



University of Sheffield

MAS61015

SCHOOL OF MATHEMATICAL AND PHYSICAL SCIENCES

**Spring Semester
2024–2025**

MAS61015 Algebraic Topology

2 hours 30 minutes

Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.

[This copy also contains solutions.](#)

No auxiliary material is provided.

MAS61015

Blank

- 1 (a) Define what it means to say that two spaces are *homeomorphic*. (3 marks)
- (b) Define what it means to say that two spaces are *homotopy equivalent*. (4 marks)
- (c) One of the conditions *homeomorphic* and *homotopy equivalent* implies the other. Prove this, and give a counterexample for the converse, with brief justification. (6 marks)
- (d) Consider the following spaces:
- $$X_0 = \{(w, x, y, z) \in \mathbb{R}^4 \mid w^2 + x^2 + y^2 + z^2 = 1\}$$
- $$X_1 = \{(w, x, y, z) \in \mathbb{R}^4 \mid w^2 + x^2 = y^2 + z^2 = 1\}$$
- $$X_2 = \{(w, x, y, z) \in \mathbb{R}^4 \mid w^2 = x^2 = y^2 = z^2 = 1\}$$
- $$X_3 = \{(w, x, y, z) \in \mathbb{R}^4 \mid w^2 + x^2 + y^2 + z^2 \leq 1\}$$
- $$X_4 = \{(w, x, y, z) \in \mathbb{R}^4 \mid w^2 + x^2 + y^2 + z^2 \geq 1\}.$$
- (i) For which pairs (i, j) is X_i homotopy equivalent to X_j ? Justify your answer briefly. In cases where X_i is homotopy equivalent to X_j you should explain why, and in cases where X_i is not homotopy equivalent to X_j , you should explain that as well. (8 marks)
- (ii) For which pairs (i, j) is X_i homeomorphic to X_j ? Justify your answer briefly. In cases where X_i is homeomorphic to X_j you should explain why, and in cases where X_i is not homeomorphic to X_j , you should explain that as well. (4 marks)

Solution.

- (a) **Bookwork** Let X and Y be topological spaces. A *homeomorphism* from X to Y is a bijective map $f : X \rightarrow Y$ such that both f and the inverse map $f^{-1} : Y \rightarrow X$ are continuous [2]. We say that X and Y are *homeomorphic* if there exists such a homeomorphism [1].
- (b) **Bookwork** Again let X and Y be topological spaces. Given continuous maps $f_0, f_1 : X \rightarrow Y$, a *homotopy* from f_0 to f_1 is a continuous map $h : [0, 1] \times X \rightarrow Y$ with $h(0, x) = f_0(x)$ and $h(1, x) = f_1(x)$ for all $x \in X$. We say that f_0 and f_1 are *homotopic* if there exists such a homotopy [2]. We say that X and Y are *homotopy equivalent* if there exist maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that gf is homotopic to id_X and fg is homotopic to id_Y [2].
- (c) **Bookwork** Claim: if spaces X and Y are homeomorphic, then they are homotopy equivalent [1]. Indeed, if X and Y are homeomorphic, then there exists a homeomorphism $f : X \rightarrow Y$ [1]. The map $g = f^{-1} : Y \rightarrow X$ then satisfies $gf = \text{id}_X$, so gf is homotopic to id_X by a constant homotopy [1]. Similarly fg is homotopic to id_Y by a constant homotopy, so f and g give a homotopy equivalence from X to Y [1]. However, the spaces $\{0\}$ and $\mathbb{R}$ are homotopy equivalent but not homeomorphic [1]. To give the details: we can define maps

$$\mathbb{R} \xrightarrow{f} \{0\} \xrightarrow{g} \mathbb{R} \xleftarrow{h} [0, 1] \times \mathbb{R}$$

1 (continued)

by $f(x) = 0$ and $g(0) = 0$ and $h(t, x) = tx$. Then $gf = \text{id}$ and h gives a homotopy from fg to id , so we have a homotopy equivalence; but clearly no map $\{0\} \rightarrow \mathbb{R}$ can be bijective, so $\{0\}$ is not homeomorphic to $\mathbb{R}$. [1] (Much less than this is required for a mark)

- (d) (i) **Similar examples have been seen, but this is a bit harder than most of them.**

The spaces X_0 and X_4 are homotopy equivalent, but there are no other coincidences. Indeed:

- * X_0 is the sphere S^3 with homology $(\mathbb{Z}, 0, 0, \mathbb{Z}, 0, \dots)$
- * X_1 is the torus $S^1 \times S^1$ with homology $(\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}, 0, \dots)$
- * X_2 is a finite discrete set of size 16 with homology $(\mathbb{Z}^{16}, 0, 0, \dots)$
- * X_3 is the closed ball B^4 and so is contractible, with contracting homotopy $h: [0, 1] \times X_4 \rightarrow X_4$ given by $h(t, x) = tx$. Thus, the homology is $(\mathbb{Z}, 0, 0, \dots)$.
- * We have an inclusion $i: X_0 \rightarrow X_4$ given by $i(u) = u$, and a retraction $r: X_4 \rightarrow X_0$ given by $r(v) = v/\|v\|$. These satisfy $r \circ i = \text{id}_{X_0}$. Next, we can define $h: [0, 1] \times X_4 \rightarrow \mathbb{R}^4$ by

$$h(t, v) = ((1-t) + t\|v\|^{-1})v.$$

This has $\|h(t, v)\| = (1-t)\|v\| + t$ and $\|v\| \geq 1$ as $v \in X_4$ so $\|h(t, v)\| = 1 + (1-t)(\|v\| - 1) \geq 1$. Thus, we actually have a map $h: [0, 1] \times X_4 \rightarrow X_4$ which is a homotopy from id_{X_4} to $i \circ r$. Thus, X_0 and X_4 are homotopy equivalent as claimed.

As homotopy equivalent spaces have isomorphic homology, it follows that there are no further coincidences. [8]

- (ii) **Similar examples have been seen.**

None of the given spaces is homeomorphic to any of the others. Indeed, homeomorphism would imply homotopy equivalence (by part (c)), and the only pair that are homotopy equivalent are X_0 and X_4 , so the only question is whether X_0 is actually homeomorphic to X_4 . The space X_0 is bounded and closed in $\mathbb{R}^4$ and so is compact. The space X_4 is unbounded in $\mathbb{R}^4$ and so is not compact and so cannot be homeomorphic to X_0 . [4]

- 2 (a) Let X and Y be topological spaces, and let $f, g: X \rightarrow Y$ be continuous maps. We might wish to define $h(t, x) = (1-t)f(x) + t g(x)$. Under what conditions is this a valid definition of a homotopy from f to g ? (2 marks)
- (b) Let X be a topological space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X / \sim$, and prove that it is indeed an equivalence relation. (8 marks)
- (c) You can assume that any continuous map $f: X \rightarrow Y$ induces a well-defined map $f_*: \pi_0(X) \rightarrow \pi_0(Y)$ with $f_*([x]) = [f(x)]$ for all $x \in X$. Show that if $f, g: X \rightarrow Y$ are continuous maps and f is homotopic to g , then $f_* = g_*: \pi_0(X) \rightarrow \pi_0(Y)$. (5 marks)
- (d) Give an example of spaces X and Y , and continuous maps $f, g: X \rightarrow Y$, where $f_* = g_*: \pi_0(X) \rightarrow \pi_0(Y)$ but f and g are not homotopic. (5 marks)
- (e) Put $X = \{z \in \mathbb{C} \mid |z| \notin \mathbb{Z}\}$ and $Y = S^2 \times S^3$. Show that there do not exist maps $X \xrightarrow{f} Y \xrightarrow{g} X$ such that $g \circ f$ is homotopic to the identity. (5 marks)

Solution.

- (a) **This point has frequently been ranted about** For this definition to be valid:
- (a) Y must be a subset of $\mathbb{R}^N$ or some other vector space, so that the expression $(1-t)f(x) + t g(x)$ is meaningful.
- (b) For each $x \in X$, the line segment from $f(x)$ to $g(x)$ must be wholly contained in Y . [2]
- (b) **Bookwork** Write $x \sim y$ iff there is a path in X from x to y , or in other words a continuous map $u: I \rightarrow X$ such that $u(0) = x$ and $u(1) = y$ [2]. I claim that this is an equivalence relation. Indeed, given $x \in X$ we can define $c_x: I \rightarrow X$ by $c_x(t) = x$ for all t . This gives a path from x to itself, showing that $\sim$ is reflexive [1]. Next, suppose that $x \sim y$, so there exists a path u from x to y in X . We can then define $\bar{u}(t) = u(1-t)$ to get a path from y to x , showing that $y \sim x$, showing that $\sim$ is symmetric [2]. Finally, suppose we have a path u from x to y , and a path v from y to z . We then define a map $w: I \rightarrow X$ by

$$w(t) = \begin{cases} u(2t) & \text{if } 0 \leq t \leq 1/2 \\ v(2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases} [2]$$

This is well-defined and continuous because $u(1) = y = v(0)$. We have $w(0) = u(0) = x$ and $w(1) = v(1) = z$, so w gives a path from x to z ; this proves that $\sim$ is transitive [1].

- (c) **Bookwork** Let $f, g: X \rightarrow Y$ be continuous maps which are homotopic. We can then choose a continuous map $h: [0, 1] \times X \rightarrow Y$ as in (b). For each $x \in X$ we then have $f_*([x]) = [f(x)]$ and $g_*([x]) = [g(x)]$ in $\pi_0(Y)$. We can define a path $v_x: [0, 1] \rightarrow Y$ by $v_x(t) = h(t, x)$. This has $v_x(0) = f(x)$ and $v_x(1) = g(x)$, so $f(x) \sim g(x)$, so $[f(x)] = [g(x)]$, i.e. $f_*([x]) = g_*([x])$. This proves that $f_* = g_*$ as required [5].
- (d) **Unseen** Define $f, g: S^1 \rightarrow S^1$ by $f(z) = z$ and $g(z) = 1$. As $\pi_0(S^1)$ is a set of size one,

2 (continued)

it is automatic that $f_* = g_*: \pi_0(S^1) \rightarrow \pi_0(S^1)$. However, on the group $H_1(S^1) = \mathbb{Z}u_1$ we have $f_*(u_1) = u_1$ and $g_*(u_1) = 0$ so f and g cannot be homotopic [5].

- (e) **Similar examples seen** Suppose (for a contradiction) that we have maps $X \xrightarrow{f} Y \xrightarrow{g} X$ as described. As $g \circ f$ is homotopic to the identity, we see that the composite

$$\pi_0(X) \xrightarrow{f_*} \pi_0(Y) \xrightarrow{g_*} \pi_0(X)$$

is the same as $(g \circ f)_*$ and so is the identity. This in particular means that f_* is injective. However, the space Y is path connected and X is not, so $|\pi_0(X)| > 1$ and $|\pi_0(Y)| = 1$, so it is impossible for f_* to be injective [5].

- 3 (a) Define what is meant by a *chain complex*, a *chain map*, and the *homology* of a chain complex. (6 marks)
- (b) Let $f: U_* \rightarrow V_*$ be a chain map between chain complexes. Define the resulting map between homology groups, and prove that what you have written is a valid, unambiguous definition. (6 marks)
- (c) Let A and B be finite abelian groups such that $|A|$ and $|B|$ are coprime, and let $\alpha: A \rightarrow B$ be a homomorphism. By considering $|\ker(\alpha)|$ and $|\text{img}(\alpha)|$ show that $\alpha = 0$. (5 marks)
- (d) Suppose we have a short exact sequence $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ of chain complexes and chain maps, such that
- $U_k = 0$ when k is odd, and $U_k = \mathbb{Z}/100$ when k is even.
 - $W_k = 0$ when k is even, and $W_k = \mathbb{Z}/99$ when k is odd.
- Find the homology groups of U_* , V_* and W_* . (8 marks)

Solution.

(a) **Bookwork**

- (1) A *chain complex* is a sequence of abelian groups U_n (for $n \in \mathbb{Z}$) together with homomorphisms $d_n: U_n \rightarrow U_{n-1}$ such that $d_n \circ d_{n+1} = 0$ for all n . [2]
- (2) Let U_* and V_* be chain complexes. A *chain map* from U_* to V_* is a sequence of homomorphisms $f_n: U_n \rightarrow V_n$ such that $d_n \circ f_n = f_{n-1} \circ d_n: U_n \rightarrow V_{n-1}$ for all n . [2]
- (3) For a chain complex U_* we define the *homology group* $H_n(U)$ to be

$$H_n(U) = \ker(d_n: U_n \rightarrow U_{n-1}) / \text{img}(d_{n+1}: U_{n+1} \rightarrow U_n).$$

(This is meaningful, because the condition $d_n \circ d_{n+1} = 0$ ensures that $\text{img}(d_{n+1}) \leq \ker(d_n)$.) [2]

3 (continued)

- (b) **Bookwork** Suppose we have a chain maps $f: U_* \rightarrow V_*$. Consider an element $a \in H_n(U)$. This can be represented as $a = [z] = z + \text{img}(d_{n+1}^U)$ for some $z \in \ker(d_n^U)$, so $d_n^U(z) = 0$. As f is a chain map we have $d_n^V(f(z)) = f(d_n^U(z)) = f(0) = 0$, so $f(z) \in \ker(d_n^V)$, so we have an element $[f(z)] \in H_n(V)$ [2]. Any other choice of z has the form $z' = z + d_{n+1}^U(x)$ for some $x \in U_{n+1}$, which gives $f(z') = f(z) + f(d_{n+1}^U(x)) = f(z) + d_{n+1}^V(f(x))$, showing that $[f(z')] = [f(z)]$ in $H_n(V)$ [2]. We therefore have a well-defined map $f_*: H_n(U) \rightarrow H_n(V)$ given by $f_*(a) = [f(z)]$ for any choice of z representing a [2].
- (c) **Unseen** Let A and B be finite abelian groups such that $|A|$ and $|B|$ are coprime, and let $\alpha: A \rightarrow B$ be a homomorphism. Then $\text{img}(\alpha)$ is a subgroup of B , so $|\text{img}(\alpha)|$ divides $|B|$ [2]. Similarly $|\ker(\alpha)|$ divides $|A|$ and $|\text{img}(\alpha)| = |A|/|\ker(\alpha)|$ by the First Isomorphism Theorem, so $|\text{img}(\alpha)|$ also divides $|A|$ [2]. As $|A|$ and $|B|$ are coprime, it follows that $|\text{img}(\alpha)| = 1$, so $\text{img}(\alpha)$ is the trivial group, so $\alpha = 0$ [1].
- (e) **Partially similar examples have been seen** Suppose we have a short exact sequence $U_* \rightarrow V_* \rightarrow W_*$ as described. We claim that

$$\begin{aligned} H_i(U) &= \begin{cases} \mathbb{Z}/100 & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases} \\ H_i(V) &= \begin{cases} \mathbb{Z}/100 & \text{if } i \text{ is even} \\ \mathbb{Z}/99 & \text{if } i \text{ is odd} \end{cases} \\ H_i(W) &= \begin{cases} 0 & \text{if } i \text{ is even} \\ \mathbb{Z}/99 & \text{if } i \text{ is odd} \end{cases} . \end{aligned}$$

Indeed, for each differential $d_i: U_i \rightarrow U_{i-1}$ either i or $i-1$ is odd so either U_i or U_{i-1} is zero so $d_i = 0$. As all differentials are zero we have $H_i(U) = U_i$ which agrees with the above table. Essentially the same argument gives $H_i(W) = W_i$ [3]. Next, the Snake Lemma gives exact sequences

$$H_{2k+1}U \xrightarrow{i_*} H_{2k+1}V \xrightarrow{p_*} H_{2k+1}(W) \xrightarrow{\delta} H_{2k}(U) \xrightarrow{i_*} H_{2k}V \xrightarrow{p_*} H_{2k}W. [2]$$

After filling in the known groups, this becomes

$$0 \rightarrow H_{2k+1}V \xrightarrow{p_*} \mathbb{Z}/99 \xrightarrow{\delta} \mathbb{Z}/100 \xrightarrow{i_*} H_{2k}V \rightarrow 0. [1]$$

Part (d) shows that $\delta = 0$. Exactness then implies that the above maps p_* and i_* are isomorphisms, so H_*V is as claimed [2].

4 For each of the following, either give an example (with justification) or prove that no example can exist. You can use results that are proved in the notes provided that you state them clearly.

(a) A space X where $|H_1(X)| = 4$. (5 marks)

(b) A contractible space X and a continuous map $f : X \rightarrow \{0, 1\}$ which is not constant. (5 marks)

(c) A pair of bounded, closed subsets $X, Y \subseteq \mathbb{R}^{10}$ and a continuous map $f : X \rightarrow Y$ such that the inverse map $f^{-1} : Y \rightarrow X$ exists but is not continuous. (5 marks)

(d) Spaces X and Y with $H_1(X) \simeq \mathbb{Z}$ and $H_1(Y) \simeq \mathbb{Z}^2$, and maps $X \xrightarrow{f} Y \xrightarrow{g} X$ such that $g \circ f$ is homotopic to the identity. (5 marks)

(e) Spaces X and Y with $H_1(X) \simeq \mathbb{Z}^2$ and $H_1(Y) \simeq \mathbb{Z}$, and maps $X \xrightarrow{f} Y \xrightarrow{g} X$ such that $g \circ f$ is homotopic to the identity. (5 marks)

Solution. Similar examples have been seen

(a) Take $X = \mathbb{R}P^2 \amalg \mathbb{R}P^2$. It is a standard calculation that $H_1(\mathbb{R}P^2) \simeq \mathbb{Z}/2$ and a standard fact that $H_1(Y \amalg Z) = H_1(Y) \oplus H_1(Z)$ so $H_1(X) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $|H_1(X)| = 4$. [5]

(b) This is not possible. Suppose that X is contractible and that $f : X \rightarrow \{0, 1\}$ is continuous. Then X is path-connected (by a theorem in the notes), so the image $f(X)$ is also path connected (by another theorem in the notes). The only path-connected subsets of $\{0, 1\}$ are singletons, so f is constant. [5]

(c) This is not possible. As X and Y are assumed to be bounded and closed in $\mathbb{R}^{10}$, they are both compact and Hausdorff. As f^{-1} is assumed to exist, the map f must be a bijection, and it is also assumed to be continuous. A theorem in the notes says that any continuous bijection from a compact space to a Hausdorff space is a homeomorphism, so the inverse is continuous. [5]

(d) We can take $X = S^1$ and $Y = S^1 \times S^1$ and $f(x) = (x, x)$ and $g(x, y) = x$. Here $g \circ f$ is equal to the identity and so is certainly homotopic to the identity. [5]

(e) This is not possible. If we had spaces and maps as specified then we would have homomorphism

$$H_1(X) = \mathbb{Z}^2 \xrightarrow{f_*} H_1(Y) = \mathbb{Z} \xrightarrow{g_*} H_1(X) = \mathbb{Z}^2$$

with $g_* \circ f_* = \text{id}$. This would imply that $f_* : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ was injective, and that $g_* : \mathbb{Z} \rightarrow \mathbb{Z}^2$ was surjective, but both of these are impossible. [5]

- 5 (a) Describe the homology groups $H_*(S^n)$ for all n , without any proofs but with careful attention to any special cases. (4 marks)
- (b) Suppose that $n, m \geq 0$ and that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$. Prove that $n = m$. You should again pay careful attention to any special cases. (6 marks)
- (c) State and prove the Brouwer Fixed Point Theorem. Your proof will probably involve the construction of a certain auxiliary map. You need to give a clear and correct definition of this map, but you need not prove that it is continuous. In this part of the question, you can also ignore low-dimensional special cases. (7 marks)
- (d) Suppose that X is homeomorphic to B^n for some n , and that $f: X \rightarrow X$ is continuous. Deduce that there is a point $x \in X$ with $f(x) = x$. (4 marks)
- (e) Give examples of
- (i) A compact path-connected space Y and a continuous map $p: Y \rightarrow Y$ with no fixed points. (2 marks)
 - (ii) A contractible space Z and a continuous map $q: Z \rightarrow Z$ with no fixed points. (2 marks)

Solution.

- (a) **Bookwork** For $n = 0$ we have $H_0(S^0) \simeq \mathbb{Z}^2$ and $H_i(S^0) = 0$ for $i \neq 0$ [2]. For $n > 0$ we have $H_i(S^n) \simeq \mathbb{Z}$ for $i \in \{0, n\}$ and $H_i(S^n) = 0$ for $i \notin \{0, n\}$ [2].

- (b) **Bookwork** Suppose we have a homeomorphism $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$. We can then define $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $g(x) = f(x) - f(0)$, and note that this is again a homeomorphism with $g^{-1}(y) = f^{-1}(y + f(0))$. We have $g(0) = 0$ so g restricts to give a homeomorphism $\mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^m \setminus \{0\}$ [1].

If $n = 0$ then $\mathbb{R}^n \setminus \{0\}$ is empty so we have a homeomorphism $g: \emptyset \rightarrow \mathbb{R}^m \setminus \{0\}$ so $\mathbb{R}^m \setminus \{0\}$ is empty so $m = 0 = n$. By applying the same logic to g^{-1} we see that if $m = 0$ then $n = 0$. We can therefore assume that $n, m > 0$ for the rest of the proof [2].

It is now standard that $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^m \setminus \{0\}$ are homotopy equivalent to S^{n-1} and S^{m-1} , so we deduce that S^{n-1} is homotopy equivalent to S^{m-1} [1], so $H_k(S^{n-1}) \simeq H_k(S^{m-1})$ for all k [1]. The largest k where $H_k(S^{n-1}) \neq 0$ is $n-1$, and the largest k where $H_k(S^{m-1}) \neq 0$ is $m-1$, so we must have $n-1 = m-1$ and so $n = m$ [1].

- (c) **Bookwork** The Brouwer Fixed Point Theorem states that if $f: B^n \rightarrow B^n$ is continuous then there exists $x \in B^n$ with $f(x) = x$ [2]. To prove this, we suppose (for a contradiction) that we have a continuous map $f: B^n \rightarrow B^n$ with $f(x) \neq x$ for all x [1]. For each $x \in B^n$ we can then draw a line segment from $f(x)$ to x and extend it until it meets the boundary S^{n-1} at a point $r(x) \in S^{n-1}$ say. This defines a map $r: B^n \rightarrow S^{n-1}$, and one can check that this is continuous [2]. It is clear by construction that if x itself lies in S^{n-1} then $r(x) = x$. In other words, if we let $i: S^{n-1} \rightarrow B^n$ denote the inclusion, we have $r \circ i = \text{id}$ [1]. This means that the composite

$$\mathbb{Z} = H_{n-1}(S^{n-1}) \xrightarrow{i_*} 0 = H_{n-1}(B^n) \xrightarrow{r_*} H_{n-1}(S^{n-1}) = \mathbb{Z}$$

5 (continued)

is the identity, which is clearly impossible. (This needs minor modification for the cases $n = 0$ and $n = 1$, but we will not worry about that.) Thus, no such map f can exist [1].

- (d) **Unseen** Suppose we have a homeomorphism $p: B^n \rightarrow X$. Given a continuous map $f: X \rightarrow X$, we have a continuous map $g = p^{-1}f p: B^n \rightarrow B^n$. By the Brouwer Fixed Point Theorem, there is a point $u \in B^n$ with $g(u) = u$, or equivalently $p^{-1}(f(p(u))) = u$, or equivalently $f(p(u)) = p(u)$. Thus, the point $x = p(u) \in X$ has $f(x) = x$ as required. [4]

(e) **Seen**

- (i) The space $Y = S^1$ is compact and path connected, and the map $p: S^1 \rightarrow S^1$ given by $p(z) = -z$ has no fixed points. [2]
- (ii) The space $Z = \mathbb{R}$ is contractible, and the map $q: \mathbb{R} \rightarrow \mathbb{R}$ given by $q(t) = t + 1$ has no fixed points. [2]

End of Question Paper

Total question marks: 125