The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2021–2022**

Algebraic Topology **2 hours 30 minutes**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 For $n \geq 3$, we put

$$X_n = \{z \in \mathbb{C} \mid |z| = 1 \text{ or } z^n \in (0, \infty)\}$$
$$Y_n = \{z \in \mathbb{C} \mid |z| = 1 \text{ or } z^n \in [0, \infty)\}.$$

(a) Sketch X_3 and Y_3. ***(2 marks)***

(b) Define the terms *homotopy* and *homotopy equivalent*. ***(5 marks)***

(c) Prove (by constructing explicit maps and homotopies, and checking their validity) that X_n and X_m are homotopy equivalent for all $n, m \geq 3$. ***(8 marks)***

(d) Prove that for all $n \neq m$, the space X_n is not homeomorphic to X_m. ***(6 marks)***

(e) Prove that for all $n \neq m$, the space Y_n is not homotopy equivalent to Y_m. ***(4 marks)***

Claims about the homology of particular spaces should be stated clearly and justified briefly, but details are not required.

2 (a) Define what is meant by a *topology* on a set X. ***(3 marks)***

(b) What does it mean to say that a topological space X is *Hausdorff*?
(If your definition relies on any other concepts, then you should define them.) ***(3 marks)***

(c) What does it mean to say that a topological space X is *compact*?
(If your definition relies on any other concepts, then you should define them.) ***(3 marks)***

(d) Let X and Y be topological spaces, and let $f\colon X \to Y$ be a continuous injective map. For each of the claims below, give a proof or a counterexample with justification.

(i) If X is Hausdorff, then Y must also be Hausdorff. ***(4 marks)***

(ii) If X is compact, then Y must also be compact. ***(4 marks)***

(iii) If Y is Hausdorff, then X must also be Hausdorff. ***(4 marks)***

(iv) If Y is compact, then X must also be compact. ***(4 marks)***

3 (a) Define the terms *chain complex*, *chain map* and *chain homotopy*. ***(8 marks)***

(b) Prove that if two chain maps are chain homotopic, then they have the same effect on homology groups. ***(5 marks)***

(c) Consider the chain complex T with $T_i = \mathbb{Z}/8$ for all i and $d(x) = 4x$ for all x. Find the homology groups of T. ***(3 marks)***

(d) Suppose we have a short exact sequence $A_* \xrightarrow{i} B_* \xrightarrow{p} C_*$ of chain complexes and chain maps. Suppose that for all $i \in \mathbb{Z}$ we have $H_{2i+1}(A) = H_{2i+1}(C) = 0$ and $|H_{2i}(A)| = 3$ and $|H_{2i}(C)| = 5$. Prove that all homology groups of B are cyclic or trivial, and determine their orders. ***(5 marks)***

(e) Let U_* be a chain complex in which all the differentials d_{2i} (for all $i \in \mathbb{Z}$) are surjective homomorphisms. What can we conclude about the homology groups of U_*? ***(4 marks)***

4 For each of the following, either give an example (with justification) or prove that no example can exist.

(a) A continuous injective map $i\colon X \to Y$ such that the map $i_*\colon H_2(X) \to H_2(Y)$ is not injective. ***(5 marks)***

(b) A continuous surjective map $p\colon X \to Y$ such that the map $p_*\colon H_2(X) \to H_2(Y)$ is not surjective. ***(5 marks)***

(c) A contractible space X and a homeomorphism $f\colon X \to X$ with no fixed points. ***(5 marks)***

(d) A continuous injective map $f\colon S^1 \to S^3$ such that $S^3 \setminus f(S^1)$ is homotopy equivalent to S^1. ***(5 marks)***

(e) A continuous injective map $f\colon S^1 \to S^3$ such that $S^3 \setminus f(S^1)$ is contractible. ***(5 marks)***

5 Put $X = \{(x, y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = 1\}$, so X is homeomorphic to S^3. Put $\omega = e^{2\pi i/3} \in \mathbb{C}$, so $\omega^3 = 1$. Define an equivalence relation on X by $(x, y) \sim (x', y')$ iff $(x', y') = \omega^k(x, y)$ for some k. Put

$$\begin{aligned} Y &= X/\sim \\ U &= \{[x, y] \in Y \mid x \neq 0\} \\ V &= \{[x, y] \in Y \mid y \neq 0\}. \end{aligned}$$

You may assume that U and V are open in Y and that $Y = U \cup V$.

(a) Show that the formula $f([x, y]) = (x^3/|x|^3, y/x)$ gives a well-defined and continuous map $f\colon U \to S^1 \times \mathbb{C}$. Do not assume any properties of the given formula without checking them. ***(6 marks)***

(b) Show that f is actually a bijection and that the inverse satisfies

$$f^{-1}(u, z) = \left[(v, z\, v)/\sqrt{1 + |z|^2}\right]$$

where v is any one of the three cube roots of u. Do not assume any properties of the given formula without checking them. ***(6 marks)***

(c) You may assume without proof that the map $f^{-1}\colon S^1 \times \mathbb{C} \to U$ is also continuous, so f is a homeomorphism. What can you conclude about the homeomorphism type of $U \cap V$? ***(3 marks)***

(d) The facts proved for U have obvious counterparts for V; you can assume these without proof. Deduce descriptions of $H_*(U)$, $H_*(V)$ and $H_*(U \cap V)$. ***(5 marks)***

(e) Use the Mayer-Vietoris sequence to compute $H_*(Y)$. You should be able to compute $H_k(Y)$ for $k = 0$ and $k \geq 3$. For $k = 1, 2$ you will need to determine a map in the Mayer-Vietoris sequence, which is possible but not so easy. If you cannot see how to do it then you should guess, and give an answer based on your guess. ***(5 marks)***

End of Question Paper