University of Sheffield

Ancillary Material:

None.

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School of Mathematics and Statistics
Spring Semester 2023–2024

Module Code and Title:
MAS61015 Algebraic Topology

Exam Duration:

2 hours 30 minutes

Exam Instructions:

Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.

Registration number from U-Card (9 digits) – to be completed by candidate

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(a) What does it mean to say that a topological space X is *Hausdorff*?
(If your definition relies on any other concepts, then you should define them.) ***(3 marks)***

(b) What does it mean to say that a topological space X is *compact*?
(If your definition relies on any other concepts, then you should define them.) ***(3 marks)***

(c) Put $X = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^4 + y^4 + z^4 = 1\}$. Prove that X is compact. You may use general theorems provided that you state them precisely. ***(5 marks)***

(d) Put $Y = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^4 + y^4 + z^4 < 1\}$. Prove that Y is not compact. Here you should argue directly from the definitions and not use any theorems. ***(5 marks)***

(e) Let Y and Z be two compact subspaces of a topological space X. Prove that $Y \cup Z$ is also compact. ***(4 marks)***

(f) Let Y and Z be topological spaces such that $Z \neq \emptyset$ and $Y \times Z$ is compact. Prove that Y is compact. You may use standard results so long as you state them clearly and verify carefully that they are applicable. ***(5 marks)***

2 (a) Let X be a topological space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is indeed an equivalence relation. ***(8 marks)***

(b) Let $f\colon X \to Y$ be a continuous map. Define the function $f_*\colon \pi_0(X) \to \pi_0(Y)$, and check that it is well-defined. ***(5 marks)***

(c) Suppose that Y is path-connected and X is not. Show that there do not exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to the identity map id_X. ***(6 marks)***

(d) Put $X = \{A \in M_2\mathbb{R} \mid A^2 = A\}$ (where $M_2\mathbb{R}$ is the space of 2×2 real matrices). What can you say about $\det(A)$ when $A \in X$? Show that X is not path-connected. ***(6 marks)***

3 (a) Define the terms *chain map*, *chain homotopy*, *chain homotopic* and *chain homotopy equivalence*. ***(8 marks)***

(b) Show that if $f, g\colon U_* \to V_*$ are chain maps that are chain homotopic to each other, then $f_* = g_*\colon H_*(U) \to H_*(V)$. ***(5 marks)***

(c) Consider the chain complex T_* with $T_i = \mathbb{Z}^2$ for all i and $d_i(x, y) = (y, 0)$ for all $(x, y) \in T_i$. Show that T_* is chain homotopy equivalent to the zero complex. ***(4 marks)***

(d) Suppose we have a short exact sequence $A_* \xrightarrow{i} B_* \xrightarrow{p} C_*$ of chain complexes and chain maps. Suppose that for all $k \in \mathbb{Z}$ we have $H_k(B) = 0$. Suppose also that $H_k(A) = \mathbb{Z}/2^k$ for $k \geq 0$ and $H_k(A) = 0$ for $k < 0$. Determine the homology groups of C_*. ***(3 marks)***

(e) Let U_* be a chain complex in which $U_k = 0$ for $k < 0$ and $|U_k| = 2^k$ for $k \geq 0$ and $d_{2i}\colon U_{2i} \to U_{2i-1}$ is surjective for all i. Find the homology groups of U_*. ***(5 marks)***

4 (a) Let X be a topological space.

(i) Define the groups $C_n(X)$ for all nonnegative integers n. ***(2 marks)***

(ii) Define the homomorphisms $\partial_n\colon C_n(X) \to C_{n-1}(X)$. ***(3 marks)***

(iii) Prove that $\partial_1 \circ \partial_2 = 0$. ***(3 marks)***

(iv) Define the groups $H_n(X)$. ***(4 marks)***

(b) Describe (without proof, but with careful attention to any special cases) the groups $H_n(\mathbb{R}^k \setminus \{0\})$ for all $n \geq 0$ and all $k \geq 1$. ***(5 marks)***

(c) Let $u = n_1 s_1 + \ldots + n_k s_k$ be an element of $Z_m(S^n)$ (where $m > 0$), and suppose that there is a point $a \in S^n$ that is not contained in any of the sets $s_1(\Delta_m), \ldots, s_k(\Delta_m)$. Prove that u is a boundary. (You may assume standard results and calculations from the course so long as you state them carefully.) ***(8 marks)***

5 Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems and calculations, provided that you state them clearly.

(a) S^3 is contractible. ***(3 marks)***

(b) $\mathbb{R}P^3$ is a homotopy retract of S^3. ***(3 marks)***

(c) If a space X is the union of two closed, path-connected subspaces A and B, then X is path-connected. ***(3 marks)***

(d) $(\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1. ***(4 marks)***

(e) $(\mathbb{R} \times \mathbb{R}^2) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1. ***(4 marks)***

(f) The space $\mathbb{C} \setminus \{0, 1\}$ is homeomorphic to $\mathbb{C} \setminus \{i, -i\}$. ***(4 marks)***

(g) The space $\mathbb{C} \setminus \{0, 1\}$ is homotopy equivalent to $\mathbb{C} \setminus \{0, 1, 2\}$. ***(4 marks)***

End of Question Paper