SCHOOL OF MATHEMATICAL AND PHYSICAL SCIENCES **Spring Semester 2024–2025**

MAS61015 Algebraic Topology **2 hours 30 minutes**

Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.

No auxiliary material is provided.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (a) Define what it means to say that two spaces are *homeomorphic*. ***(3 marks)***

(b) Define what it means to say that two spaces are *homotopy equivalent*. ***(4 marks)***

(c) One of the conditions *homeomorphic* and *homotopy equivalent* implies the other. Prove this, and give a counterexample for the converse, with brief justification. ***(6 marks)***

(d) Consider the following spaces:

$$
\begin{aligned}
X_0 &= \{(w,x,y,z) \in \mathbb{R}^4 \mid w^2+x^2+y^2+z^2 = 1\} \\
X_1 &= \{(w,x,y,z) \in \mathbb{R}^4 \mid w^2+x^2 = y^2+z^2 = 1\} \\
X_2 &= \{(w,x,y,z) \in \mathbb{R}^4 \mid w^2 = x^2 = y^2 = z^2 = 1\} \\
X_3 &= \{(w,x,y,z) \in \mathbb{R}^4 \mid w^2+x^2+y^2+z^2 \leq 1\} \\
X_4 &= \{(w,x,y,z) \in \mathbb{R}^4 \mid w^2+x^2+y^2+z^2 \geq 1\}.
\end{aligned}
$$

(i) For which pairs (i,j) is X_i homotopy equivalent to X_j? Justify your answer briefly. In cases where X_i is homotopy equivalent to X_j you should explain why, and in cases where X_i is not homotopy equivalent to X_j, you should explain that as well. ***(8 marks)***

(ii) For which pairs (i,j) is X_i homeomorphic to X_j? Justify your answer briefly. In cases where X_i is homeomorphic to X_j you should explain why, and in cases where X_i is not homeomorphic to X_j, you should explain that as well. ***(4 marks)***

2 (a) Let X and Y be topological spaces, and let $f, g\colon X \to Y$ be continuous maps. We might wish to define $h(t,x) = (1-t)f(x) + t\,g(x)$. Under what conditions is this a valid definition of a homotopy from f to g? ***(2 marks)***

(b) Let X be a topological space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is indeed an equivalence relation. ***(8 marks)***

(c) You can assume that any continuous map $f\colon X \to Y$ induces a well-defined map $f_*\colon \pi_0(X) \to \pi_0(Y)$ with $f_*([x]) = [f(x)]$ for all $x \in X$. Show that if $f, g\colon X \to Y$ are continuous maps and f is homotopic to g, then $f_* = g_*\colon \pi_0(X) \to \pi_0(Y)$. ***(5 marks)***

(d) Give an example of spaces X and Y, and continuous maps $f, g\colon X \to Y$, where $f_* = g_*\colon \pi_0(X) \to \pi_0(Y)$ but f and g are not homotopic. ***(5 marks)***

(e) Put $X = \{z \in \mathbb{C} \mid |z| \notin \mathbb{Z}\}$ and $Y = S^2 \times S^3$. Show that there do not exist maps $X \xrightarrow{f} Y \xrightarrow{g} X$ such that $g \circ f$ is homotopic to the identity. ***(5 marks)***

3 (a) Define what is meant by a *chain complex*, a *chain map*, and the *homology* of a chain complex. ***(6 marks)***

(b) Let $f\colon U_* \to V_*$ be a chain map between chain complexes. Define the resulting map between homology groups, and prove that what you have written is a valid, unambiguous definition. ***(6 marks)***

(c) Let A and B be finite abelian groups such that $|A|$ and $|B|$ are coprime, and let $\alpha\colon A \to B$ be a homomorphism. By considering $|\ker(\alpha)|$ and $|\mathrm{img}(\alpha)|$ show that $\alpha = 0$. ***(5 marks)***

(d) Suppose we have a short exact sequence $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ of chain complexes and chain maps, such that

- $U_k = 0$ when k is odd, and $U_k = \mathbb{Z}/100$ when k is even.
- $W_k = 0$ when k is even, and $W_k = \mathbb{Z}/99$ when k is odd.

Find the homology groups of U_*, V_* and W_*. ***(8 marks)***

4 For each of the following, either give an example (with justification) or prove that no example can exist. You can use results that are proved in the notes provided that you state them clearly.

(a) A space X where $|H_1(X)| = 4$. ***(5 marks)***

(b) A contractible space X and a continuous map $f : X \to \{0, 1\}$ which is not constant. ***(5 marks)***

(c) A pair of bounded, closed subsets $X, Y \subseteq \mathbb{R}^{10}$ and a continuous map $f : X \to Y$ such that the inverse map $f^{-1}: Y \to X$ exists but is not continuous. ***(5 marks)***

(d) Spaces X and Y with $H_1(X) \simeq \mathbb{Z}$ and $H_1(Y) \simeq \mathbb{Z}^2$, and maps $X \xrightarrow{f} Y \xrightarrow{g} X$ such that $g \circ f$ is homotopic to the identity. ***(5 marks)***

(e) Spaces X and Y with $H_1(X) \simeq \mathbb{Z}^2$ and $H_1(Y) \simeq \mathbb{Z}$, and maps $X \xrightarrow{f} Y \xrightarrow{g} X$ such that $g \circ f$ is homotopic to the identity. ***(5 marks)***

5 (a) Describe the homology groups $H_*(S^n)$ for all n, without any proofs but with careful attention to any special cases. ***(4 marks)***

(b) Suppose that $n, m \geq 0$ and that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$. Prove that $n = m$. You should again pay careful attention to any special cases. ***(6 marks)***

(c) State and prove the Brouwer Fixed Point Theorem. Your proof will probably involve the construction of a certain auxiliary map. You need to give a clear and correct definition of this map, but you need not prove that it is continuous. In this part of the question, you can also ignore low-dimensional special cases. ***(7 marks)***

(d) Suppose that X is homeomorphic to B^n for some n, and that $f : X \to X$ is continuous. Deduce that there is a point $x \in X$ with $f(x) = x$. ***(4 marks)***

(e) Give examples of

(i) A compact path-connected space Y and a continuous map $p: Y \to Y$ with no fixed points. ***(2 marks)***

(ii) A contractible space Z and a continuous map $q: Z \to Z$ with no fixed points. ***(2 marks)***

End of Question Paper