Algebraic Topology

(1)

(a) What does it mean to say that a topological space X is *Hausdorff*?
(If your definition relies on any other concepts, then you should define them.) **(3 marks)**

(b) What does it mean to say that a topological space X is *compact*?
(If your definition relies on any other concepts, then you should define them.) **(3 marks)**

(c) Put $X = \{(x, y, z) \in \mathbb{R}^3 : x^4 + y^4 + z^4 = 1\}$. Prove that X is compact. You may use general theorems provided that you state them precisely. **(5 marks)**

(d) Put $Y = \{(x, y, z) \in \mathbb{R}^3 : x^4 + y^4 + z^4 < 1\}$. Prove that Y is not compact. Here you should argue directly from the definitions and not use any theorems. **(5 marks)**

(e) Let Y and Z be two compact subspaces of a topological space X. Prove that $Y \cup Z$ is also compact. **(4 marks)**

(f) Let Y and Z be topological spaces such that $Z \neq \emptyset$ and $Y \times Z$ is compact. Prove that Y is compact. You may use standard results so long as you state them clearly and verify carefully that they are applicable. **(5 marks)**

Solution:

(a) **Bookwork** Let X be a topological space. Given $a, b \in X$ with $a \neq b$, a *Hausdorff separation* for (a, b) is a pair of open sets $U, V \subseteq X$ with $a \in U$ and $b \in V$ and $U \cap V = \emptyset$[2]. We say that X is *Hausdorff* if every pair of distinct points has a Hausdorff separation [1].

(b) **Bookwork** Let X be a topological space. An *open cover* of X is a family $(U_i)_{i \in I}$ of open sets whose union is all of X [1]. Given such a cover, a *finite subcover* is a subfamily $(U_i)_{i \in J}$ where $J \subseteq I$ is finite and the union is still all of X [1]. We say that X is *compact* if every open cover has a finite subcover [1].

(c) **Similar examples seen** If $(x, y, z) \in X$ then $x^4 \leq x^4 + y^4 + z^4 = 1$ so $|x| \leq 1$. Similarly, we see that $|y| \leq 1$ and $|z| \leq 1$, which implies that X is bounded [2]. Also, we can define $f : \mathbb{R}^4 \to \mathbb{R}$ by $f(x, y, z) = x^4 + y^4 + z^4$. This is continuous (because it is polynomial) and $\{1\}$ is closed in $\mathbb{R}$ so the set $X = f^{-1}\{1\}$ is closed in $\mathbb{R}^4$ [2]. Any bounded closed subset of $\mathbb{R}^n$ is compact, so we deduce that X is compact as claimed [1].

(d) **Similar examples seen** For $n > 0$ put $U_n = \{(x, y, z) \in \mathbb{R}^3 \mid x^4 + y^4 + z^4 < 1 - n^{-1}\}$, so these sets are form an open cover of Y [2]. However, U_n is not all of Y, because the point $((1 - 1/(n + 1))^{1/4}, 0, 0)$ lies in $Y \setminus U_n$ [1]. If Y was compact then we would have a finite subcover, say $Y = U_{n_1} \cup \cdots \cup U_{n_p}$ and this would give $Y = U_n$ where $n = \max(n_1, \ldots, n_p)$, which is a contradiction; so Y is not compact. [2]

(e) Suppose that Y and Z are compact subsets of X; we claim that $Y \cup Z$ is also compact, To see this, let $(U_i)_{i \in I}$ be a family of open subsets of X that covers $Y \cup Z$; we must show that there is a finite subcover [1]. As the family covers $Y \cup Z$, it certainly covers Y, and Y is compact, so we can choose indices $i_1, \ldots, i_p$ with $Y \subseteq U_{i_1} \cup \cdots \cup U_{i_p}$ [1]. Similarly, we can choose indices $i_{p+1}, \ldots, i_{p+q}$ such that $Z \subseteq U_{i_{p+1}} \cup \cdots \cup U_{i_{p+q}}$. It follows that $Y \cup Z \subseteq U_{i_1} \cup \cdots \cup U_{i_{p+q}}$, so we have the required finite subcover [2].

(f) Let Y and Z be topological spaces such that $Z \neq \emptyset$ (so we can choose $z_0 \in Z$). Suppose that $Y \times Z$ is compact; we claim that Y is also compact. Because $\pi(y, z_0) = y$, we see that the projection $Y \times Z \to Y$ is surjective (and also continuous, by the definition of the product topology) [2]. A standard theorem says that if $f : A \to B$ is continuous and surjective and A is compact then B is also compact [2]. Using this, we see that Y is compact as claimed [1]. (It is also not hard to prove this directly by consideration of open covers.)

(2)

(a) Let X be a topological space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is indeed an equivalence relation. **(8 marks)**

(b) Let $f\colon X \to Y$ be a continuous map. Define the function $f_*\colon \pi_0(X) \to \pi_0(Y)$, and check that it is well-defined. **(5 marks)**

(c) Suppose that Y is path-connected and X is not. Show that there do not exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to the identity map id_X. **(6 marks)**

(d) Put $X = \{A \in M_2\mathbb{R} \mid A^2 = A\}$ (where $M_2\mathbb{R}$ is the space of 2×2 real matrices). What can you say about $\det(A)$ when $A \in X$? Show that X is not path-connected. **(6 marks)**

Solution:

(a) **Bookwork** Write $x \sim y$ iff there is a path in X from x to y [1], or in other words a continuous map $u\colon I \to X$ such that $u(0) = x$ and $u(1) = y$ [1]. I claim that this is an equivalence relation. Indeed, given $x \in X$ we can define $c_x\colon I \to X$ by $c_x(t) = x$ for all t. This gives a path from x to itself, showing that $\sim$ is reflexive [1]. Next, suppose that $x \sim y$, so there exists a path u from x to y in X. We can then define $\overline{u}(t) = u(1-t)$ to get a path from y to x, showing that $y \sim x$, showing that $\sim$ is symmetric [2]. Finally, suppose we have a path u from x to y, and a path v from y to z. We then define a map $w\colon I \to X$ by

$$w(t) = \begin{cases} u(2t) & \text{if } 0 \leq t \leq 1/2 \\ v(2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases} \text{[2]}$$

This is well-defined and continuous because $u(1) = y = v(0)$. We have $w(0) = u(0) = x$ and $w(1) = v(1) = z$, so w gives a path from x to z; this proves that $\sim$ is transitive [1].

(b) **Bookwork** Let $f\colon X \to Y$ be a continuous map. We define $f_*\colon \pi_0(X) \to \pi_0(Y)$ by $f_*([x]) = [f(x)]$ [1](where $[x]$ is the equivalence class of x under the relation $\sim$). To see that this is well-defined, suppose that $[x_0] = [x_1]$ in $\pi_0(X)$ [1]. This means that $x_0 \sim x_1$, so there is a path $u\colon I \to X$ from x_0 to x_1 [1]. The function $f \circ u\colon I \to Y$ gives a path from $f(x_0)$ to $f(x_1)$ in Y [1], so $[f(x_0)] = [f(x_1)]$ as required [1].

(c) **Slightly disguised bookwork** Suppose that Y is path-connected, so $\pi_0(Y)$ has only a single element, which we will call b. Then $f_*\colon \pi_0(X) \to \pi_0(Y)$ must be the constant map with value b, so $g_*f_*\colon \pi_0(X) \to \pi_0(X)$ must be the constant map with value $g_*(b)$. On the other hand, if $gf \simeq 1$ then g_*f_* is the identity. Thus, the identity map of $\pi_0(X)$ is constant, so $\pi_0(X)$ can only have a single element. This means that X is path-connected, contrary to assumption. [6]

(d) **Similar examples seen** Put $X = \{A \in M_2\mathbb{R} \mid A^2 = A\}$. For $A \in X$ we have $\det(A)^2 = \det(A)$ so $\det(A) \in \{0, 1\}$ [2]. We can thus regard det as a continuous map $X \to \mathbb{R}$ such that $\det(A) \neq 1/2$ for all A. The zero matrix and the identity matrix lie in X, with $\det(0) = 0 < 1/2$ and $\det(I) = 1 > 1/2$. It follows that 0 cannot be connected to I by a path in X, so X is not path-connected. [4]

(3)

(a) Define the terms *chain map*, *chain homotopy*, *chain homotopic* and *chain homotopy equivalence*. **(8 marks)**

(b) Show that if $f, g\colon U_* \to V_*$ are chain maps that are chain homotopic to each other, then $f_* = g_*\colon H_*(U) \to H_*(V)$. **(5 marks)**

(c) Consider the chain complex T_* with $T_i = \mathbb{Z}^2$ for all i and $d_i(x, y) = (y, 0)$ for all $(x, y) \in T_i$. Show that T_* is chain homotopy equivalent to the zero complex. **(4 marks)**

(d) Suppose we have a short exact sequence $A_* \xrightarrow{i} B_* \xrightarrow{p} C_*$ of chain complexes and chain maps. Suppose that for all $k \in \mathbb{Z}$ we have $H_k(B) = 0$. Suppose also that $H_k(A) = \mathbb{Z}/2^k$ for $k \geq 0$ and $H_k(A) = 0$ for $k < 0$. Determine the homology groups of C_*. **(3 marks)**

(e) Let U_* be a chain complex in which $U_k = 0$ for $k < 0$ and $|U_k| = 2^k$ for $k \geq 0$ and $d_{2i}\colon U_{2i} \to U_{2i-1}$ is surjective for all i. Find the homology groups of U_*. **(5 marks)**

Solution:

(a) **Bookwork**

(1) Let U_* and V_* be chain complexes. A *chain map* from U_* to V_* is a sequence of homomorphisms $f_i\colon U_i \to V_i$ [1]such that $d_i \circ f_i = f_{i-1} \circ d_i\colon U_i \to V_{i-1}$ for all $i \in \mathbb{Z}$ (or more briefly, $df = fd$) [1].

(2) Let $f, g\colon U_* \to V_*$ be chain maps [1]. A *chain homotopy* between f and g is a sequence of homomorphisms $s_i\colon U_i \to V_{i+1}$ [1]with $ds + sd = g - f$ [1].

(3) We say that chain maps $f, g\colon U_* \to V_*$ are *chain homotopic* if there exists a chain homotopy as in (2). [1]

(4) A chain map $f\colon U_* \to V_*$ is a *chain homotopy equivalence* if there is a chain map $g\colon V_* \to U_*$ [1]such that $g \circ f\colon U_* \to U_*$ and $f \circ g\colon V_* \to V_*$ are chain homotopic to the corresponding identity maps [1].

(b) **Bookwork** Suppose we have chain maps $f, g\colon U_* \to V_*$ and a chain homotopy s as above. Consider an element $u \in H_n(U)$, so $u = [z]$ for some cycle $z \in U_n$ with $d(z) = 0$ [1]. As s is a chain homotopy from f to g, we have $g(z) - f(z) = d(s(z)) + s(d(z))$ [1]. As $d(z) = 0$ this becomes $g(z) - f(z) = d(s(z)) \in \operatorname{img}(d)_n = B_n(V)$ [1], so the cosets $[g(z)] = g(z) + B_n(V)$ and $[f(z)] = f(z) + B_n(V)$ are the same [1], or in other words $g_*(u) = f_*(u)$ as required [1].

(c) **Unseen** Let $i\colon 0 \to T_*$ and $r\colon T_* \to 0$ be the zero maps, so $r \circ i\colon 0 \to 0$ is the identity, and $i \circ r = 0\colon T_* \to T_*$. Define $s_k\colon T_k \to T_{k+1}$ by $s_k(x, y) = (0, x)$ [2]. Then

$$(ds + sd)(x, y) = d(0, x) + s(y, 0) = (x, 0) + (0, y) = (x, y),$$

so $ds + ds = 1 = 1 - 0 = 1 - i \circ r$, so $i \circ r$ is chain homotopic to the identity. This means that i and r are chain homotopy equivalences [2].

(d) **Unseen** Let $A_* \xrightarrow{i} B_* \xrightarrow{p} C_*$ be as described. The Snake Lemma then gives exact sequences

$$0 = H_k(B) \to H_k(C) \xrightarrow{\delta} H_{k-1}(A) \to H_{k-1}(B) = 0,$$

which means that the map δ is an isomorphism [2]. It follows that when $k > 0$ we have $H_k(C) \simeq H_{k-1}(A) \simeq \mathbb{Z}/2^{k-1}$ and when $k \leq 0$ we have $H_k(C) = 0$ [1].

(e) **Unseen** Let U_* be a chain complex as described. As $d_{2i}\colon U_{2i} \to U_{2i-1}$ is surjective, we see that $B_{2i-1}(U) = U_{2i-1}$. This means that every element $u \in U_{2i-1}$ can be expressed as $u = d(u')$ for some u', so $d(u) = d^2(u') = 0$. Thus, the homomorphism $d_{2i-1}\colon U_{2i-1} \to U_{2i-2}$ is zero [2]. We now have $Z_{2i-1}(U) = B_{2i-1}(U) = U_{2i-1}$, so $H_{2i-1}(U) = U_{2i-1}/U_{2i-1} = 0$. We also have $B_{2i}(U) = 0$ and so $H_{2i}(U) \simeq Z_{2i}(U) = \ker(d_{2i}\colon U_{2i} \to U_{2i-1})$ [1]. As d_{2i} is surjective with $|U_{2i}| = 2^{2i}$ and $|U_{2i-1}| = 2^{2i-1}$ we see that $|\ker(d_{2i})| = 2$ and so $\ker(d_{2i}) \simeq \mathbb{Z}/2$. In summary, we have

$$H_k(U) = \begin{cases} \mathbb{Z}/2 & \text{if } k \text{ is even and } k > 0 \\ 0 & \text{otherwise. [2]} \end{cases}$$

(4)

(a) Let X be a topological space.

(i) Define the groups $C_n(X)$ for all nonnegative integers n. **(2 marks)**

(ii) Define the homomorphisms $\partial_n : C_n(X) \to C_{n-1}(X)$. **(3 marks)**

(iii) Prove that $\partial_1 \circ \partial_2 = 0$. **(3 marks)**

(iv) Define the groups $H_n(X)$. **(4 marks)**

(b) Describe (without proof, but with careful attention to any special cases) the groups $H_n(\mathbb{R}^k \setminus \{0\})$ for all $n \geq 0$ and all $k \geq 1$. **(5 marks)**

(c) Let $u = n_1 s_1 + \ldots + n_k s_k$ be an element of $Z_m(S^n)$ (where $m > 0$), and suppose that there is a point $a \in S^n$ that is not contained in any of the sets $s_1(\Delta_m), \ldots, s_k(\Delta_m)$. Prove that u is a boundary. (You may assume standard results and calculations from the course so long as you state them carefully.) **(8 marks)**

Solution:

(a) (i) **Bookwork** The group $C_n(X)$ is the free Abelian group [1]generated by the set of continuous maps $s : \Delta_n \to X$ [1], where $\Delta_n = \{t \in \mathbb{R}^{n+1} \mid t_i \geq 0,\ \sum_i t_i = 1\}$.

(ii) **Bookwork** We define continuous maps $\delta_0, \ldots, \delta_n : \Delta_{n-1} \to \Delta_n$ by

$$\delta_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})[1].$$

For any continuous map $s : \Delta_n \to X$ we define

$$\partial_n(s) = \sum_{k=0}^{n} (-1)^k (s \circ \delta_k) \in C_{n-1}(X)[1].$$

This can be extended in a unique way to give a homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$. [1]

(iii) **Bookwork** From the definitions, we have

$$\begin{aligned}\partial_1\partial_2[s] &= \partial_1([s\delta_0] - [s\delta_1] + [s\delta_2]) \\ &= [s\delta_0\delta_0] - [s\delta_0\delta_1] - [s\delta_1\delta_0] + [s\delta_1\delta_1] + [s\delta_2\delta_0] - [s\delta_2\delta_1] \\ &= ([s\delta_0\delta_0] - [s\delta_1\delta_0]) - ([s\delta_0\delta_1] - [s\delta_2\delta_0]) + ([s\delta_1\delta_1] - [s\delta_2\delta_1]).[1]\end{aligned}$$

Whenever $k \leq l$ we have $\delta_k\delta_l = \delta_{l+1}\delta_k$; this shows that each of the bracketed terms is zero [1]. Thus $\partial_2\partial_1$ vanishes on all singular 2-simplices, so it vanishes on all singular 2-chains [1].

(iv) **Bookwork** We define $Z_n(X) = \ker(\partial_n : C_n(X) \to C_{n-1}(X))$ [1]and $B_n(X) = \operatorname{img}(\partial_{n+1} : C_{n+1}(X) \to C_n(X))$ [1]. We have $\partial_n\partial_{n+1} = 0$, which implies that $B_n(X) \leq Z_n(X)$ [1], so we can define a quotient group $H_n(X) = Z_n(X)/B_n(X)$ [1].

(b) **Bookwork** As $\mathbb{R}^k \setminus \{0\}$ is homotopy equivalent to S^{k-1}, we have

$$H_n(\mathbb{R}^k \setminus \{0\}) = \begin{cases} \mathbb{Z}^2 & \text{if } n = 0, k = 1[2] \\ \mathbb{Z} & \text{if } n = 0, k > 1[1] \text{ or } n = k - 1 > 0[1] \\ 0 & \text{otherwise } [1]. \end{cases}$$

(c) **Unseen** The space $S^n \setminus \{a\}$ [2] is homeomorphic to $\mathbb{R}^n$ [1]by stereographic projection, and thus is contractible [1]. This implies that $H_m(S^n \setminus \{a\}) = 0$ for $m > 0$ [1], so every m-cycle in $S^n \setminus \{a\}$ is a boundary [1]. We can regard u as an m-cycle in $S^m \setminus \{a\}$, so it is a boundary in $S^n \setminus \{a\}$ [1]and thus in S^n [1], as required.

(5) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems and calculations, provided that you state them clearly.

(a) S^3 is contractible. **(3 marks)**

(b) $\mathbb{R}P^3$ is a homotopy retract of S^3. **(3 marks)**

(c) If a space X is the union of two closed, path-connected subspaces A and B, then X is path-connected. **(3 marks)**

(d) $(\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1. **(4 marks)**

(e) $(\mathbb{R} \times \mathbb{R}^2) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1. **(4 marks)**

(f) The space $\mathbb{C} \setminus \{0, 1\}$ is homeomorphic to $\mathbb{C} \setminus \{i, -i\}$. **(4 marks)**

(g) The space $\mathbb{C} \setminus \{0, 1\}$ is homotopy equivalent to $\mathbb{C} \setminus \{0, 1, 2\}$. **(4 marks)**

Solution:

(a) False [1]. We have $H_3(S^3) = \mathbb{Z}$ but H_3 of a point is zero, so S^3 is not homotopy equivalent to a point [2].

(b) False [1]. If $\mathbb{R}P^3$ was a homotopy retract of S^3 then the group $H_1(\mathbb{R}P^3) = \mathbb{Z}/2$ would be isomorphic to a subgroup of the group $H_1(S^3) = 0$, which is clearly not true [2].

(c) False [1]. Put $X = \{0, 1\}$ and $A = \{0\}$ and $B = \{1\}$. Then A and B are closed path connected subsets of X with $X = A \cup B$, but X is not path connected [2]. (You would not be required to say this, but I remark that if $X = A \cup B$ where A and B are path connected (not necessarily closed) and $A \cap B \neq \emptyset$ then X is path connected.)

(d) False [1]. Write

$$X = (\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{R} \times \{0\}) = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}.$$

We can then define a map $f\colon X \to \mathbb{R}$ by $f(x, y) = y$. This is never zero and it is positive at $(0, 1)$ and negative at $(0, -1)$, so $(0, 1)$ cannot be joined to $(0, -1)$ by a path in X, so X is not path connected [1]. However, S^1 is path connected [1]and anything homotopy equivalent to a path connected space is again path connected so X is not homotopy equivalent to S^1 [1].

(e) True [1]. Write $Y = (\mathbb{R} \times \mathbb{R}^2) \setminus (\mathbb{R} \times \{0\})$, and define maps as follows

$$f\colon Y \to S^1 \qquad f(x, y, z) = (y, z)/\sqrt{y^2 + z^2}\text{[1]}$$
$$g\colon S^1 \to Y \qquad g(y, z) = (0, y, z)\text{[1]}.$$

Then $fg = 1_{S^1}$, and gf is linearly homotopic to 1_Y [1].

(f) True [1]. We can define a homeomorphism $f\colon \mathbb{C} \setminus \{0, 1\} \to \mathbb{C} \setminus \{i, -i\}$ by $f(z) = 2iz - i$, with inverse $f^{-1}(w) = (w + i)/2i$ [3].

(g) False [1]. We have $H_1(\mathbb{C} \setminus \{0, 1\}) \simeq \mathbb{Z}^2$, and this is not isomorphic to $H_1(\mathbb{C} \setminus \{0, 1, 2\}) \simeq \mathbb{Z}^3$, so $\mathbb{C} \setminus \{0, 1\}$ is not homotopy equivalent to $\mathbb{C} \setminus \{0, 1, 2\}$ [3].