $d(u, v) = \|x - y\|$. Next, choose $z \in S^2$ such that $q(z) = w$. After replacing z by $-z$ if necessary, we may assume that $\|y - z\| \leq \|y + z\|$, so that $d(v, w) = \|y - z\|$. We then have

$$\begin{aligned} d(u, w) &= \min(\|x - z\|, \|x + z\|) \\ &\leq \|z - x\| \\ &\leq \|y - x\| + \|z - y\| \\ &= d(u, v) + d(v, w), \end{aligned}$$

as required.

(c) The space Δ_2 is $\{(t_1, t_2) \in \mathbb{R}^2 \mid 0 \leq t_1 \leq t_2 \leq 1\}$. Given a point $a = (u, v, w) \in S^2$ we have $u^2, v^2, w^2 \geq 0$ so $0 \leq u^2 \leq u^2 + v^2 \leq 1 = u^2 + v^2 + w^2$, so $(u^2, u^2 + v^2) \in \Delta_2$. We can thus define a map $g\colon S^2 \to \Delta_2$ by $g(u, v, w) = (u^2, u^2 + v^2)$. The components of g are polynomial functions, so g is continuous. Moreover, $g(-u, -v, -w) = g(u, v, w)$, or in other words $g(-a) = g(a)$, or in other words $g(a) = g(b)$ whenever $a \sim b$. Thus, there is a well-defined function $f\colon \mathbb{R}P^2 \to \Delta_2$ defined by $f(q(a)) = g(a)$. Any function $h\colon \mathbb{R}P^2 \to Y$ is continuous iff $hq\colon S^2 \to Y$ is continuous. We have seen that $g = fq$ is continuous, so f is continuous. Moreover, for any $(t_1, t_2) \in \Delta_2$ we have $(\sqrt{t_1}, \sqrt{t_2 - t_1}, \sqrt{1 - t_2}) \in S^2$ and $fq(\sqrt{t_1}, \sqrt{t_2 - t_1}, \sqrt{1 - t_2}) = (t_1, t_2)$, so f is surjective.

(56)

(a) Define the set $\mathbb{R}P^n$, and write down a metric on it, proving that your formula is well-defined. (You need not show that it is a metric.) **(6 marks)**

(b) Define what it means for a metric space X to be *sequentially compact*. **(3 marks)**

(c) Define the set $\pi_0(X)$, and say what it means for X to be *path-connected*. **(6 marks)**

(d) Prove that the space $\mathbb{R}P^n$ is sequentially compact and path-connected. State clearly any general theorems or results that you use. **(10 marks)**

Solution:

(a) The set $\mathbb{R}P^n$ is the quotient of S^n by the equivalence relation $\sim$, where $x \sim y$ iff ($x = y$ or $x = -y$); we write q for the obvious map $S^n \to \mathbb{R}P^n$. We can define a function $e\colon S^n \times S^n \to [0, \infty)$ by

$$e(x, y) = \min(\|x - y\|_2, \|x + y\|_2) = \min(d_2(x, y), d_2(x, -y)).[2]$$

It is easy to see that

$$e(x, y) = e(-x, y) = e(x, -y) = e(-x, -y).[2]$$

Now suppose we have $a, b \in \mathbb{R}P^n$. We can choose $x, y \in S^n$ such that $q(x) = a$ and $q(y) = b$; these elements are unique up to sign. It follows from the above equation that the value of $e(x, y)$ is independent of the signs, so we may define $d(a, b) = e(x, y)$. This gives a metric on $\mathbb{R}P^n$. [2] [**bookwork**]

(b) A space X is *sequentially compact* if every sequence in X has a convergent subsequence. [3] [**bookwork**]

(c) We define a relation on X by $x \sim y$ iff there is a path in X joining x to y, in other words a continuous map $s\colon I \to X$ with $s(0) = x$ and $s(1) = y$ [2]. Using constant paths we see that this is reflexive, using path reversal we see that it is symmetric, and using path join we see that it is transitive. It is thus an equivalence relation [1], so we can define a quotient set $X/\sim$; this is called $\pi_0(X)$ [1].

We say that X is path-connected if $\pi_0(X)$ is a one-point set, or equivalently if $x \sim y$ for all $x, y \in X$. [2] [**bookwork**]

(d) We have a surjective [1]continuous [1]map $q\colon S^n \to \mathbb{R}P^n$. The set S^n is bounded and closed in $\mathbb{R}^{n+1}$, so it is sequentially compact [2]. A continuous image of a sequentially compact set is sequentially compact [1], so $\mathbb{R}P^n$ is sequentially compact [1]. Also, the space S^n is path-connected (by using great circles, say) [2] and a continuous image of a path-connected set is path-connected [1], so $\mathbb{R}P^n$ is path-connected [1]. [**unseen**]

10 Multipart questions

(57)

(a) What is a *metric space*? What is a *continuous function*?

(b) Define the discrete metric on a set X.

(c) Let X be a space with a discrete metric. Show that any path $s\colon \Delta_1 \to X$ is constant, and deduce that $\pi_0(X) = X$.

(d) Consider the space $Y = \{(x, y) \in \mathbb{R}^2 \mid xy \neq 0\}$ and show that $\pi_0(Y)$ has precisely four elements. If $f\colon Y \to Y$ denotes reflection in the line $x = y$, describe the map $f_*\colon \pi_0(Y) \to \pi_0(Y)$. Is f homotopic to the identity map?

Solution:

(a) A metric space is a set X equipped with a metric, ie a function $d\colon X \times X \to \mathbb{R}$ such that

- $d(x, y) \geq 0$ for all $x, y \in X$, with equality iff $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A function $f\colon X \to Y$ between metric spaces is continuous if for each sequence (x_n) in X that converges to a point $x \in X$, the resulting sequence $(f(x_n))$ in Y converges to the point $f(x)$.

(b) The discrete metric on a set X is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

(c) Let $s\colon \Delta_1 \to X$ be a path. Define $f\colon \Delta_1 \to \mathbb{R}$ by $f(t) = d(s(t), s(0))$, so f is continuous and $f(0) = 0$. As d can only take the values 0 and 1, we see that f can only take the vales 0 and 1, so by the Intermediate Value Theorem it must be constant. As $f(0) = 0$ we see that $f(t) = 0$ for all t. As $d(s(0), s(t)) = 0$ we see that $s(t) = s(0)$ for all t, in other words s is constant.

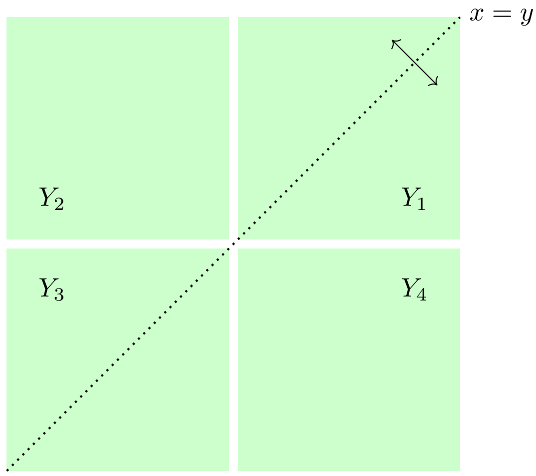
As usual we write $x \sim y$ if x can be connected to y by a path, so $\sim$ is an equivalence relation and $\pi_0(X) = X/\sim$. As the only paths are constant, if $x \sim y$ we must have $x = y$. Thus, each equivalence class consists of just a single point, so $\pi_0(X)$ can be identified with X.

(d) Define

$$\begin{aligned} Y_1 &= \text{ 1st quadrant } = \{(x,y) \mid x > 0, y > 0\} \\ Y_2 &= \text{ 2nd quadrant } = \{(x,y) \mid x < 0, y > 0\} \\ Y_3 &= \text{ 3rd quadrant } = \{(x,y) \mid x < 0, y < 0\} \\ Y_4 &= \text{ 4th quadrant } = \{(x,y) \mid x > 0, y < 0\}. \end{aligned}$$

These are all nonempty convex sets and thus path-connected, and clearly they are open and disjoint. If $(x,y) \in Y$ then $xy \neq 0$ so $x \neq 0$ and $y \neq 0$ so $x < 0$ or $x > 0$ and $y < 0$ or $y > 0$. It follows that (x,y) lies in one of the sets Y_i, so $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$. A path in Y has the form $s(t) = (u(t), v(t))$, where for all t we have $u(t) \neq 0$ and $v(t) \neq 0$. By the intermediate value theorem, we see that $u(0)$ has the same sign as $u(1)$, and $v(0)$ has the same sign as $v(1)$, so if $s(0) \in Y_i$ then $s(1) \in Y_i$ also. It follows that the sets Y_i are the path components of Y, so $\pi_0(Y) = \{Y_1, Y_2, Y_3, Y_4\}$. The formula for the map f is $f(x,y) = (y,x)$, and it follows easily that

$$f_*(Y_1) = Y_1 \qquad f_*(Y_2) = Y_4 \qquad f_*(Y_3) = Y_3 \qquad f_*(Y_4) = Y_2.$$

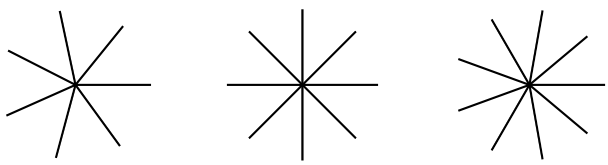


As f_* is not the identity map, we see that f is not homotopic to the identity.

(58) 2018-19 Q1:

(a) Given a topological space X, define the set $\pi_0(X)$. You should include a proof that the relevant equivalence relation is in fact an equivalence relation. **(8 marks)**

(b) Consider $[0,1]$ as a based space with 0 as the basepoint. For $n \geq 3$ we define $X_n = \{z \in \mathbb{C} \mid z^n \in [0,1]\}$:



X_7 X_8 X_9

(i) For which n and m (with $n, m \geq 3$) is X_n homotopy equivalent to X_m? **(3 marks)**

(ii) For which n and m (with $n, m \geq 3$) is X_n homeomorphic to X_m? **(4 marks)**

Justify your answers carefully.

(c) Give examples as follows, with justification:

(1) A based space W with $|\pi_1(W)| = 8$. **(3 marks)**

(2) A space X with two points $a, b \in X$ such that $\pi_1(X, a)$ is not isomorphic to $\pi_1(X, b)$. **(3 marks)**

(3) A space Y such that $H_0(Y) \simeq H_2(Y) \simeq H_4(Y) \simeq H_6(Y) \simeq \mathbb{Z}$ and all other homology groups are trivial. **(4 marks)**

Solution:

(a) We define a relation on X by declaring that $x \sim y$ if there is a continuous path $u \colon [0, 1] \to X$ with $u(0) = x$ and $u(1) = y$. [1]

- For any $x \in X$ we can define $c \colon [0, 1] \to X$ by $c(t) = x$ for all t. Using this we see that $x \sim x$, so or relation is reflexive. [1]
- Suppose that $x \sim y$, as witnessed by a path u from x to y. The reversed path $\overline{u}(t) = u(1 - t)$ is also continuous, with $\overline{u}(0) = y$ and $\overline{u}(1) = x$, which shows that $y \sim x$. This shows that our relation is symmetric. [2]
- Suppose that $x \sim y$ and $y \sim z$, as witnessed by a path u from x to y and a path v from y to z. We can define the concatenated path $u*v \colon [0, 1] \to X$ by $(u*v)(t) = u(2t)$ for $0 \leq t \leq 1/2$ and $(u*v)(t) = v(2t-1)$ for $1/2 \leq t \leq 1$ [2] (so in particular $(u * v)(1/2) = y = u(1) = v(0)$). This is continuous on the closed sets $[0, 1/2]$ and $[1/2, 1]$, which cover $[0, 1]$, so it is continuous on $[0, 1]$. As $(u * v)(0) = u(0) = x$ and $(u * v)(1) = v(1) = z$ we see that $x \sim z$. This shows that our relation is transitive. [1]

We now see that we have an equivalence relation, so we can define $\pi_0(X) = X/\sim$. [1]**[All bookwork]**

(b) (i) For any n we have a contraction of X_n to 0 given by $h(t, z) = tz$ for $0 \leq t \leq 1$. Thus, all the spaces X_n are homotopy equivalent to a point and thus to each other. [3] **[Unseen but easy]**

(ii) Note that $|\pi_0(X_n \setminus \{a\})|$ is 2 for most values of a, but it is n if $a = 0$, and 1 if $|a| = 1$. If we have a homeomorphism $f \colon X_n \to X_m$ then we get a homeomorphism $X_n \setminus \{0\} \to X_m \setminus \{f(0)\}$ so

$$n = |\pi_0(X_n \setminus \{0\})| = |\pi_0(X_m \setminus \{f(0)\})| \in \{1, 2, m\}.$$

As $n, m \geq 3$ this can only occur if $n = m$. Thus, no two of the spaces X_n are homeomorphic. [4] **[Unseen, but the general technique has been seen.]**

(c) (1) We can take $W = (\mathbb{R}P^2)^3$ [2], so $\pi_1(W) = \pi_1(\mathbb{R}P^2)^3 = (\mathbb{Z}/2)^3$, so $|\pi_1(W)| = 8$. [1]**[Unseen, but $\mathbb{R}P^2$ is a standard example.]**

(2) We can take $X = S^1 \cup \{0\} \subset \mathbb{C}$ and $a = 0$ and $b = 1$, so $\pi_1(X, a) = 0$ and $\pi_1(X, b) = \mathbb{Z}$. [3] **[Unseen]**

(3) We can take $Y = S^2 \vee S^4 \vee S^6$. This is connected, so $H_0(Y) = \mathbb{Z}$. For $i > 0$ we have $H_i(Y) = H_i(S^2) \oplus H_i(S^4) \oplus H_i(S^6)$. We also have $H_i(S^i) = \mathbb{Z}$, and $H_i(S^j) = 0$ for $j \neq i$; it follows that $H_*(Y)$ is as required. [4] Alternatively, we can take $Y = \mathbb{C}P^3$. **[Similar examples have been seen.]**

(59) 2021-22 Mock Q1: For $n \geq 3$, we put $X_n = \mathbb{R}^2 \setminus \{(1, 0), (2, 0), \ldots, (n, 0)\}$.

(a) Define the following terms: *topology*, *topological space*, *continuous map*, *homeomorphism*. **(7 marks)**

(b) Find a space Y_n consisting of a finite number of straight line segments that is homotopy equivalent to X_n. Give a brief justification for the claim that Y_n is homotopy equivalent to X_n. **(6 marks)**

(c) Prove that X_n is not homeomorphic to Y_n. **(3 marks)**

(d) Prove that X_n is not homotopy equivalent to S^m for any m. **(4 marks)**

(e) Find contractible open sets $U_n, V_n \subseteq \mathbb{C}$ such that $X_n = U_n \cup V_n$. Give a careful proof that U_n and V_n are contractible. **(5 marks)**

Claims about the homology of particular spaces should be stated clearly and justified briefly, but details are not required. **Solution:**

(a) A *topology* on a set X is a family τ of subsets of X (called *open sets*) [1]such that

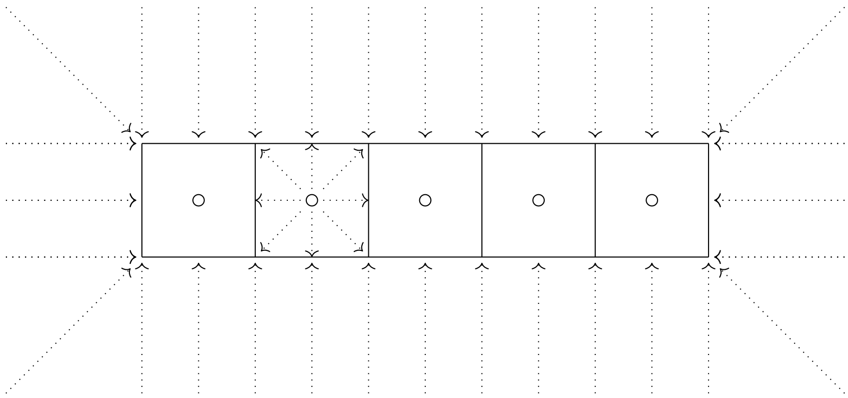
(1) The empty set and the whole set X are open [1]

(2) The union of any family of open sets is open [1]

(3) The intersection of any finite list of open sets is open. [1]

A *topological space* is a st equipped with a topology. If X and Y are topological spaces, a *continuous map* from X to Y is a function $f\colon X \to Y$ such that for every open set $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X [2]. A *homeomorphism* from X to Y is a bijective map $f\colon X \to Y$ with the property that both $f\colon X \to Y$ and $f^{-1}\colon Y \to X$ are continuous [1].

(b) We define Y_n to be the union of line segments from $[\frac{1}{2}, n+\frac{1}{2}] \times \{\pm\frac{1}{2}\}$ and $\{i+\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}]$ for $0 \leq i \leq n$ [3]:



Let $i\colon Y_n \to X_n$ be the inclusion. The dotted arrows indicate a continuous map $r\colon X_n \to Y_n$ such that $ri = \mathrm{id}$ and ir is homotopic to the identity by a straight line homotopy; this proves that Y_n is homotopy equivalent to X_n. [3]

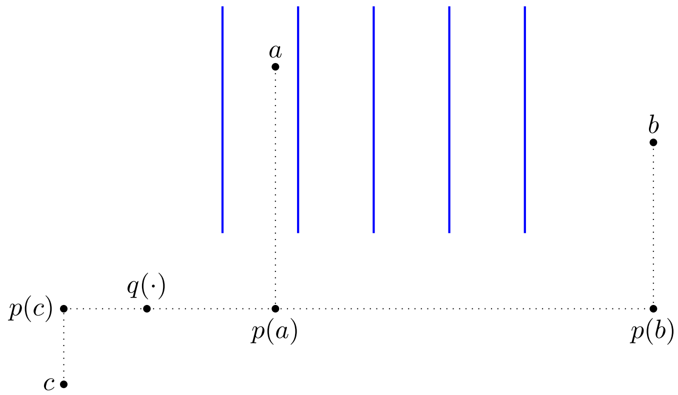
(c) The space Y_n is a bounded and closed subspace of $\mathbb{R}^2$, so it is compact. The space X_n is unbounded and so is not compact. It follows that X_n cannot be homeomorphic to Y_n. [3]

(d) It was proved in the notes that

$$H_i(X_n) \simeq \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^n & \text{if } i = 1 \\ 0 & \text{otherwise .[2]} \end{cases}$$

In particular, the total rank of all the homology groups of X_n is $n+1 \geq 4$, whereas the total rank of all homology groups of S^m is 2. Homotopy equivalent spaces have isomorphic homology, so X_n cannot be homotopy equivalent to S^m. [2]

(e) Put $A_n = \{1, \dots, n\} \times [0, \infty)$ and $B_n = \{1, \dots, n\} \times (-\infty, 0]$. These are closed subsets of $\mathbb{R}^2$ with $A_n \cap B_n = \{(1, 0), \dots, (n, 0)\}$. It follows that the sets $U_n = \mathbb{R}^2 \setminus A_n$ and $V_n = \mathbb{R}^2 \setminus B_n$ are open with $U_n \cup V_n = \mathbb{R}^2 \setminus (A_n \cap B_n) = X_n$ [2]. Define $p, q\colon U_n \to U_n$ by $p(x, y) = (x, -1)$ and $q(x, y) = (0, -1)$. If $(x, y) \in U_n$ then the line segment from (x, y) to $p(x, y)$ is vertical, and the line segment from $p(x, y)$ to $q(x, y)$ is horizontal, and neither segment touches A_n. Thus, we have straight line homotopies from the identity to p and then from p to the constant map q, proving that U_n is contractible.



Essentially the same argument (using $r(x,y) = (x,1)$ and $s(x,t) = (0,1)$) proves that V_n is contractible. [3]

(60) 2021-22 Q1: For $n \geq 3$, we put

$$X_n = \{z \in \mathbb{C} \mid |z| = 1 \text{ or } z^n \in (0,\infty)\}$$
$$Y_n = \{z \in \mathbb{C} \mid |z| = 1 \text{ or } z^n \in [0,\infty)\}.$$

(a) Sketch X_3 and Y_3. **(2 marks)**

(b) Define the terms *homotopy* and *homotopy equivalent.* **(5 marks)**

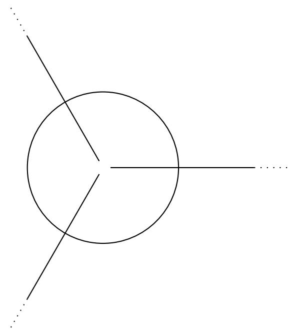
(c) Prove (by constructing explicit maps and homotopies, and checking their validity) that X_n and X_m are homotopy equivalent for all $n, m \geq 3$. **(8 marks)**

(d) Prove that for all $n \neq m$, the space X_n is not homeomorphic to X_m. **(6 marks)**

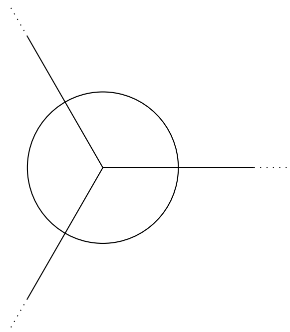
(e) Prove that for all $n \neq m$, the space Y_n is not homotopy equivalent to Y_m. **(4 marks)**

Claims about the homology of particular spaces should be stated clearly and justified briefly, but details are not required. **Solution: This has many ideas in common with Q1 from 2018-19**

(a) The spaces X_3 and Y_3 are as follows:



X_3 [1]



Y_3 [1]

(b) **Bookwork** Let A and B be topological spaces. If p and q are continuous maps from A to B, then a *homotopy* from p to q is a continuous map $h\colon [0,1] \times A \to B$ such that $h(0,a) = p(a)$ and $h(1,a) = q(a)$ for all $a \in A$ [2]. We say that A and B are *homotopy equivalent* if there exist continuous maps $A \xrightarrow{f} B \xrightarrow{g} A$ and a homotopy from $g \circ f$ to id_A and a homotopy from $f \circ g$ to id_B. [3]

(c) For $z \in X_p$ we have $z^p \neq 0$ so $z \neq 0$ so it is legitimate to divide by $|z|$. We can therefore define $f\colon X_n \to X_m$ and $g\colon X_m \to X_n$ by $f(z) = z/|z| \in S^1 \subset X_m$ and $g(w) = w/|w| \in S^1 \subset X_n$ [4]. For $z \in X_n$ we have $g(f(z)) = z/|z|$. If z lies on the unit circle then $g(f(z)) = z$. If z lies on one of the rays of X_n then $g(f(z))$ lies on the same ray so the straight line from z to $g(f(z))$ is wholly contained in X_n. It follows that $g \circ f$ is homotopic to the identity by a linear homotopy $h(t,z) = (1-t)z + t\,z/|z|$ [3]. The same argument shows that $f \circ g$ is homotopic to the identity, so f and g are homotopy equivalences [1].

(d) Say that a point is *special* if its removal separates the space into three path components [2]. The space X_n has precisely n special points, namely the points $e^{2k\pi i/n}$ for $0 \leq k < n$ [2]. If $n \neq m$ then X_n and X_m have different numbers of special points, so they are not homeomorphic [2].

(e) The space Y_n is path connected and has n holes, so $H_1(Y_n) \simeq \mathbb{Z}^n$ [2]. If $n \neq m$ then $H_1(Y_n)$ is not isomorphic to $H_1(Y_m)$, so Y_n cannot be homotopy equivalent to Y_m [2].

(61) 2018-19 Q3: Let K and L be abstract simplicial complexes.

(a) Define what is meant by a *simplicial map* from K to L. **(3 marks)**

(b) Let $s, t\colon K \to L$ be simplicial maps. Define what it means for s and t to be *directly contiguous*. **(3 marks)**

(c) Prove that if s and t are directly contiguous, then the resulting maps $|s|, |t|\colon |K| \to |L|$ are homotopic. **(3 marks)**

(d) Prove that if s and t are directly contiguous, then the resulting maps $s_*, t_*\colon H_*(K) \to H_*(L)$ are the same. (You can prove the main formula just for $n = 3$ rather than general n.) **(9 marks)**

(e) How many injective simplicial maps are there from $\partial\Delta^2$ to itself? Show that no two of them are directly contiguous. **(7 marks)**

Solution:

(a) A simplicial map from K to L is a function $s\colon \mathrm{vert}(K) \to \mathrm{vert}(L)$ such that whenever $\sigma = \{v_0, \dots, v_n\}$ is a simplex of K, the image $s(\sigma) = \{\sigma(v_0), \dots, \sigma(v_n)\}$ is a simplex of L. [3]

(b) We say that s and t are directly contiguous if whenever $\sigma = \{v_0, \dots, v_n\}$ is a simplex of K, the set

$$s(\sigma) \cup t(\sigma) = \{s(v_0), \dots, s(v_n), t(v_0), \dots, t(v_n)\}$$

is a simplex of L. [3] [**Bookwork**]

(c) Suppose that s and t are directly contiguous. Consider a point $x \in |K|$, so $x \in |\sigma|$ for some $\sigma \in \mathrm{simp}(K)$. Put $\tau = s(\sigma) \cup t(\sigma)$, which is a simplex of L because of the contiguity condition. Both $|s|(x)$ and $|t|(x)$ lie in $|\tau|$, so the whole line segment from $|s|(x)$ to $|t|(x)$ lies in $|\tau|$. We can therefore define a linear homotopy $h\colon [0,1] \times |K| \to |L|$ from $|s|$ to $|t|$ by $h(r, x) = (1-r)|s|(x) + r\,|t|(x)$. [3] [**Bookwork**]

(d) Suppose again that s and t are directly contiguous. Define $u\colon C_n K \to C_{n+1} L$ by

$$u\langle v_0, \dots, v_n\rangle = \sum_{i=0}^{n} (-1)^i \langle s(v_0), \dots, s(v_i), t(v_i), \dots, t(v_n)\rangle.[2]$$

We claim that $du + ud = t_\# - s_\#$ [1]. We will prove this for a generator $x = \langle v_0, v_1, v_2, v_3\rangle \in C_3(K)$, using the abbreviated notation i for v_i or $s(v_i)$, and $\bar{i}$ for $t(v_i)$. We have

$$u(x) = \; +0\overline{0123} \quad -01\overline{123} \quad +012\overline{23} \quad -0123\overline{3} \qquad\qquad d(x) = \; +123 \; -023 \; +013 \; -012$$

$$du(x) = \begin{array}{cccc} +\overline{0123} & -1\overline{123} & +12\overline{23} & -123\overline{3} \\ -0\overline{123} & +0\overline{123} & -02\overline{23} & +023\overline{3} \\ +0\overline{023} & -01\overline{23} & +01\overline{23} & -013\overline{3} \\ -0\overline{013} & +01\overline{13} & -012\overline{3} & +012\overline{3} \\ +0\overline{012} & -01\overline{12} & +012\overline{2} & -0123 \end{array} \qquad ud(x) = \begin{array}{ccc} +1\overline{123} & -12\overline{23} & +123\overline{3} \\ -0\overline{023} & +02\overline{23} & -023\overline{3} \\ +0\overline{013} & -01\overline{13} & +013\overline{3} \\ -0\overline{012} & +01\overline{12} & -012\overline{2} \end{array}$$

Most terms cancel in the indicated groups, leaving $du(x) + ud(x) = \overline{0123} - 0123$. In the original notation, this says that

$$(du + ud)(x) = \langle t(v_0), t(v_1), t(v_2), t(v_3)\rangle - \langle s(v_0), s(v_1), s(v_2), s(v_3)\rangle = t_\#(x) - s_\#(x),$$

which means that u is a chain homotopy between $s_\#$ and $t_\#$ [5]. As these maps are chain-homotopic, they induce the same homomorphism between homology groups. [1][**Bookwork**]

(f) The injective simplicial maps from $\partial\Delta^2$ to itself are just given by permuting the three vertices, so there are $3! = 6$ such maps [2]. Suppose that f and g are permutations that are contiguous. Then the set $f(\{0,1\}) \cup g(\{0,1\})$ must be a simplex, so it has size at most two. However, $f(\{0,1\})$ and $g(\{0,1\})$ both have size two already, so this is only possible if $f(\{0,1\}) = g(\{0,1\})$. As f and g are permutations, it follows that $f(2) = g(2)$. By applying the same logic to $\{0,2\}$ and then $\{1,2\}$, we also see that $f(1) = g(1)$ and $f(0) = g(0)$. Thus, we actually have $f = g$ [5]. **[Unseen]**

(62) 2018-19 Q5: Consider a simplicial complex K with subcomplexes L and M such that $K = L \cup M$. Use the following notation for the inclusion maps:

$$\begin{tikzcd} L \cap M \arrow[r, "i"] \arrow[d, "j"'] & L \arrow[d, "f"] \\ M \arrow[r, "g"'] & K. \end{tikzcd}$$

(a) State the Seifert-van Kampen Theorem (in a form applicable to simplicial complexes and subcomplexes as above). **(4 marks)**

(b) State the Mayer-Vietoris Theorem. **(5 marks)**

(c) State a theorem about the relationship between π_1 and H_1. **(3 marks)**

(d) Suppose that $|L|$, $|M|$ and $|L \cap M|$ are all homotopy equivalent to S^1. Suppose that the maps i and j both have degree two.

  (1) Find a presentation for $\pi_1(|K|)$. **(3 marks)**

  (2) Find $H_*(K)$. In particular, you should express each nonzero group as a direct sum of terms like $\mathbb{Z}$ or $\mathbb{Z}/n$. **(10 marks)**

Solution:

(a) Suppose that $|L \cap M|$ is connected and that we have presentations

$$\pi_1(|L|) = \langle x_1, \dots, x_p \mid u_1 = \dots = u_k = 1 \rangle$$
$$\pi_1(|M|) = \langle y_1, \dots, y_q \mid v_1 = \dots = v_l = 1 \rangle$$
$$\pi_1(|L \cap M|) = \langle z_1, \dots, z_r \mid w_1 = \dots = w_m = 1 \rangle.$$

Then we have a presentation of $\pi_1(|K|)$ with generators $x_1, \dots, x_p, y_1, \dots, y_q$ and relations $u_1 = \dots = u_r = v_1 = \dots = v_l = 1$ and $i_*(z_t) = j_*(z_t)$ for all t. [4] **[Bookwork]**

(b) There is a natural map $\delta\colon H_n(K) = H_n(L \cup M) \to H_{n-1}(L \cap M)$ such that the resulting sequence

$$H_{n+1}(L \cup M) \xrightarrow{\delta} H_n(L \cap M) \xrightarrow{\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]} H_n(L) \oplus H_n(M) \xrightarrow{[\, f_* \; g_* \,]} H_n(L \cup M) \xrightarrow{\delta} H_{n-1}(L \cap M)$$

is exact for all n [5]. **[Bookwork]**

(c) If $|K|$ is connected [1], then $H_1(K)$ is naturally isomorphic to the abelianisation of $\pi_1(|K|)$ [2]. **[Bookwork]**

(d) (1) As $|L \cap M| \simeq S^1$, we can choose a generator z for $\pi_1(|L \cap M|)$. As i has degree two we see that there is a generator x of $\pi_1(|L|)$ with $i_*(z) = x^2$. As j has degree two we see that there is a generator y of $\pi_1(|M|)$ with $j_*(z) = y^2$. The Seifert-van Kampen Theorem now gives $\pi_1(|K|) = \langle x, y \mid x^2 = y^2 \rangle$. [3] **[Similar examples have been seen.]**

  (2) We have a Mayer-Vietoris sequence as follows:

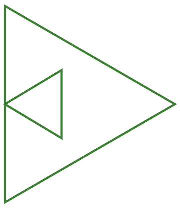
$$\begin{tikzcd} H_2(L \cap M) \arrow[r, "\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]"] & H_2(L) \oplus H_2(M) \arrow[r, "{[\, f_* \; g_* \,]}"] & H_2(K) \arrow[dll, rounded corners] \\ H_1(L \cap M) \arrow[r, "\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]"] & H_1(L) \oplus H_1(M) \arrow[r, "{[\, f_* \; g_* \,]}"] & H_1(K) \arrow[dll, rounded corners] \\ H_0(L \cap M) \arrow[r, "\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]"] & H_0(L) \oplus H_0(M) \arrow[r, two heads, "{[\, f_* \; g_* \,]}"] & H_0(K). \end{tikzcd}$$ [3]

The spaces $|L \cap M|$, $|L|$ and $|M|$ are all homotopy equivalent to S^1 and so have $H_0 = H_1 = \mathbb{Z}$ and all other homology groups are zero. We also know that i_* and j_* act as the identity on H_0, and as multiplication by 2 on H_1. The sequence therefore has the following form:

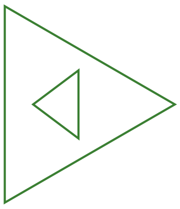
$$\begin{gathered}
0 \xrightarrow{\ 0\ } 0 \xrightarrow{\ 0\ } H_2(K) \\
\to \mathbb{Z} \xrightarrow{\left[\begin{smallmatrix}2\\-2\end{smallmatrix}\right]} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{[\,f_*\ g_*\,]} H_1(K) \\
\to \mathbb{Z} \xrightarrow{\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{[\,f_*\ g_*\,]} H_0(K).
\end{gathered}$$ [3]

From this we can read off that $H_2(K) = 0$ and $H_0(K) = \mathbb{Z}$ [1]and that $H_1(K) = \mathbb{Z}^2/\mathbb{Z}.(2,-2)$ [1]. If we use the basis $\{(1,0),(1,-1)\}$ for $\mathbb{Z}^2$ we get $H_1(K) \simeq \mathbb{Z}\oplus\mathbb{Z}/2$ [1]. By extending the sequence further upwards, it is also clear that $H_n(K) = 0$ for $n > 2$ [1]. **[Similar examples have been seen.]**

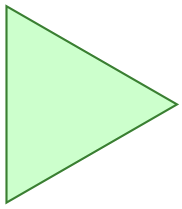
(63) 2019-20 Q1: Consider the following spaces:



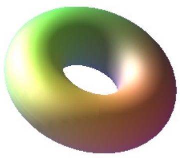
X_0



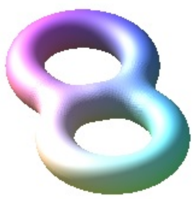
X_1



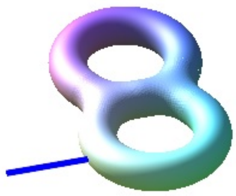
X_2



X_3



X_4



X_5

$$X_6 = (S^1 \times S^1) \setminus \{(1,1)\} \qquad X_7 = GL_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) \mid \det(A) \neq 0\}$$
$$X_8 = \mathbb{R} \qquad X_9 = \{(u,v) \in \mathbb{C}^2 \mid 1 \leq |u| \leq 2 \leq |v| \leq 3\}.$$

(Here X_3 and X_4 are closed orientable surfaces, and X_5 is the union of X_4 with a line segment with one endpoint lying on X_4. Everything else should be clear.)

(a) These 10 spaces can be grouped into 5 pairs $\{X_i, X_j\}$ such that X_i is homotopy equivalent to X_j. Find these pairs, and justify your answers. In each case you should prove that X_i is homotopy equivalent to X_j, and also that it is not homotopy equivalent to any of the other spaces. **(25 marks)**

(b) For each pair $\{X_i, X_j\}$ as in (a), prove that X_i is not homeomorphic to X_j. (In one case you may need to appeal to some geometric intuition, but you should be able to give a more formal proof in the other four cases.) **(15 marks)**

Solution:

(a) **This will need to be marked as a whole. There will be 5 marks for correct identification of the pairs, 10 marks for justifying why they are homotopy equivalent, and a further 10 marks for explaining why there are no further equivalences.** [15]

- X_0 consists of two circles meeting at a single point and so is homeomorphic to the figure eight. This is in turn homotopy equivalent to the punctured torus X_6, as explained in Example 15.26 and the associated interactive demonstration.
- X_1 is homeomorphic to the union of two disjoint circles. On the other hand, Example 4.9 shows that the space $X_7 = GL_2(\mathbb{R})$ is homeomorphic to $\mathbb{R}^3 \times S^1 \times \{1, -1\}$, so it is homotopy equivalent to $S^1 \times \{1, -1\}$, which is again a union of two disjoint circles. Thus, X_1 is homotopy equivalent to X_7.
- X_2 and X_8 are both contractible and so are homotopy equivalent to each other.
- X_3 is just the torus $S^1 \times S^1$. There is a homeomorphism

$$p\colon [0,1]^2 \times X_3 = [0,1]^2 \times S^1 \times S^1 \to X_9$$

  given by $p(s,t,u,v) = ((1+s)u,\ (2+t)v)$, and $[0,1]^2$ is contractible, so X_3 is homotopy equivalent to X_9.
- The spaces X_4 and X_5 are homotopy equivalent. Indeed, the extra interval in X_5 can be parametrised as $\{u(t) \mid 0 \leq t \leq 1\}$, with $u(0)$ being the end lying in X_4. We have an evident inclusion $i\colon X_4 \to X_5$ and a retraction $r\colon X_5 \to X_4$ given by $r(u(t)) = u(0)$ and $r(x) = x$ for all $x \in X_4$. Then $r \circ i$ is equal to the identity. We can also define $h\colon [0,1] \times X_5 \to X_5$ by $h(s, u(t)) = u(st)$ and $h(s,x) = x$ for all $x \in X_4$. This gives a homotopy $i \circ r \simeq \mathrm{id}$, so we have a homotopy equivalence as claimed.

If two spaces are homotopy equivalent, then they have isomorphic homology. We can tabulate the homology groups of the X_i as follows:

	H_0	H_1	H_2
X_0, X_6	$\mathbb{Z}$	$\mathbb{Z}^2$	0
X_1, X_7	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
X_2, X_8	$\mathbb{Z}$	0	0
X_3, X_9	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
X_4, X_5	$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}$

As all the lines are different, there are no additional homotopy equivalences [10]. **There are also valid approaches using π_0 and π_1. They are less clear and efficient, but can also be given full marks if done correctly.**

(b) The space X_0 is compact but X_6 is not, so X_0 is not homeomorphic to X_6 [3]. Similarly X_1 is compact but X_7 is not [3], and X_2 is compact but X_8 is not [3]. Next, X_5 can be disconnected by removing a single point, but X_4 cannot, so X_4 and X_5 are not homeomorphic [3]. Finally X_3 and X_9 are not homeomorphic because X_3 is 2-dimensional and X_9 is 4-dimensional [3]. (This is not quite a complete proof, because we have not given a formal definition of dimensionality. The Invariance of Domain Theorem does most of what we need, but a bit more discussion would be required.)

(64) 2019-20 Q2:

(a) Let A and B be finite abelian groups such that $|A|$ and $|B|$ are coprime.

  (i) What can you say about homomorphisms from A to B? **(10 marks)**

  (ii) Now suppose we have a short exact sequence $A \to U \to B$ of abelian groups. By considering the classification of finite abelian groups, or otherwise, what can you say about U? **(15 marks)**

(b) Let X be a topological space, with open subspaces U and V such that $X = U \cup V$. Suppose that U, V, X and $U \cap V$ are all path-connected, and that for all $k > 0$ we have $H_k(U \cap V) = \mathbb{Z}/2^k$ and $H_k(U) = \mathbb{Z}/3^k$ and $H_k(V) = \mathbb{Z}/5^k$. Calculate $H_*(X)$. **(15 marks)**

Solution:

(a) (i) The only homomorphism from A to B is the zero homomorphism [3]. Indeed, if $\phi\colon A \to B$ is a homomorphism then $\phi(A)$ is a subgroup of B and so has order dividing $|B|$. On the other hand, the First Isomorphism Theorem says that $|\phi(A)| = |A|/|\ker(\phi)|$, and this is a divisor of $|A|$. As $|A|$ and $|B|$ are coprime, we conclude that $|\phi(A)| = 1$, so $\phi(A) = \{0\}$, so $\phi = 0$. [7]

(ii) If $A \xrightarrow{f} U \xrightarrow{g} B$ is a short exact sequence, we claim that $U \simeq A \oplus B$ [3]. Indeed, we have $|U| = |A|.|B|$. We can write U as a direct sum of groups of the form $\mathbb{Z}/p^k$. As $|U| = |A|.|B|$ with $|A|$ and $|B|$ coprime, we see that p must divide $|A|$ or $|B|$ but not both. Let A' be the sum of all the factors where p divides $|A|$, and let B' be the sum of all the factors where p divides $|B|$, so $U = A' \oplus B'$. The homomorphism $f\colon A \to A' \oplus B'$ can be decomposed into a pair of homomorphisms $f_0\colon A \to A'$ and $f_1\colon A \to B'$. The homomorphism $g\colon A' \oplus B' \to B$ can be decomposed into a pair of homomorphisms $g_0\colon A' \to B$ and $g_1\colon B' \to B$. Here f_1 and g_0 are zero by part (i). As $f_1 = 0$ we have $\mathrm{img}(f) \leq A'$, and as $g_0 = 0$ we have $\ker(g) \geq A'$. As the sequence is exact we have $\mathrm{img}(f) = \ker(g)$, so this group must be equal to A'. Also, as f is injective we see that f_0 is injective, and as g is surjective we see that g_1 is surjective. It now follows that f_0 and g_1 are isomorphisms, and thus that $U = A' \oplus B; \simeq A \oplus B$ as claimed. [12]

(b) The connectivity assumptions mean that $H_0(X) = \mathbb{Z}$ and that we have a truncated Mayer-Vietoris sequence [2]. For $k > 1$ this takes the form

$$\mathbb{Z}/2^k \xrightarrow{e} \mathbb{Z}/3^k \oplus \mathbb{Z}/5^k \xrightarrow{f} H_k(X) \xrightarrow{g} \mathbb{Z}/2^{k-1} \xrightarrow{e} \mathbb{Z}/3^{k-1} \oplus \mathbb{Z}/5^{k-1}.[3]$$

The maps marked e are zero by (a)(i) [2], so f is injective and g is surjective by exactness [4], which means that the middle three terms form a short exact sequence. Thus, (a)(ii) tells us that

$$H_k(X) = \mathbb{Z}/3^k \oplus \mathbb{Z}/5^k \oplus \mathbb{Z}/2^{k-1} = \mathbb{Z}/(30^k/2)[4]$$

(where we have used the Chinese Remainder Theorem to tidy up the final answer a little). This formula remains valid for $k = 1$, although the argument is a tiny bit different.

(65) 2020-21 Q2: Fix $n \geq 2$. Define an equivalence relation on the disc $B^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ by $z_0 \sim z_1$ iff $z_0 = z_1$, or ($|z_0| = |z_1| = 1$ and $z_0^n = z_1^n$). Put $X = B^2/\sim$ and

$$Y = \{(u, v) \in \mathbb{C}^2 \mid |u| \leq 1, \quad v^n = (1 - |u|)^n u\}.$$

Note that when $n = 2$ we just have $X = \mathbb{R}P^2$; this should guide your thinking about the general case.

(a) Show carefully that there is a homeomorphism $f\colon X \to Y$ such that $f([z]) = (z^n, (1 - |z|^n)z)$ for all $z \in B^2$. You should prove in particular that f is well-defined, injective and surjective, and that both f and f^{-1} are continuous. You may assume that polynomials and the absolute value function are continuous, but beyond that you should not assume any properties of the given formula without proof. **(13 marks)**

(b) For the boundary $S^1 \subset B^2$, explain briefly why $S^1/\sim$ is homeomorphic to S^1 again. **(3 marks)**

(c) By adapting the method used for $\mathbb{R}P^2$, calculate $H_*(X)$. **(14 marks)**

Solution:

(a) Suppose that $z \in B^2$ (so $|z| \leq 1$) and put $u = z^n$ and $v = (1 - |z|^n)z = (1 - |u|)z$. We then have $|u| = |z|^n \leq 1$ and $v^n = (1 - |u|)^n z^n = (1 - |u|)^n u$, so $(u, v) \in Y$ [1]. We can thus define a continuous map $f_0\colon B^2 \to Y$ by $f_0(z) = (z^n, (1 - |z|^n)z)$. Now suppose we have $z_0, z_1 \in B^2$ with $z_0 \sim z_1$; we claim that $f(z_0) = f(z_1)$ [1]. If $z_0 = z_1$ then this is clear. Otherwise, we must have $|z_0| = |z_1| = 1$ (which means that $f_0(z_i) = (z_i^n, 0)$) and $z_0^n = z_1^n$, so $f_0(z_0) = f_0(z_1)$ as required [1]. By the universal property of quotients (Corollary 8.20) there is a unique continuous map $f\colon X \to Y$ such that $f([z]) = f_0(z)$ for all z [1].

Now suppose we have $(u, v) \in Y$, so $v^n = (1-|u|)^n u$. If $|u| \neq 1$ then $0 < 1-|u| \leq 1$ and we put $z = v/(1-|u|) \in \mathbb{C}$. The relation $v^n = (1 - |u|)^n u$ becomes $z^n = u$. It follows that $|z|^n = |u| < 1$ so $|z| < 1$ so $z \in B^2$, and we find that $f([z]) = f_0(z) = u$. On the other hand, if $|u| = 1$ then the relation $v^n = (1 - |u|)^n u$ gives $v = 0$. We can let z be any one of the n'th roots of u and we get $|z| = 1$ and $f([z]) = f_0(z) = (u, 0)$. This shows that f is surjective. [3]

Now suppose we have $z_0, z_1 \in B^2$ with $f([z_0]) = f([z_1])$, or in other words $z_0^n = z_1^n$ and $(1 - |z_0|^n)z_0 = (1 - |z_1|^n)z_1$. Put $r = |z_0| \in [0, 1]$. Using $z_0^n = z_1^n$ we get $r^n = |z_1|^n$ so $|z_1|$ is also equal to r. Thus, the equation $(1 - |z_0|^n)z_0 = (1 - |z_1|^n)z_1$ becomes $(1 - r^n)(z_0 - z_1) = 0$. If $r < 1$ this gives $z_0 = z_1$, so certainly $[z_0] = [z_1]$. On the other hand, if $r = 1$ then the relation $z_0^n = z_1^n$ gives $z_0 \sim z_1$ (from the definition of the equivalence relation) and so $[z_0] = [z_1]$. Either way, we have $[z_0] = [z_1]$, so we conclude that f is injective. [3]

Note also that X is a quotient of the compact space B^2, so it is again compact. Moreover, Y is a metric space and so is Hausdorff. As f is a continuous bijection from a compact space to a Hausdorff space, it is a homeomorphism by Proposition 9.28. [3]

(b) For $z \in S^1$ we have $(1-|z|^n)z = 0$, so f restricts to give a homeomorphism $S^1/\sim \to S^1 \times \{0\} \simeq S^1$. Alternatively, on S^1 the equivalence relation is just $z_0 \sim z_1 \iff z_0^n = z_1^n$, so the map $[z] \mapsto z^n$ gives the required homeomorphism. [3]

(c) Put $\widetilde{U} = B^2 \setminus \{0\}$ and $\widetilde{V} = B^2 \setminus S^1 = OB^2$. Let U and V be the images of $\widetilde{U}$ and $\widetilde{V}$ in X. These are open sets which cover X, so they give a Mayer-Vietoris sequence. [3]

The equivalence relation does not do anything to $\widetilde{V}$, so V is just an open disc, which is contractible. Thus, the only nontrivial homology group is $H_0(V) = \mathbb{Z}$ [2]. Next, we can deform $\widetilde{U}$ radially outward onto S^1, and this is compatible with the equivalence relation, so U is homotopy equivalent to $S^1/\sim$, which is homeomorphic to S^1 by (b). Thus, we have $H_0(U) = H_1(U) = \mathbb{Z}$ and all other homology groups are zero [2]. Also, $U \cap V$ is an annulus so $H_0(U \cap V) = H_1(U \cap V) = \mathbb{Z}$ and again all other homology groups are zero [1]. As U, V and $U \cap V$ are connected we can use the truncated version of the Mayer-Vietoris sequence:

$$H_2(U) \oplus H_2(V) \to H_2(X) \to H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(X) \to H_1(U \cap V) \to 0.\text{[2]}$$

Using the above determination of the homology groups, this becomes

$$0 \to H_2(X) \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \to H_1(X) \to 0.\text{[1]}$$

The standard circle in the annulus $U \cap V$ gets wrapped n times around the boundary circle $S^1/\sim$, so i_* is multiplication by n, which is injective [1]. It follows that $H_2(X) = 0$ and $H_1(X) = \mathbb{Z}/n$. As X is connected, we have $H_0(X) = \mathbb{Z}$ [1]. For $k > 2$ is is clear from the Mayer-Vietoris sequence that $H_k(X) = 0$. [1]

Feedback:

(a) Very few people checked that $(z^n, (1-|z|^n)z) \in Y$, despite my ranting about this sort of thing in connection with Problem Sheet 10. Very few people distinguished clearly between f_0 and f; in particular, many people claimed to be proving that f is continuous, but actually proved that f_0 is continuous. Attempts to prove that f is well-defined and injective were of variable quality. For surjectivity, many people claimed that $f([u^{1/n}]) = (u, v)$ for all $(u, v) \in Y$. Here everything is complex so we usually have n different choices of z with $z^n = u$, i.e. n different possible values of $u^{1/n}$. If you choose the right one then you will get $f([z]) = (u, v)$, but if you choose the wrong one then you will instead get $f([z]) = (u, e^{2\pi ik/n}v)$ for some $k \neq 0$. Thus, a more detailed argument needs to be given. These issues also mean that f^{-1} is not given by a simple and well-defined formula, so the only reasonable way to prove that f^{-1} is continuous is to use Proposition 9.28. This is all similar to Examples 8.24, 8.26, 9.29 and 9.30 in the notes.

(b) Most people gave answers that were along the right lines.

(c) Most people who made a serious attempt at this got it roughly right; but some people gave up.

(66) 2021-22 Mock Q5: Let X be a path connected space, and put

$$U = \{(t, x) \in S^1 \times X \mid t \neq (0, 1)\}$$
$$V = \{(t, x) \in S^1 \times X \mid t \neq (0, -1)\}.$$

We use the usual notation for inclusion maps:

$$\begin{tikzcd} U \cap V \arrow[r, "i"] \arrow[d, "j"'] & U \arrow[d, "k"] \\ V \arrow[r, "l"'] & S^1 \times X. \end{tikzcd}$$

(a) Define maps $f, g \colon X \to U \cap V$ such that f gives a homotopy equivalence from X to one path component of $U \cap V$, and g gives a homotopy equivalence from X to the other path component of $U \cap V$. **(4 marks)**

(b) Prove that the map $i' = i \circ f \colon X \to U \times X$ is homotopic to $i \circ g$, and also that i' is a homotopy equivalence. (You can then assume without further argument that the map $j' = j \circ f \colon X \to V \times X$ is homotopic to $j \circ g$, and that j' is a homotopy equivalence.) **(6 marks)**

(c) Deduce descriptions of the homology groups $H_p(U \cap V)$, $H_p(U)$ and $H_p(V)$, and the homomorphism

$$\alpha = \begin{bmatrix} i_* \\ -j_* \end{bmatrix} : H_p(U \cap V) \to H_p(U) \oplus H_p(V).$$

Find the kernel and image of α. **(8 marks)**

(d) Show that every element of $H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a), 0) + \alpha(b)$ for a unique pair $(a, b) \in H_p(X)^2$. **(3 marks)**

(e) Deduce that there is a short exact sequence $H_p(X) \to H_p(S^1 \times X) \to H_{p-1}(X)$. **(4 marks)**

Solution:

(a) The path components of $S^1 \setminus \{(0, 1), (0, -1)\}$ are $A = [(-1, 0)] = \{(x, y) \in S^1 \mid x < 0\}$ and $B = [(+1, 0)] = \{(x, y) \in S^1 \mid x > 0\}$, so the path components of $U \cap V$ are $A \times X$ and $B \times X$ [2]. Here A is contractible and contains $(-1, 0)$ so the map $f(x) = ((-1, 0), x)$ gives a homotopy equivalence from X to $A \times X$. Similarly, the map $g(x) = ((1, 0), x)$ gives a homotopy equivalence from X to $B \times X$ [2].

(b) We can define $h(t, x) = ((-\cos(\pi t), -\sin(\pi t)), x)$ for $0 \leq t \leq 1$. As $(-\cos(\pi t), -\sin(\pi t))$ lies on the bottom half of S^1, this does not pass through $(0, 1) \times X$ and so gives a continuous map $[0, 1] \times X \to U$. It satisfies $h(0, x) = ((-1, 0), x) = i(f(x)) = i'(x)$ and $h(1, x) = ((1, 0), x) = i(g(x))$, so this gives a homotopy between i' and $i \circ g$ [3]. We can also define $r: U \times X \to X$ by $r(t, x) = x$. Then $r \circ i' = \text{id}$, and contractibility of $S^1 \setminus \{(0, 1)\}$ ensures that $i'r$ is homotopic to the identity [3].

(c) As $f: X \to A \times X$ and $g: X \to B \times X$ are homotopy equivalences, we see that every element of $H_p(U \cap V)$ can be written as $f_*(a) + g_*(b)$ for a unique pair $(a, b) \in H_p(X)^2$. [2] Similarly, any element of $H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a), j'_*(b))$ for a unique pair $(a, b) \in H_p(X)^2$[2]. As $i_* f_* = i_* g_* = i'_*$ and $j_* f_* = j_* g_* = j'_*$ we see that

$$\alpha(f_*(a) + g_*(b)) = (i'_*(a + b),\ -j'_*(a + b)[2].)$$

This means that

$$\ker(\alpha) = \{f_*(a) - g_*(a) \mid a \in H_p(X)\} \simeq H_p(X)[1]$$
$$\operatorname{img}(\alpha) = \{(i'_*(c), -j'_*(c)) \mid c \in H_p(X)\} \simeq H_p(X)[1].$$

(d) We now see that every element $(i'_*(a), j'_*(b)) \in H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a + b), 0) + (i'_*(-b), j'_*(-b))$ with the second term lying in $\operatorname{img}(\alpha)$, and this decomposition is unique [3].

(e) From the exact sequence

$$H_p(U \cap V) \xrightarrow{\alpha} H_p(U) \oplus H_p(V) H_p(S^1 \times X) \xrightarrow{\delta} H_{p-1}(U \cap V) \xrightarrow{\alpha} H_{p-1}(U) \oplus H_{p-1}(V)$$

we get a short exact sequence

$$(H_p(U) \oplus H_p(V))/\operatorname{img}(\alpha_p) \to H_p(S^1 \times X) \to \ker(\alpha_{p-1})[2]$$

Part (d) gives an isomorphism $(H_p(U) \oplus H_p(V))/\operatorname{img}(\alpha_p) \simeq H_p(X)$ [1]. Part (c) gives an isomorphism $\ker(\alpha_{p-1}) \simeq H_{p-1}(X)$ [1]. We therefore have a short exact sequence

$$H_p(X) \to H_p(S^1 \times X) \to H_{p-1}(X)$$

as claimed.

(67) 2021-22 Q5: Put $X = \{(x, y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = 1\}$, so X is homeomorphic to S^3. Put $\omega = e^{2\pi i/3} \in \mathbb{C}$, so $\omega^3 = 1$. Define an equivalence relation on X by $(x, y) \sim (x', y')$ iff $(x', y') = \omega^k(x, y)$ for some k. Put

$$Y = X/\sim$$
$$U = \{[x, y] \in Y \mid x \neq 0\}$$
$$V = \{[x, y] \in Y \mid y \neq 0\}.$$

You may assume that U and V are open in Y and that $Y = U \cup V$.

(a) Show that the formula $f([x, y]) = (x^3/|x|^3, y/x)$ gives a well-defined and continuous map $f: U \to S^1 \times \mathbb{C}$. Do not assume any properties of the given formula without checking them. **(6 marks)**

(b) Show that f is actually a bijection and that the inverse satisfies

$$f^{-1}(u, z) = \left[(v, z\,v)/\sqrt{1 + |z|^2}\right]$$

where v is any one of the three cube roots of u. Do not assume any properties of the given formula without checking them. **(6 marks)**

(c) You may assume without proof that the map $f^{-1}: S^1 \times \mathbb{C} \to U$ is also continuous, so f is a homeomorphism. What can you conclude about the homeomorphism type of $U \cap V$? **(3 marks)**

(d) The facts proved for U have obvious counterparts for V; you can assume these without proof. Deduce descriptions of $H_*(U)$, $H_*(V)$ and $H_*(U \cap V)$. **(5 marks)**

(e) Use the Mayer-Vietoris sequence to compute $H_*(Y)$. You should be able to compute $H_k(Y)$ for $k = 0$ and $k \geq 3$. For $k = 1, 2$ you will need to determine a map in the Mayer-Vietoris sequence, which is possible but not so easy. If you cannot see how to do it then you should guess, and give an answer based on your guess. **(5 marks)**

Solution: Somewhat similar examples have been seen

(a) For $(x, y) \in U'$ we have $x \neq 0$ so it is meaningful to define $f_0(x, y) = (x^3/|x|^3,\ y/x) \in \mathbb{C}^2$ [1]. Basic complex analysis shows that this gives a continuous map $f_0: U' \to \mathbb{C}^2$ [1]. As $|x^3/|x|^3| = |x|^3/|x|^3 = 1$ we see that $f_0(x, y) \in S^1 \times \mathbb{C}$ [1]. If $(x, y) \sim (x', y')$ then $(x', y') = (\omega^k x, \omega^k y)$ for some k so

$$f_0(x', y') = (\omega^3 x^3/|\omega x|^3, (\omega y)/(\omega x)) = (x^3/|x|^3,\ y/x) = f_0(x, y).\text{[1]}$$

This proves that f_0 is saturated, so it induces a well-defined and continuous map $f: U = U'/\sim \to S^1 \times \mathbb{C}$ given by $f([x, y]) = f_0(x, y) = (x^3/|x|^3,\ y/x)$ [2].

(b) Consider a point $(u, z) \in S^1 \times \mathbb{C}$. Let v be a cube root of u, so $|v| = 1$. Put $x = v/\sqrt{1 + |z|^2}$ and $y = zx = zv/\sqrt{1 + |z|^2}$. We then have $x \neq 0$ and

$$|x|^2 + |y|^2 = (1 + |z|^2)^{-1}(|v|^2 + |z|^2|v|^2|) = 1,$$

so $(x, y) \in X$. We also have $x/|x| = v$ and $y/x = (zx)/x = z$ so

$$f([x, y]) = f_0(x, y) = (v^3, z) = (u, z).$$

This proves that f is surjective [3].

Now suppose we have another element $[x', y'] \in U$ with $f([x', y']) = (u, z)$, so $(x'/|x'|)^3 = u$ and $y'/x' = z$. This gives $y' = zx'$ so $(1 + |z|^2)|x'|^2 = |x'|^2 + |y'|^2 = 1$ so $|x'| = (1 + |z|)^{-1/2} = |x|$. Together with the relation $(x/|x|)^3 = u = (x'/|x'|)^3$ this gives $(x'/x)^3 = 1$, so $x' = \omega^k x$ for some k. This in turn gives $y' = zx' = \omega^k zx = \omega^k y$, so $(x', y') = \omega^k(x, y)$, so $[x', y'] = [x, y]$. This shows that f is also injective, and therefore bijective [3].

(c) As $f: U \to S^1 \times \mathbb{C}$ is a homeomorphism, it also gives a homeomorphism from $U \cap V$ to $f(U \cap V)$ [1]. From the formulae in (b) and (c) we see that $f^{-1}(u, z)$ lies in $U \cap V$ iff $z \neq 0$, so f gives a homeomorphism from $U \cap V$ to $S^1 \times \mathbb{C}^\times$ [2].

(d) We now see that U is homotopy equivalent to S^1, and V is also homotopy equivalent to S^1 by a symmetric argument [1]. On the other hand, $U \cap V$ is homotopy equivalent to $S^1 \times S^1$ [1]. This gives $H_0(U) = H_0(V) = H_0(U \cap V) = \mathbb{Z}$ and $H_1(U) = H_1(V) = \mathbb{Z}$ and $H_1(U \cap V) = \mathbb{Z}^2$ and $H_2(U \cap V) = \mathbb{Z}$ [3].

(e) It is easy to see that U, V, $U \cap V$ and Y are all path connected, so $H_0(Y) = \mathbb{Z}$. The interesting parts of the truncated Mayer-Vietoris sequence are now

$$0 \to H_3(Y) \xrightarrow{\delta} H_2(U \cap V) = \mathbb{Z} \to 0\text{[1]}$$

and

$$0 \to H_2(Y) \xrightarrow{\delta} \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^2 \xrightarrow{\beta} H_1(Y) \to 0.\text{[1]}$$

From the first of these we get $H_3(Y) \simeq \mathbb{Z}$ [1](and similar arguments give $H_n(Y) = 0$ for $n > 3$) [1]. One can check that α has the form $(i, j) \mapsto (i, -i - 3j)$ so it is injective with image of index 3 in $\mathbb{Z}^2$; this gives $H_2(Y) = 0$ and $H_1(Y) = \mathbb{Z}/3$ [1].

(68) 2022-23 Q1

(a) Explain the terms *homeomorphism* and *homeomorphic*. **(3 marks)**

(b) Explain the terms *homotopy*, *homotopic* and *homotopy equivalent*, distinguishing carefully between them. **(5 marks)**

(c) Consider the following spaces:

$$\begin{aligned} X_0 &= \{z \in \mathbb{C} \mid \mathrm{Re}(z) \notin \mathbb{Z}\} \\ X_1 &= \{z \in \mathbb{C} \mid \mathrm{Im}(z) \in \mathbb{Z}\} \\ X_2 &= \{z \in \mathbb{C} \mid z \notin \mathbb{Z}\} \\ X_3 &= \{z \in \mathbb{R} \mid z \notin \mathbb{Z}\} \\ X_4 &= \{z \in \mathbb{C} \mid |z| \in \mathbb{Z}\}. \end{aligned}$$

(i) Sketch all these spaces. **(5 marks)**

(ii) For which pairs (i, j) is X_i homotopy equivalent to X_j? Justify your answer briefly. In cases where X_i is homotopy equivalent to X_j you should explain why, and in cases where X_i is not homotopy equivalent to X_j, you should explain that as well. **(6 marks)**

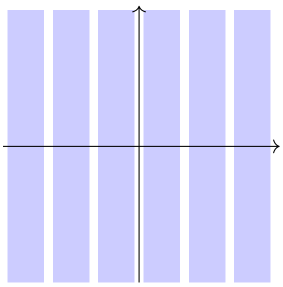
(iii) For which pairs (i, j) is X_i homeomorphic to X_j? Justify your answer briefly. In cases where X_i is homeomorphic to X_j you should explain why, and in cases where X_i is not homeomorphic to X_j, you should explain that as well. **(6 marks)**

Solution:

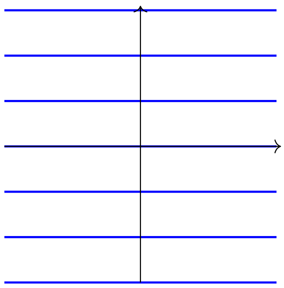
(a) **Bookwork** Let X and Y be topological spaces. A *homeomorphism* from X to Y is a bijective map $f\colon X \to Y$ such that both f and the inverse map $f^{-1}\colon Y \to X$ are continuous [2]. We say that X and Y are *homeomorphic* if there exists such a homeomorphism [1].

(b) **Bookwork** Again let X and Y be topological spaces. Given continuous maps $f_0, f_1\colon X \to Y$, a *homotopy* from f_0 to f_1 is a continuous map $h\colon [0,1] \times X \to Y$ with $h(0, x) = f_0(x)$ and $h(1, x) = f_1(x)$ for all $x \in X$ [2]. We say that f_0 and f_1 are *homotopic* if there exists such a homotopy [1]. We say that X and Y are *homotopy equivalent* if there exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to id_X and fg is homotopic to id_Y [2].

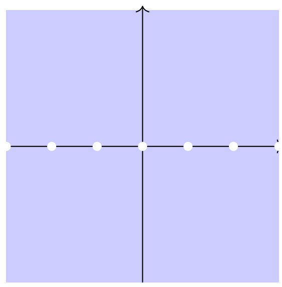
(c) (i) **Similar examples seen** The spaces X_i can be sketched as follows [5]:



X_0

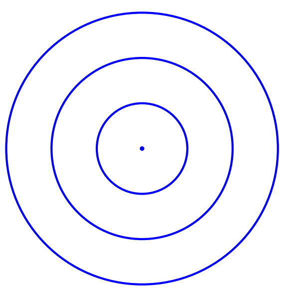


X_1



X_2

X_3



X_4

(ii) **Similar examples have been seen, but this is a bit harder than most of them.**
The spaces X_0, X_1 and X_3 are all homotopy equivalent to $\mathbb{Z}$ and thus to each other. [1]Indeed, we can define maps $\mathbb{Z} \xrightarrow{f_i} X_i \xrightarrow{g_i} \mathbb{Z}$ by

$$\begin{aligned} f_0(n) &= n + \tfrac{1}{2} & g_0(z) &= \lfloor \mathrm{Re}(z) \rfloor \\ f_1(n) &= in & g_1(z) &= \mathrm{Im}(z) \\ f_3(n) &= n + \tfrac{1}{2} & g_3(z) &= \lfloor z \rfloor. \end{aligned}$$

These are all continuous, because the floor function is continuous away from integer arguments. In each case we have $g_i f_i = \mathrm{id}$ and $f_i g_i$ is homotopic to the identity by a linear homotopy [2]. The spaces X_2 and X_4 have nontrivial H_1 and so cannot be homotopy equivalent to X_0, X_1 and X_3 [2]. The space X_2 is path-connected but X_4 is not, so X_2 is not homotopy equivalent to X_4 [1].

(iii) **Similar examples have been seen, but this is a bit harder than most of them.**
If we remove a point from X_0 we obtain a space with nontrivial H_1 but the same path components. However, if we remove a point from X_1 or X_3, we obtain a space with trivial H_1 and an extra path component. It follows that X_0 is not homeomorphic to X_1 or X_3 [2]. However, X_1 is a disjoint union of countably many copies of $\mathbb{R}$, and X_3 is a disjoint union of countably many copies of $(0, 1)$, and $\mathbb{R}$ is homeomorphic to $(0, 1)$, so X_1 is homeomorphic to X_3 [2]. Explicitly, we can define a homeomorphism $f \colon X_1 \to X_3$ by $f(x+ni) = n+\frac{1}{2}+x/(2\sqrt{1+x^2})$. As homeomorphism implies homotopy equivalence, part (ii) implies that there can be no further homeomorphisms. [2]

(69) 2022-23 Q5: Consider S^1 as the unit circle in $\mathbb{R}^2$ as usual. Let X be a path connected space, and put

$$\begin{aligned} U &= \{(t, x) \in S^1 \times X \mid t \neq (0, 1)\} \\ V &= \{(t, x) \in S^1 \times X \mid t \neq (0, -1)\}. \end{aligned}$$

We use the usual notation for inclusion maps:

$$\begin{tikzcd} U \cap V \arrow[r, "i"] \arrow[d, "j"'] & U \arrow[d, "k"] \\ V \arrow[r, "l"'] & S^1 \times X. \end{tikzcd}$$

(a) Define maps $f, g \colon X \to U \cap V$ such that f gives a homotopy equivalence from X to one path component of $U \cap V$, and g gives a homotopy equivalence from X to the other path component of $U \cap V$. **(4 marks)**

(b) Prove that the map $i' = i \circ f \colon X \to U$ is homotopic to $i \circ g$, and also that i' is a homotopy equivalence. (You can then assume without further argument that the map $j' = j \circ f \colon X \to V$ is homotopic to $j \circ g$, and that j' is a homotopy equivalence.) **(6 marks)**

(c) Deduce descriptions (in terms of $H_*(X)$) of the homology groups $H_p(U \cap V)$, $H_p(U)$ and $H_p(V)$, and the homomorphism

$$\alpha = \begin{bmatrix} i_* \\ -j_* \end{bmatrix} : H_p(U \cap V) \to H_p(U) \oplus H_p(V).$$

Find the kernel and image of α. **(8 marks)**

(d) Show that every element of $H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a), 0) + \alpha(b)$ for a unique pair $(a, b) \in H_p(X) \oplus H_p(X)$. **(3 marks)**

(e) Deduce that there is a short exact sequence $H_p(X) \to H_p(S^1 \times X) \to H_{p-1}(X)$. **(4 marks)**

Solution:

(a) The path components of $S^1 \setminus \{(0, 1), (0, -1)\}$ are $A = [(-1, 0)] = \{(x, y) \in S^1 \mid x < 0\}$ and $B = [(+1, 0)] = \{(x, y) \in S^1 \mid x > 0\}$, so the path components of $U \cap V$ are $A \times X$ and $B \times X$ [2]. Here A is contractible and contains $(-1, 0)$ so the map $f(x) = ((-1, 0), x)$ gives a homotopy equivalence from X to $A \times X$. Similarly, the map $g(x) = ((1, 0), x)$ gives a homotopy equivalence from X to $B \times X$ [2].

(b) We can define $h(t,x) = ((-\cos(\pi t), -\sin(\pi t)), x)$ for $0 \leq t \leq 1$. As $(-\cos(\pi t), -\sin(\pi t))$ lies on the bottom half of S^1, this does not pass through $(0,1) \times X$ and so gives a continuous map $[0,1] \times X \to U$. It satisfies $h(0,x) = ((-1,0),x) = i(f(x)) = i'(x)$ and $h(1,x) = ((1,0),x) = i(g(x))$, so this gives a homotopy between i' and $i \circ g$ [3]. We can also define $r\colon U \to X$ by $r(t,x) = x$. Then $r \circ i' = \mathrm{id}$, and contractibility of $S^1 \setminus \{(0,1)\}$ ensures that $i'r$ is homotopic to the identity [3].

(c) As $f\colon X \to A \times X$ and $g\colon X \to B \times X$ are homotopy equivalences, we see that every element of $H_p(U \cap V)$ can be written as $f_*(a) + g_*(b)$ for a unique pair $(a,b) \in H_p(X) \oplus H_p(X)$. [2] Similarly, any element of $H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a), j'_*(b))$ for a unique pair $(a,b) \in H_p(X) \oplus H_p(X)$[2]. As $i_* f_* = i_* g_* = i'_*$ and $j_* f_* = j_* g_* = j'_*$ we see that

$$\alpha(f_*(a) + g_*(b)) = (i'_*(a+b),\ -j'_*(a+b)).\text{[2]}$$

This means that

$$\ker(\alpha) = \{f_*(a) - g_*(a) \mid a \in H_p(X)\} \simeq H_p(X)\text{[1]}$$
$$\operatorname{img}(\alpha) = \{(i'_*(c), -j'_*(c)) \mid c \in H_p(X)\} \simeq H_p(X)\text{[1]}.$$

(d) We now see that every element $(i'_*(a), j'_*(b)) \in H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a+b), 0) + (i'_*(-b), -j'_*(-b))$ with the second term lying in $\operatorname{img}(\alpha)$, and this decomposition is unique [3].

(e) From the exact sequence

$$H_p(U \cap V) \xrightarrow{\alpha} H_p(U) \oplus H_p(V) \to H_p(S^1 \times X) \xrightarrow{\delta} H_{p-1}(U \cap V) \xrightarrow{\alpha} H_{p-1}(U) \oplus H_{p-1}(V)$$

we get a short exact sequence

$$(H_p(U) \oplus H_p(V))/\operatorname{img}(\alpha_p) \to H_p(S^1 \times X) \to \ker(\alpha_{p-1})\text{[2]}$$

Part (d) gives an isomorphism $(H_p(U) \oplus H_p(V))/\operatorname{img}(\alpha_p) \simeq H_p(X)$ [1]. Part (c) gives an isomorphism $\ker(\alpha_{p-1}) \simeq H_{p-1}(X)$ [1]. We therefore have a short exact sequence

$$H_p(X) \to H_p(S^1 \times X) \to H_{p-1}(X)$$

as claimed.