Algebraic Topology Exam Questions

This is a collection of questions taken from Algebraic Topology exams over a number of years. To some extent, they have been modified to be compatible with the version of the course taught in 2021-22, but some differences remain.

1 Compactness

Some questions in this section use ideas that are specific to metric spaces rather than general topological spaces. These ideas are not developed in the current version of the course.

(1) Let X be a metric space.

(a) Let Y be a compact subspace of X. Prove that Y is closed in X.

(b) Let Y and Z be two compact subspaces of X. Prove that $Y \cup Z$ is compact.

(c) Deduce (or prove otherwise) that every finite space is compact.

(d) Let Y and Z be compact metric spaces. Prove that $Y \times Z$ is compact.

(e) Conversely, let Y and Z be metric spaces such that $Z \neq \emptyset$ and $Y \times Z$ is compact. Prove that Y is compact.

(f) Put $X = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^4 + y^4 + z^4 = 1\}$. Prove that X is compact. You may use general theorems provided that you state them precisely.

(2) Let X be a metric space.

(a) Let Y be a compact subspace of X. Prove that Y is closed in X.

(b) Let Y and Z be two compact subspaces of X. Prove that $Y \cup Z$ is compact.

(c) Deduce (or prove otherwise) that every finite space is compact.

(d) Let Y and Z be compact metric spaces. Prove that $Y \times Z$ is compact.

(e) Conversely, let Y and Z be metric spaces such that $Z \neq \emptyset$ and $Y \times Z$ is compact. Prove that Y is compact.

(f) Put $X = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^4 + y^4 + z^4 = 1\}$. Prove that X is compact. You may use general theorems provided that you state them precisely.

(3)

(a) What does it mean to say that a metric space X is *compact*? **(3 marks)**

(b) Let $f \colon X \to Y$ be a continuous surjective map of metric spaces, where X is compact. Prove that Y is compact. **(6 marks)**

(c) Let Z be a closed subset of a compact space X. Prove that Z is compact. **(6 marks)**

(d) Put $U = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq 1\}$, and define $g \colon \mathbb{C} \to \mathbb{C}$ by $g(z) = e^z$.

  (i) Is U compact? **(2 marks)**

  (ii) Is $g(U)$ compact? **(4 marks)**

  (iii) Is $g(g(U))$ compact? **(4 marks)**

Justify your answers.

(4)

(a) What does it mean to say that a metric space X is *compact*? **(3 marks)**

(b) Let X and Y be compact metric spaces. Prove that $X \times Y$ is compact. **(8 marks)**

(c) Let $f\colon I \to Y$ be a continuous map (where $I = [0, 1]$). Prove that $f(I)$ is closed in Y. **(7 marks)**

(d) Put $X = \mathbb{Z} \times \mathbb{Z}$ and $Y = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$, considered as subspaces of the plane $\mathbb{R}^2$.

  (i) Is X compact? **(2 marks)**

  (ii) Is Y compact? **(2 marks)**

  (iii) Is $X \cap Y$ compact? **(3 marks)**

Justify your answers.

2 Path components

(5)

(a) What does it mean to say that a topological space X is *path-connected*?

(b) Prove that the space S^n is path-connected for all $n > 0$.

(c) Let X be a subset of $\mathbb{R}^n$, and let a be a point in X. What does it mean to say that X is *star-shaped* around a? Show that if X is star-shaped around a, then it is path-connected.

(d) Suppose that $f\colon X \to \mathbb{R}$ is continuous, $f(x)$ is nonzero for all x, and there exist $x_0, x_1 \in X$ with $f(x_0) < 0 < f(x_1)$. Prove that X is not path-connected.

(e) Recall that $GL_3(\mathbb{R})$ is the space of 3×3 invertible matrices over $\mathbb{R}$. Prove that this space is not path-connected.

(6)

(a) Let X be a topological space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is an equivalence relation.

(b) Let $f\colon X \to Y$ be a continuous map. Define the induced map $f_*\colon \pi_0(X) \to \pi_0(Y)$, and prove that it is well-defined.

(c) Show that if $f, g\colon X \to Y$ are homotopic maps then $f_* = g_*\colon \pi_0(X) \to \pi_0(Y)$.

(d) Put $X = [-3, -2] \cup [-1, 1] \cup [2, 3]$ and $Y = [0, 1] \cup [2, 10]$, and define $f\colon X \to Y$ by $f(x) = x^2$. Describe the sets $\pi_0(X)$ and $\pi_0(Y)$ and the map $f_*\colon \pi_0(X) \to \pi_0(Y)$.

(7)

(a) Let X be a metric space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is indeed an equivalence relation. **(8 marks)**

(b) Let $f\colon X \to Y$ be a continuous map. Define the function $f_*\colon \pi_0(X) \to \pi_0(Y)$, and check that it is well-defined. **(5 marks)**

(c) Suppose that Y is path-connected and X is not. Show that there do not exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to the identity map id_X. **(6 marks)**

(d) Put $X = \{A \in M_2\mathbb{R} \mid A^2 = A\}$. What can you say about $\det(A)$ when $A \in X$? Show that X is not path-connected. **(6 marks)**

3 The fundamental group

These questions involve material that is not covered in the current version of the course.

(8)

(a) Let X be a metric space, and let x_0 and x_1 be points in X. What does it mean to say that two paths from x_0 to x_1 are *pinned homotopic*? Define the set $\pi_1(X; x_0, x_1)$.

(b) Let X be path-connected. Prove that the group $\pi_1(X; x_0)$ is isomorphic to the group $\pi_1(X; x_1)$.

(c) Put $X = \{(w, x, y, z) \in \mathbb{C}^4 \mid w \neq x\ ,\ x \neq y\ ,\ y \neq z\}$, and take $x_0 = (0, 1, 2, 3)$ as the basepoint in X. Calculate $\pi_1(X)$. (You may wish to consider the expression $f(w, x, y, z) = (w, x - w, y - x, z - y)$.)

(9)

(a) Let X be a based topological space, and let Y be a subspace of X containing the basepoint. What does it mean to say that Y is a *retract* of X?

(b) Prove that if Y is a retract of X, then $|\pi_1(Y)| \leq |\pi_1(X)|$.

(c) Recall that $\mathbb{R}P^3$ is a subspace of the space $M_4(\mathbb{R})$ of all 4×4 matrices over $\mathbb{R}$, which is homeomorphic to $\mathbb{R}^{16}$. Prove that $\mathbb{R}P^3$ is not a retract of $M_4(\mathbb{R})$.

(d) Recall that $U(3)$ is the space of 3×3 matrices A over $\mathbb{C}$ such that $A^\dagger A = I$. You may assume that for such A we have $\det(A) \in S^1$. Define $j \colon S^1 \to U(3)$ by

$$j(z) = \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

What is $\det(j(z))$? Deduce that $\pi_1(U(3))$ is infinite.

4 Homotopy equivalence

(10)

(a) Let $f, g \colon X \to Y$ be continuous maps between topological spaces. What does it mean to say that f is *homotopic* to g?

(b) Let X and Y be topological spaces. What does it mean to say that X and Y are *homotopy equivalent*?

(c) Show that if X and Y are homotopy equivalent then there is a bijection between the sets of path-components $\pi_0(X)$ and $\pi_0(Y)$.

(d) Consider the cross $X = \{(x, 0) \mid -1 \leq x \leq 1\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$, and let $C = \mathbb{R}^2 \setminus X$ be its complement. Prove that C is homotopy equivalent to S^1.

(11) Consider a metric space X.

(a) (i) What does it mean to say that a subset U of X is open?

(ii) What does it mean to say that a subset F of X is closed?

(b) Show that a subset $F \subseteq X$ is closed iff for every sequence (x_n) in F that converges to a point $x \in X$, we actually have $x \in F$.

(c) Explain what it means for a subset $A \subseteq X$ to be compact. Show that if A is compact and $f \colon X \to Y$ is continuous then $f(A)$ is compact.

(d) Prove that the space $[0,1]$ is compact. Show that there is a continuous bijection $g\colon [-1,-1/2)\cup[1/2,1]\to[0,1]$; can it be chosen to be a homeomorphism?

(12)

(a) What does it mean to say that a topological space X is *homotopy equivalent* to a metric space Y? Show that the relation of homotopy equivalence is an equivalence relation.

(b) What does it mean for a space to be (a) *contractible* and (b) *path connected*? Show that any contractible space is path connected. Is the reverse implication true?

(c) Consider the rational comb space

$$X=\{(x,y)\in\mathbb{R}^2\mid y\geq 0 \text{ or } x\in\mathbb{Q}\}.$$

Show that X is homotopy equivalent to the upper half plane $Y=\{(x,y)\in\mathbb{R}^2\mid y\geq 0\}$, and deduce that X is contractible.

(13)

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(14)

(a) Let X be a subspace of $\mathbb{R}^n$, and let a be a point in X.

(i) Explain what it means for X to be *star-shaped* around a. **(4 marks)**

(ii) Prove that if X is star-shaped around a, then X is contractible. **(4 marks)**

(b) (i) Suppose that $\alpha,\beta>0$ and that $0\leq t\leq 1$. Show that $\alpha t+\beta(1-t)$ is strictly greater than zero. **(3 marks)**

(ii) Suppose that $\gamma,\delta,\epsilon>0$ and that $0\leq t\leq 1$. Show that $\gamma t^2+\delta t(1-t)+\epsilon(1-t)^2$ is strictly greater than zero. **(3 marks)**

(iii) Consider a matrix $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2\mathbb{R}$. Put $\lambda=\mathrm{trace}(A)$ and $\mu=\det(A)$. Express $\mathrm{trace}((1-t)I+tA)$ and $\det((1-t)I+tA)$ in terms of λ, μ and t. **(6 marks)**

(iv) Put $X=\{A\in M_2\mathbb{R}\mid \det(A)>0 \text{ and } \mathrm{trace}(A)>0\}$. Prove that X is contractible. **(5 marks)**

(15)

(a) Given metric spaces X,Y and continuous maps $f,g\colon X\to Y$, what does it mean for f and g to be homotopic? **(3 marks)**

(b) Show that if Y is contractible, then any two maps $f,g\colon X\to Y$ are homotopic. **(7 marks)**

(c) Show that if X is contractible and Y is path-connected, then any two maps $f,g\colon X\to Y$ are homotopic. **(10 marks)**

(d) Regard S^1 as $\{z \in \mathbb{C} \mid |z| = 1\}$, and put $T = S^1 \times S^1$. Define $f\colon T \to T$ by $f(z, w) = (iz, -iw)$. Prove that f is homotopic to the identity map. **(5 marks)**

(16) Let E be the figure eight space, so $E = E_- \cup E_+$ where $E_\pm$ is the circle of radius one centred at $(\pm 1, 0)$.

(a) Prove that E is not homotopy equivalent to the torus. **(4 marks)**

(b) Put $A = \{(1, 0), (-1, 0)\}$ and $X = \mathbb{R}^2 \setminus A$. Sketch a proof that X is homotopy equivalent to E. **(5 marks)**

(c) Put $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 = 1\ ,\ y = 0\}$ and $Y = \mathbb{R}^3 \setminus B$. Deduce that Y is homotopy equivalent to E. **(4 marks)**

(d) Put $C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 = 1\ ,\ y = xz\}$ and $Z = \mathbb{R}^3 \setminus C$. Deduce that Z is homotopy equivalent to E. You may wish to consider the expression

$$(x, \mathrm{rot}_{\pi x/4}(y, z)) = (x, \cos(\pi x/4)y - \sin(\pi x/4)z, \sin(\pi x/4)y + \cos(\pi x/4)z).$$

(12 marks)

5 Abelian groups and chain complexes

(17)

(a) In the context of Abelian groups, define the terms

- homomorphism **(2 marks)**
- subgroup **(2 marks)**
- kernel **(2 marks)**
- image. **(2 marks)**

(b) Let A and B be Abelian groups, and let $\phi\colon A \to B$ be a homomorphism. Prove that

(i) The kernel of ϕ is a subgroup of A **(3 marks)**

(ii) The kernel of ϕ is a subgroup of the kernel of the homomorphism 2ϕ. **(2 marks)**

(c) Let $\phi\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}/12$ be the homomorphism defined by

$$\phi(n, m) = (3n, (2n + 4m \bmod 12)).$$

Give an isomorphism $\psi\colon \mathbb{Z} \to \ker(\phi)$. **(6 marks)**

(d) Let A be a finite Abelian group, and let B be a free Abelian group. Prove that if $\phi\colon A \to B$ is a homomorphism, then $\phi = 0$. **(6 marks)**

(18) **2018-19 Q4:** Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain complexes and chain maps.

(a) Define what is meant by saying that the above sequence is short exact. **(3 marks)**

Now recall that a *snake* for the above sequence is a system (c, w, v, u, a) such that

- $c \in H_n(W)$;
- $w \in Z_n(W)$ is a cycle such that $c = [w]$;
- $v \in V_n$ is an element with $p(v) = w$;
- $u \in Z_{n-1}(U)$ is a cycle with $i(u) = d(v) \in V_{n-1}$;

- $a = [u] \in H_{n-1}(U)$.

(b) Prove that for each $c \in H_n(W)$ there is a snake starting with c. **(8 marks)**

(c) Prove that if two snakes have the same starting point, then they also have the same endpoint. **(10 marks)**

(d) Suppose that the differential $d\colon V_{n+1} \to V_n$ is surjective. Show that any snake starting in $H_n(W)$ ends with zero. **(4 marks)**

6 Singular chains

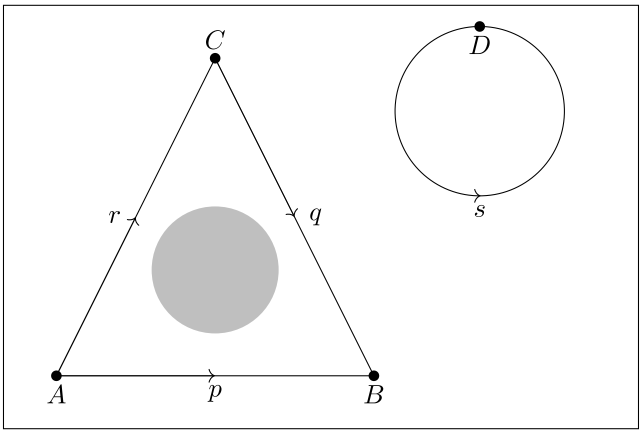
(19) Let X be a topological space.

(a) Let $c\colon \Delta_1 \to X$ be a constant path. Prove that c is homologous to 0.

(b) Let $s\colon \Delta_1 \to X$ be a path. Define the reversed path $\overline{s}$, and prove that $\overline{s}$ is homologous to $-s$.

(c) Let $r, s\colon \Delta_1 \to X$ be paths such that $r(e_1) = s(e_0)$. Write down a path $u\colon \Delta_1 \to X$ and prove that u is homologous to $r + s$.

(d) Let X be the complement of the shaded disc in the diagram below. Write down a path $u\colon \Delta_1 \to X$ such that u is homologous to $2p - 2q - 2r + s$.



(20)

(a) Let X be a topological space.

(i) Define the groups $C_n(X)$ for all nonnegative integers n. **(2 marks)**

(ii) Define the homomorphisms ∂_n. **(3 marks)**

(iii) Prove that $\partial_1 \circ \partial_2 = 0$. **(3 marks)**

(iv) Define the groups $H_n(X)$. **(4 marks)**

(b) Describe (without proof, but with careful attention to any special cases) the groups $H_n(\mathbb{R}^k \setminus \{0\})$ for all $n \geq 0$ and all $k \geq 1$. **(5 marks)**

(c) Let $u = n_1 s_1 + \ldots + n_k s_k$ be an m-cycle in S^n (where $m > 0$), and suppose that there is a point $a \in S^n$ that is not contained in any of the sets $s_1(\Delta_m), \ldots, s_k(\Delta_m)$. Prove that u is a boundary. **(8 marks)**

(21)

(a) Let X be a topological space.

(i) Define the groups $C_0(X)$ and $C_1(X)$, and the homomorphism $\partial_1 \colon C_1(X) \to C_0(X)$.

(ii) Define the subdivision homomorphism $\mathrm{sd}\colon C_1(X) \to C_1(X)$.

(iii) Prove that $\partial_1 \mathrm{sd}^n(u) = \partial_1(u)$ for all $n \geq 1$.

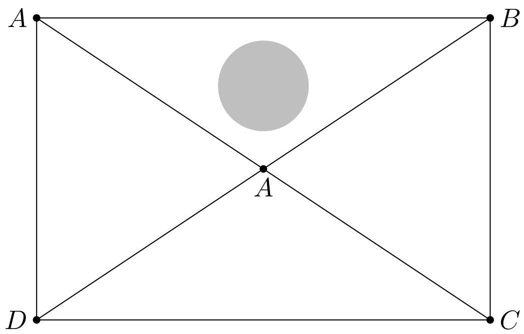
(iv) Prove that if $u \in B_1(X)$ then $\mathrm{sd}(u) \in B_1(X)$.

(v) Let A and B be points in a vector space V. Give an expression for $\mathrm{sd}\langle A, B\rangle$ in terms of paths of the form $\langle C, D\rangle$.

(b) Describe without proof the groups $H_1(S^1)$, $H_1(S^1 \times S^1)$, $H_1(\mathbb{R}P^2)$ and $H_1(\mathbb{R}^3 \setminus \{0\})$.

(c) For each element $u \in H_1(\mathbb{R}P^2)$, give a path s in $\mathbb{R}P^2$ such that $u = [s]$.

(22) Consider the following diagram.



Let X be the complement in $\mathbb{R}^2$ of the shaded disc. Define $u, v, w \in C_1(X)$ by

$$\begin{aligned} u &= \langle A, B\rangle + \langle B, E\rangle + \langle E, A\rangle \\ v &= \langle A, B\rangle + \langle B, C\rangle + \langle C, D\rangle + \langle D, A\rangle \\ w &= \langle A, E\rangle + \langle E, D\rangle + \langle D, A\rangle. \end{aligned}$$

(a) Prove that u is a cycle. **(2 marks)**

(b) Prove that $\langle B, B\rangle$ is homologous to 0 in X. **(3 marks)**

(c) Prove that $\langle E, B\rangle$ is homologous to $-\langle B, E\rangle$ in X. **(4 marks)**

(d) Prove in detail that u is homologous to v in X, justifying each step. **(8 marks)**

(e) Write down a basic 1-chain s that is homologous in X to $\langle A, B\rangle + \langle B, C\rangle$ **(5 marks)**

(f) Is u homologous to w? Give a brief reason for your answer. **(3 marks)**

7 True or false

(23) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(a) The punctured disc $X = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$ is compact.

(b) The circle S^1 is homeomorphic to $S^1 \times I$.

(c) The circle S^1 is homotopy equivalent to $S^1 \times I$.

(d) $\mathbb{C} \setminus S^1$ is homotopy equivalent to $Y = \{z \in \mathbb{C} \mid z = 0 \text{ or } |z| = 1\}$.

(e) Every continuous bijection from $[0, 1] \cup (2, 3]$ to $[0, 1]$ is a homeomorphism.

(24) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(a) S^3 is contractible.

(b) If a space X is the union of two closed, path-connected subspaces A and B, then X is path-connected.

(c) $(\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1.

(d) $(\mathbb{R} \times \mathbb{R}^2) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1.

(e) The space $\mathbb{C} \setminus \{0, 1\}$ is homeomorphic to $\mathbb{C} \setminus \{i, -i\}$.

(f) The space $\mathbb{C} \setminus \{0, 1\}$ is homotopy equivalent to $\mathbb{C} \setminus \{0, 1, 2\}$.

(25) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(a) The identity map of the unit circle is homotopic to the constant map $c\colon S^1 \to S^1$ defined by $c(z) = 1$ for all z.

(b) Let $f_n\colon S^1 \to S^1$ be defined by $f_n(z) = z^n$. Then f_n is not homotopic to f_m when $n \neq m$.

(c) $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^3$.

(d) If $f\colon X \to X$ is a homotopy equivalence, then $f_*\colon H_1(X) \to H_1(X)$ is the identity map.

(26) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(a) The torus $T = S^1 \times S^1$ is homotopy equivalent to S^2.

(b) There is a map $r\colon B^4 \to S^3$ such that rj is homotopic to id_{S^3}, where $j\colon S^3 \to B^4$ is the inclusion map.

(c) $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^3$.

(d) Every continuous function $f\colon S^2 \to \mathbb{R}^3$ is homotopic to a constant function.

(e) Let $K \subset S^3$ be a trefoil knot. Then $S^3 \setminus K$ is homotopy equivalent to $\mathbb{R}P^2$.

(27) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems or calculations of homology groups without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) $\mathbb{R}P^1$ is homeomorphic to S^1.

(b) The Möbius strip is homotopy equivalent to S^2.

(c) $S^2 \setminus S^1$ is homotopy equivalent to $\mathbb{R} \setminus \{0\}$.

(d) The letter A is homeomorphic to the letter D.

(e) Any compact convex subset of $\mathbb{R}^2$ is homeomorphic to B^2.

(28) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems or calculations of homology groups without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) $\mathbb{R}P^1$ is homeomorphic to S^1.

(b) The Möbius strip is homotopy equivalent to S^2.

(c) $SO(3)$ is homeomorphic to $\mathbb{R}P^3$.

(d) $S^2 \setminus S^1$ is homotopy equivalent to $\mathbb{R} \setminus \{0\}$.

(e) The letter A is homeomorphic to the letter D.

(f) Any compact convex subset of $\mathbb{R}^2$ is homeomorphic to B^2.

(29) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems or calculations of homology groups without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) S^1 is homotopy equivalent to S^2. **(3 marks)**

(b) S^1 is homotopy equivalent to the Möbius strip. **(4 marks)**

(c) S^1 is homeomorphic to the Möbius strip. **(4 marks)**

(d) $\mathbb{R}P^2$ is homeomorphic to $S^1 \times S^1$. **(4 marks)**

(e) $SU(2) \setminus \{I\}$ is homeomorphic to $\mathbb{R}^3$. **(5 marks)**

(f) $\Delta_n \times \Delta_m$ is homeomorphic to Δ_{n+m}. **(5 marks)**

(30) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) There is a continuous surjective map from $S^1 \times S^1$ to $\mathbb{R}$

(b) $\mathbb{C} \setminus \{2\}$ is homotopy equivalent to S^1

(c) $\mathbb{C} \setminus \{-1, 1\}$ is homotopy equivalent to S^1

(d) $S^2 \setminus \{\text{the north pole}\}$ is homeomorphic to $\mathbb{C}$.

(e) The letter X (considered as a subspace of $\mathbb{R}^2$) is homeomorphic to the letter Y.

(f) The letter X (considered as a subspace of $\mathbb{R}^2$) is homotopy equivalent to the letter Y.

(31) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) There is a continuous surjective map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R} \setminus \{0\}$ **(5 marks)**

(b) $S^2 \setminus S^1$ is homeomorphic to $\mathbb{R}^2$ **(5 marks)**

(c) $SO(2)$ is homotopy equivalent to the Möbius strip **(5 marks)**

(d) $SO(3)$ is homotopy equivalent to the torus **(5 marks)**

(e) The space $X = S^1 \cup \{(x, 0) \mid x \in \mathbb{R}\}$ is homeomorphic to $Y = S^1 \cup \{(x, 1) \mid x \in \mathbb{R}\}$. **(5 marks)**

(32) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) If X and Y are both path-connected subsets of $\mathbb{R}^2$, then $X \cap Y$ is also path-connected. **(5 marks)**

(b) The torus is homotopy equivalent to S^2. **(5 marks)**

(c) If $f\colon X \to Y$ and $g\colon Y \to X$ are continuous, based maps and $gf = \mathrm{id}_X$ then $\pi_1(X) \simeq \pi_1(Y)$. **(5 marks)**

(d) If two letters of the alphabet, considered as subspaces of $\mathbb{R}^2$, both have infinite H_1, then they are homotopy equivalent. **(5 marks)**

(e) The space $GL_3(\mathbb{R})$ is path-connected. **(5 marks)**

(33) 2018-19 Q2: Are the following true or false? Justify your answers.

(a) S^5 is a Hausdorff space. **(4 marks)**

(b) The Klein bottle is a retract of $S^1 \times S^1 \times S^1$. **(4 marks)**

(c) There is a connected space X with $\pi_1(X) \simeq \mathbb{Z}/2$ and $H_1(X) \simeq \mathbb{Z}$. **(4 marks)**

(d) There is a short exact sequence $\mathbb{Z}/9 \to \mathbb{Z}/99 \to \mathbb{Z}/11$. **(4 marks)**

(e) If K is a simplicial complex and L is a subcomplex and $H_3(K) = 0$ then $H_3(L) = 0$. **(4 marks)**

(f) If K and L are simplicial complexes and $f\colon |K| \to |L|$ is a continuous map then there is a simplicial map $s\colon K \to L$ such that f is homotopic to $|s|$. **(5 marks)**

8 Examples

(34) Give examples of the following things, with careful justification.

(a) A noncompact metric space X with a sequence of compact subspaces $Y_1 \subset Y_2 \subset \ldots$ such that the union of all the sets Y_n is equal to X.

(b) A metric space X with two noncompact subsets Y, Z such that $Y \cup Z$ is compact.

(c) A sequence in $\mathbb{R}$ with no convergent subsequence.

(d) A non-surjective map $f\colon X \to Y$ such that $f_*\colon \pi_0(X) \to \pi_0(Y)$ is surjective.

(e) An injective map $f\colon X \to Y$ such that $f_*\colon \pi_0(X) \to \pi_0(Y)$ is not injective.

(35) Give examples of the following things, with careful justification.

(a) A continuous bijection that is not a homeomorphism. **(3 marks)**

(b) An infinite sequence of open sets whose intersection is not open. **(3 marks)**

(c) Two metric spaces X, Y such that X is bounded, Y is unbounded, and X is homeomorphic to Y. **(4 marks)**

(d) A sequence in $(0, 1)$ such that no subsequence converges in $(0, 1)$. **(5 marks)**

(e) Two contractible subsets of $\mathbb{R}^2$ whose intersection is not contractible. **(5 marks)**

(f) Two metric spaces X, Y and points $x \in X$, $y \in Y$ such that X is homotopy equivalent to Y but $X \setminus \{x\}$ is not homotopy equivalent to $Y \setminus \{y\}$. **(5 marks)**

(36) Give examples of the following things, with justification.

(a) Connected sets $X, Y \subseteq \mathbb{R}^2$ such that $X \cap Y$ is not connected.

(b) A sequence of open sets $U_n \subseteq \mathbb{R}$ such that the set $X = U_1 \cap U_2 \cap \ldots = \bigcap_n U_n$ is not open.

(c) A surjective map $f\colon X \to Y$ of topological spaces such that the homomorphism $f_*\colon H_1(X) \to H_1(Y)$ is not surjective.

(d) A path connected space X that is homotopy equivalent to $X \times X$.

(e) A path connected space X that is not homotopy equivalent to $X \times X$.

(37) Give examples of the following things.

(a) A space X and a point $x \in X$ such that X is not contractible but $X \setminus \{x\}$ is contractible. **(3 marks)**

(b) A subspace $X \subseteq \mathbb{R}^2$ that is homotopy equivalent to $S^4 \setminus S^2$. (You need not give a proof.) **(4 marks)**

(c) Spaces X and Y, a discontinuous map $f\colon X \to Y$, and an open subset $V \subseteq Y$ such that $f^{-1}V$ is not open in X. (You should justify your answer carefully.) **(6 marks)**

(d) A space X and a point $x \in X$ such that $\pi_1(X)$ is abelian and $\pi_1(X \setminus \{x\})$ is nonabelian. (You should state what $\pi_1(X)$ and $\pi_1(X \setminus \{x\})$ are, but no further justification is required.) **(6 marks)**

(e) A space X such that $a(X) = 2$ and $b(X) = 2$, where as usual

$$\begin{aligned} a(X) &= \max\{|S| \mid S \text{ is a finite subset of } X \text{ and } X \setminus S \text{ is path-connected } \} \\ &= \text{the largest number of points that can be} \\ &\quad\ \text{removed from } X \text{ without disconnecting it} \\ b(X) &= \min\{|S| \mid S \text{ is a finite subset of } X \text{ and } X \setminus S \text{ is not path-connected } \} \\ &= \text{the smallest number of points that have to be} \\ &\quad\ \text{removed from } X \text{ to disconnect it} \end{aligned}$$

(You should justify your answer, but complete rigour is not required.) **(6 marks)**

(38) Give one example of each of the following things, with justification.

(a) A path connected space X with $H_1(X) = \mathbb{Z} \oplus (\mathbb{Z}/2)$. **(4 marks)**

(b) A path-connected space X and points $a, b, c \in X$ such that $X \setminus \{a, b, c\}$ is still path-connected. **(3 marks)**

(c) A path-connected space X and a point $a \in X$ such that $H_1(X)$ and $H_1(X \setminus \{a\})$ are both trivial. **(5 marks)**

(d) A continuous, surjective map $f\colon X \to Y$, where Y is compact but X is not. **(3 marks)**

(e) A space X and points $a, b \in X$ such that $\pi_1(X)$ is nonabelian but the space $Y = X \setminus \{a, b\}$ is simply connected. **(5 marks)**

(f) A continuous bijection that is not a homeomorphism. **(5 marks)**

(39) 2020-21 Q1: Give examples as follows, justifying your answers.

(a) Topological spaces X and Y, together with injective functions $f\colon X \to Y$ and $g\colon Y \to X$ such that f, $f \circ g$ and $g \circ f$ are all continuous, but g is not continuous. **(4 marks)**

(b) A compact, path-connected space X together with a continuous map $f\colon X \to X$ with no fixed points. **(4 marks)**

(c) A space X such that $H_1(X)$ is not a free abelian group. (Note here that the zero group is free abelian with no generators, so in particular $H_1(X)$ must be nonzero.) **(4 marks)**

(d) A space X together with points $a, b, c \in X$ such that $|\Pi(X; a, b)| \neq |\Pi(X; b, c)|$. **(4 marks)**

(e) A space X such that $\pi_1(X)$ is a free group with 3 generators, and $H_2(X) = \mathbb{Z}$. **(4 marks)**

9 Real projective space

In the current version of the course, in the Introduction we define $\mathbb{R}P^n = S^n/(x \sim -x)$ and

$$P_n = \{A \in M_{n+1}(\mathbb{R}) \mid A^2 = A^T = A,\ \mathrm{trace}(A) = 1\},$$

and we mention that $\mathbb{R}P^n$ is homeomorphic to P_n. A proof is given in Problem Sheet 5. In some earlier versions of the course, $\mathbb{R}P^n$ was just defined to be the same as P_n. Problems in this section should be approached from that point of view.

(40)

(a) Define the set $\mathbb{R}P^2$ and the map $q\colon S^2 \to \mathbb{R}P^2$.

(b) Define the usual metric on $\mathbb{R}P^2$, and prove that it is a metric.

(c) Define the space Δ_2, and prove carefully that there is a surjective continuous map $f\colon \mathbb{R}P^2 \to \Delta_2$ satisfying $fq(u,v,w) = (u^2, u^2+v^2)$ for all $(u,v,w) \in \Delta_2$. You may use general theorems provided that you state them precisely.

(41)

(a) Define the set $\mathbb{R}P^n$, and write down a metric on it, proving that your formula is well-defined. (You need not show that it is a metric.) **(6 marks)**

(b) Define what it means for a metric space X to be *sequentially compact*. **(3 marks)**

(c) Define the set $\pi_0(X)$, and say what it means for X to be *path-connected*. **(6 marks)**

(d) Prove that the space $\mathbb{R}P^n$ is sequentially compact and path-connected. State clearly any general theorems or results that you use. **(10 marks)**

10 Multipart questions

(42)

(a) What is a *metric space*? What is a *continuous function*?

(b) Define the discrete metric on a set X.

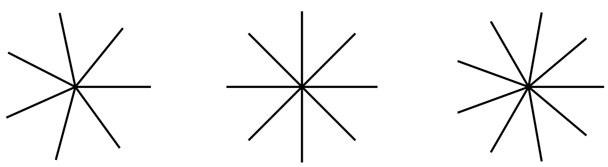
(c) Let X be a space with a discrete metric. Show that any path $s\colon \Delta_1 \to X$ is constant, and deduce that $\pi_0(X) = X$.

(d) Consider the space $Y = \{(x,y) \in \mathbb{R}^2 \mid xy \neq 0\}$ and show that $\pi_0(Y)$ has precisely four elements. If $f\colon Y \to Y$ denotes reflection in the line $x = y$, describe the map $f_*\colon \pi_0(Y) \to \pi_0(Y)$. Is f homotopic to the identity map?

(43) 2018-19 Q1:

(a) Given a topological space X, define the set $\pi_0(X)$. You should include a proof that the relevant equivalence relation is in fact an equivalence relation. **(8 marks)**

(b) Consider $[0,1]$ as a based space with 0 as the basepoint. For $n \geq 3$ we define $X_n = \{z \in \mathbb{C} \mid z^n \in [0,1]\}$:



X_7 X_8 X_9

(i) For which n and m (with $n, m \geq 3$) is X_n homotopy equivalent to X_m? **(3 marks)**

(ii) For which n and m (with $n, m \geq 3$) is X_n homeomorphic to X_m? **(4 marks)**

Justify your answers carefully.

(c) Give examples as follows, with justification:

(1) A based space W with $|\pi_1(W)| = 8$. **(3 marks)**

(2) A space X with two points $a, b \in X$ such that $\pi_1(X, a)$ is not isomorphic to $\pi_1(X, b)$. **(3 marks)**

(3) A space Y such that $H_0(Y) \simeq H_2(Y) \simeq H_4(Y) \simeq H_6(Y) \simeq \mathbb{Z}$ and all other homology groups are trivial. **(4 marks)**

(44) 2018-19 Q3: Let K and L be abstract simplicial complexes.

(a) Define what is meant by a *simplicial map* from K to L. **(3 marks)**

(b) Let $s, t \colon K \to L$ be simplicial maps. Define what it means for s and t to be *directly contiguous*. **(3 marks)**

(c) Prove that if s and t are directly contiguous, then the resulting maps $|s|, |t| \colon |K| \to |L|$ are homotopic. **(3 marks)**

(d) Prove that if s and t are directly contiguous, then the resulting maps $s_*, t_* \colon H_*(K) \to H_*(L)$ are the same. (You can prove the main formula just for $n = 3$ rather than general n.) **(9 marks)**

(e) How many injective simplicial maps are there from $\partial\Delta^2$ to itself? Show that no two of them are directly contiguous. **(7 marks)**

(45) 2018-19 Q5: Consider a simplicial complex K with subcomplexes L and M such that $K = L \cup M$. Use the following notation for the inclusion maps:

$$\begin{tikzcd} L \cap M \arrow[r, "i"] \arrow[d, "j"'] & L \arrow[d, "f"] \\ M \arrow[r, "g"'] & K. \end{tikzcd}$$

(a) State the Seifert-van Kampen Theorem (in a form applicable to simplicial complexes and subcomplexes as above). **(4 marks)**

(b) State the Mayer-Vietoris Theorem. **(5 marks)**

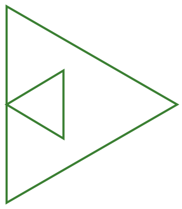
(c) State a theorem about the relationship between π_1 and H_1. **(3 marks)**

(d) Suppose that $|L|$, $|M|$ and $|L \cap M|$ are all homotopy equivalent to S^1. Suppose that the maps i and j both have degree two.

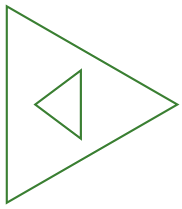
(1) Find a presentation for $\pi_1(|K|)$. **(3 marks)**

(2) Find $H_*(K)$. In particular, you should express each nonzero group as a direct sum of terms like $\mathbb{Z}$ or $\mathbb{Z}/n$. **(10 marks)**

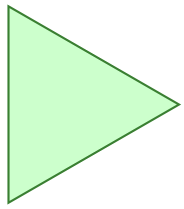
(46) 2019-20 Q1: Consider the following spaces:



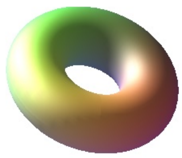
X_0



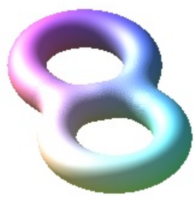
X_1



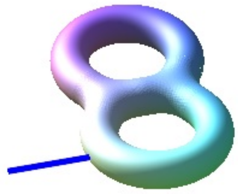
X_2



X_3



X_4



X_5

$X_6 = (S^1 \times S^1) \setminus \{(1,1)\}$ $\quad$ $X_7 = GL_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) \mid \det(A) \neq 0\}$

$X_8 = \mathbb{R}$ $\quad$ $X_9 = \{(u,v) \in \mathbb{C}^2 \mid 1 \leq |u| \leq 2 \leq |v| \leq 3\}$.

(Here X_3 and X_4 are closed orientable surfaces, and X_5 is the union of X_4 with a line segment with one endpoint lying on X_4. Everything else should be clear.)

(a) These 10 spaces can be grouped into 5 pairs $\{X_i, X_j\}$ such that X_i is homotopy equivalent to X_j. Find these pairs, and justify your answers. In each case you should prove that X_i is homotopy equivalent to X_j, and also that it is not homotopy equivalent to any of the other spaces. **(25 marks)**

(b) For each pair $\{X_i, X_j\}$ as in (a), prove that X_i is not homeomorphic to X_j. (In one case you may need to appeal to some geometric intuition, but you should be able to give a more formal proof in the other four cases.) **(15 marks)**

(47) 2019-20 Q2:

(a) Let A and B be finite abelian groups such that $|A|$ and $|B|$ are coprime.

(i) What can you say about homomorphisms from A to B? **(10 marks)**

(ii) Now suppose we have a short exact sequence $A \to U \to B$ of abelian groups. By considering the classification of finite abelian groups, or otherwise, what can you say about U? **(15 marks)**

(b) Let X be a topological space, with open subspaces U and V such that $X = U \cup V$. Suppose that U, V, X and $U \cap V$ are all path-connected, and that for all $k > 0$ we have $H_k(U \cap V) = \mathbb{Z}/2^k$ and $H_k(U) = \mathbb{Z}/3^k$ and $H_k(V) = \mathbb{Z}/5^k$. Calculate $H_*(X)$. **(15 marks)**

(48) 2020-21 Q2: Fix $n \geq 2$. Define an equivalence relation on the disc $B^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ by $z_0 \sim z_1$ iff $z_0 = z_1$, or ($|z_0| = |z_1| = 1$ and $z_0^n = z_1^n$). Put $X = B^2/\sim$ and

$$Y = \{(u,v) \in \mathbb{C}^2 \mid |u| \leq 1, \quad v^n = (1-|u|)^n u\}.$$

Note that when $n = 2$ we just have $X = \mathbb{R}P^2$; this should guide your thinking about the general case.

(a) Show carefully that there is a homeomorphism $f \colon X \to Y$ such that $f([z]) = (z^n, (1-|z|^n)z)$ for all $z \in B^2$. You should prove in particular that f is well-defined, injective and surjective, and that both f and f^{-1} are continuous. You may assume that polynomials and the absolute value function are continuous, but beyond that you should not assume any properties of the given formula without proof. **(13 marks)**

(b) For the boundary $S^1 \subset B^2$, explain briefly why $S^1/\sim$ is homeomorphic to S^1 again. **(3 marks)**

(c) By adapting the method used for $\mathbb{R}P^2$, calculate $H_*(X)$. **(14 marks)**