Algebraic Topology

(1) Give examples as follows, justifying your answers.

(a) Topological spaces X and Y, together with injective functions $f\colon X \to Y$ and $g\colon Y \to X$ such that f, $f \circ g$ and $g \circ f$ are all continuous, but g is not continuous. **(4 marks)**

(b) A compact, path-connected space X together with a continuous map $f\colon X \to X$ with no fixed points. **(4 marks)**

(c) A space X such that $H_1(X)$ is not a free abelian group. (Note here that the zero group is free abelian with no generators, so in particular $H_1(X)$ must be nonzero.) **(4 marks)**

(d) A space X together with points $a, b, c \in X$ such that $|\Pi(X; a, b)| \neq |\Pi(X; b, c)|$. **(4 marks)**

(e) A space X such that $\pi_1(X)$ is a free group with 3 generators, and $H_2(X) = \mathbb{Z}$. **(4 marks)**

Solution: In each case, two marks will be awarded for a correct example, and two further marks for justifying it. Up to two marks may also be awarded for intelligent discussion of an incorrect example. Note that in addition to the main lecture notes, students have access to a two-page summary of examples.

(a) We can use the standard example of a continuous bijection that is not a homeomorphism (Example 4.8):

$$X = (-\infty, 0] \cup (1, \infty) \qquad\qquad Y = \mathbb{R}$$

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x - 1 & \text{if } x > 1 \end{cases} \qquad\qquad g(y) = \begin{cases} y & \text{if } y \leq 0 \\ y + 1 & \text{if } y > 0. \end{cases}$$

Here f is continuous because the domains of the two clauses are both open in X, and $f \circ g$ and $g \circ f$ are identity maps so they are certainly continuous, but g is discontinuous at $y = 0$. [4]

(b) We can take $X = S^n$ for any $n > 0$, and $f(x) = -x$. (Example 9.15 mentions that S^n is compact, as an easy application of Proposition 9.14. It is path-connected by Proposition 5.11. This example of a fixed-point-free endomorphism is mentioned in the solution to Exercise 3 of Problem Sheet 9.) [4]

(c) We can take $X = \mathbb{R}P^2$, then $H_1(X) = \mathbb{Z}/2$, which is not free abelian. (Example 12.15 shows that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2$, and Theorem 18.18 shows that $H_1(\mathbb{R}P^2)$ is the abelianisation of this, which is $\mathbb{Z}/2$ again.) [4]

(d) We can take $X = \{0\} \amalg \mathbb{R}P^2$, with $a = 0$ and $b = c =$ basepoint of $\mathbb{R}P^2$. Then $\Pi(X; a, b) = \emptyset$ and $\Pi(X; b, c) = \pi_1(\mathbb{R}P^2, b) = C_2$ so $|\Pi(X; a, b)| = 0$ but $|\Pi(X; b, c)| = 2$. [4]

(e) We can take $X = S^1 \vee S^1 \vee S^1 \vee S^2$. Using Corollary 15.20 (a special case of the van Kampen Theorem) we see that $\pi_1(X)$ is the free product of three copies of $\pi_1(S^1) = \mathbb{Z}$ together with one copy of $\pi_1(S^2) = 1$, so it is free on three generators. Similarly, we can use Lemma 21.4 (a special case of the Mayer-Vietoris Theorem) to show that $H_2(X) = 0 \oplus 0 \oplus 0 \oplus \mathbb{Z} = \mathbb{Z}$ as required. [4]

Feedback: For part (a), another good answer (given by several students) is to define $f\colon [0, 2\pi) \to S^1$ by $f(x) = e^{ix}$, note that this is bijective, and take $g = f^{-1}$. Most people answered (b) correctly, using the same example as in the solution above. Some people gave answers for (c) where they claimed that $H_1(X)$ was not abelian, but homology groups are always abelian. Most people answered (d) correctly (but sometimes with inadequate justification); correct answers for (e) were rare.

(2) Fix $n \geq 2$. Define an equivalence relation on the disc $B^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ by $z_0 \sim z_1$ iff $z_0 = z_1$, or ($|z_0| = |z_1| = 1$ and $z_0^n = z_1^n$). Put $X = B^2/\sim$ and

$$Y = \{(u, v) \in \mathbb{C}^2 \mid |u| \leq 1, \quad v^n = (1 - |u|)^n u\}.$$

Note that when $n = 2$ we just have $X = \mathbb{R}P^2$; this should guide your thinking about the general case.

(a) Show carefully that there is a homeomorphism $f\colon X \to Y$ such that $f([z]) = (z^n, (1-|z|^n)z)$ for all $z \in B^2$. You should prove in particular that f is well-defined, injective and surjective, and that both f and f^{-1} are continuous. You may assume that polynomials and the absolute value function are continuous, but beyond that you should not assume any properties of the given formula without proof. **(13 marks)**

(b) For the boundary $S^1 \subset B^2$, explain briefly why $S^1/\sim$ is homeomorphic to S^1 again. **(3 marks)**

(c) By adapting the method used for $\mathbb{R}P^2$, calculate $H_*(X)$. **(14 marks)**

Solution:

(a) Suppose that $z \in B^2$ (so $|z| \leq 1$) and put $u = z^n$ and $v = (1-|z|^n)z = (1-|u|)z$. We then have $|u| = |z|^n \leq 1$ and $v^n = (1-|u|)^n z^n = (1-|u|)^n u$, so $(u, v) \in Y$ [1]. We can thus define a continuous map $f_0\colon B^2 \to Y$ by $f_0(z) = (z^n, (1-|z|^n)z)$. Now suppose we have $z_0, z_1 \in B^2$ with $z_0 \sim z_1$; we claim that $f(z_0) = f(z_1)$ [1]. If $z_0 = z_1$ then this is clear. Otherwise, we must have $|z_0| = |z_1| = 1$ (which means that $f_0(z_i) = (z_i^n, 0)$) and $z_0^n = z_1^n$, so $f_0(z_0) = f_0(z_1)$ as required [1]. By the universal property of quotients (Corollary 8.20) there is a unique continuous map $f\colon X \to Y$ such that $f([z]) = f_0(z)$ for all z [1].

Now suppose we have $(u, v) \in Y$, so $v^n = (1-|u|)^n u$. If $|u| \neq 1$ then $0 < 1-|u| \leq 1$ and we put $z = v/(1-|u|) \in \mathbb{C}$. The relation $v^n = (1-|u|)^n u$ becomes $z^n = u$. It follows that $|z|^n = |u| < 1$ so $|z| < 1$ so $z \in B^2$, and we find that $f([z]) = f_0(z) = u$. On the other hand, if $|u| = 1$ then the relation $v^n = (1-|u|)^n u$ gives $v = 0$. We can let z be any one of the n'th roots of u and we get $|z| = 1$ and $f([z]) = f_0(z) = (u, 0)$. This shows that f is surjective. [3]

Now suppose we have $z_0, z_1 \in B^2$ with $f([z_0]) = f([z_1])$, or in other words $z_0^n = z_1^n$ and $(1-|z_0|^n)z_0 = (1-|z_1|^n)z_1$. Put $r = |z_0| \in [0, 1]$. Using $z_0^n = z_1^n$ we get $r^n = |z_1|^n$ so $|z_1|$ is also equal to r. Thus, the equation $(1-|z_0|^n)z_0 = (1-|z_1|^n)z_1$ becomes $(1-r^n)(z_0 - z_1) = 0$. If $r < 1$ this gives $z_0 = z_1$, so certainly $[z_0] = [z_1]$. On the other hand, if $r = 1$ then the relation $z_0^n = z_1^n$ gives $z_0 \sim z_1$ (from the definition of the equivalence relation) and so $[z_0] = [z_1]$. Either way, we have $[z_0] = [z_1]$, so we conclude that f is injective. [3]

Note also that X is a quotient of the compact space B^2, so it is again compact. Moreover, Y is a metric space and so is Hausdorff. As f is a continuous bijection from a compact space to a Hausdorff space, it is a homeomorphism by Proposition 9.28. [3]

(b) For $z \in S^1$ we have $(1-|z|^n)z = 0$, so f restricts to give a homeomorphism $S^1/\sim \to S^1 \times \{0\} \simeq S^1$. Alternatively, on S^1 the equivalence relation is just $z_0 \sim z_1 \iff z_0^n = z_1^n$, so the map $[z] \mapsto z^n$ gives the required homeomorphism. [3]

(c) Put $\widetilde{U} = B^2 \setminus \{0\}$ and $\widetilde{V} = B^2 \setminus S^1 = OB^2$. Let U and V be the images of $\widetilde{U}$ and $\widetilde{V}$ in X. These are open sets which cover X, so they give a Mayer-Vietoris sequence. [3]

The equivalence relation does not do anything to $\widetilde{V}$, so V is just an open disc, which is contractible. Thus, the only nontrivial homology group is $H_0(V) = \mathbb{Z}$ [2]. Next, we can deform $\widetilde{U}$ radially outward onto S^1, and this is compatible with the equivalence relation, so U is homotopy equivalent to $S^1/\sim$, which is homeomorphic to S^1 by (b). Thus, we have $H_0(U) = H_1(U) = \mathbb{Z}$ and all other homology groups are zero [2]. Also, $U \cap V$ is an annulus so $H_0(U \cap V) = H_1(U \cap V) = \mathbb{Z}$ and again all other homology groups are zero [1]. As U, V and $U \cap V$ are connected we can use the truncated version of the Mayer-Vietoris sequence:

$$H_2(U) \oplus H_2(V) \to H_2(X) \to H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(X) \to H_1(U \cap V) \to 0.$$ [2]

Using the above determination of the homology groups, this becomes

$$0 \to H_2(X) \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \to H_1(X) \to 0.$$ [1]

The standard circle in the annulus $U \cap V$ gets wrapped n times around the boundary circle $S^1/\sim$, so i_* is multiplication by n, which is injective [1]. It follows that $H_2(X) = 0$ and $H_1(X) = \mathbb{Z}/n$. As X is connected, we have $H_0(X) = \mathbb{Z}$ [1]. For $k > 2$ is is clear from the Mayer-Vietoris sequence that $H_k(X) = 0$. [1]

Feedback:

(a) Very few people checked that $(z^n, (1-|z|^n)z) \in Y$, despite my ranting about this sort of thing in connection with Problem Sheet 10. Very few people distinguished clearly between f_0 and f; in particular, many people claimed to be proving that f is continuous, but actually proved that f_0 is continuous. Attempts to prove that f is well-defined and injective were of variable quality. For surjectivity, many people claimed that $f([u^{1/n}]) = (u, v)$ for all $(u, v) \in Y$. Here everything is complex so we usually have n different choices of z with $z^n = u$, i.e. n different possible values of $u^{1/n}$. If you choose the right one then you will get $f([z]) = (u, v)$, but if you choose the wrong one then you will instead get $f([z]) = (u, e^{2\pi ik/n}v)$ for some $k \neq 0$. Thus, a more detailed argument needs to be given. These issues also mean that f^{-1} is not given by a simple and well-defined formula, so the only reasonable way to prove that f^{-1} is continuous is to use Proposition 9.28. This is all similar to Examples 8.24, 8.26, 9.29 and 9.30 in the notes.

(b) Most people gave answers that were along the right lines.

(c) Most people who made a serious attempt at this got it roughly right; but some people gave up.