Parts (a) & (b) are bookwork. Parts (c) are seen.

A1: (1)(a) X is compact if every open cover has a finite subcover. 2

X is Hausdorff if for every pair of distinct points $x, y \in X$ there are open sets $U \ni x$, $V \ni y$ with $U \cap V = \emptyset$ 2

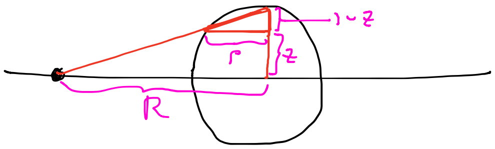
(b) Suppose X is compact
Y is Hausdorff
& $f : X \longrightarrow Y$ is a continuous bijection. We may define $g : Y \to X$ by $g = f^{-1}$ & it remains to &
show that g is continuous.
Suppose then $V \subseteq X$ is closed, we must show $g^{-1}(V) = f(V)$ is closed.

However V is a closed subset of the compact space Y & so compact. 2
Hence fV is the continuous image of a compact space & so compact. 2
Finally fV is a compact subset of a Hausdorff space & so closed. 2

(c) Now take $X = \overline{B}^2 / S^2$ (Compact)
$Y = S^2$ (Hausdorff)

& define $f : X \longrightarrow Y$ by $f([S^2]) = (0,0,1)$ & $f(r, \theta) = (\rho, \theta, z)$ 2

$$R(r) = \frac{r}{1-r}, \quad z(r) = \frac{R^2-1}{R^2+1}, \quad \rho(r) = (1-z)R$$

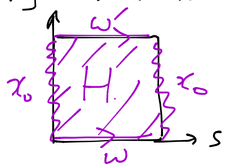


$$\frac{\rho}{R} = \frac{1-z}{1}$$

Now f is a continuous bijection & hence a homeomorphism. 1

(ii)(a) A path from a to b in X is a continuous function $f : [0,1] \longrightarrow X$ with $f(0) = a$, $f(1) = b$. 2

(b) $\pi_1(X, x_0) = \{\omega : [0,1] \to X \mid \omega(0) = \omega(1) = x_0\} / \simeq$ where two loops ω, ω' are path homotopic ~~if~~ 2
there is a continuous function $H : [0,1] \times [0,1] \longrightarrow X$ so that $H(s,0) = \omega(s)$ $H(0,t) = x_0$
$H(s,1) = \omega'(s)$ $H(1,t) = x_0$



The group operation is
$[\omega][\sigma] = [\omega \cdot \sigma]$

where $(\omega \cdot \sigma)(s) = \begin{cases} \omega(2s) & 0 \leq s \leq \frac{1}{2} \\ \sigma(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$ 2

This is well defined since if $H : \omega \simeq \omega'$
$K : \sigma \simeq \sigma'$ are path homotopies then

$H \bullet K : \omega \bullet \sigma \simeq \omega' \bullet \sigma'$ where $(H \bullet K)(s,t) = \begin{cases} H(2s,t) & 0 \leq s \leq 1/2 \\ K(2s-1,t) & \frac{1}{2} \leq s \leq 1. \end{cases}$ 2

(c) We claim the maps $\theta : \pi_1(X, Y; (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$
$\varphi : \pi_1(X, x_0) \times \pi(Y, y_0) \longrightarrow \pi_1(X, Y; (x_0, y_0))$

defined by $\theta = \{(\pi_X)_*, (\pi_Y)_*\}$
$\varphi([\omega], [\sigma]) = [\{\omega, \sigma\}]$ are inverse isomorphisms 2

Indeed
$$\theta\varphi([\omega],[\sigma]) = \theta([\{\omega,\sigma\}]) = ([\pi_X \circ \{\omega,\sigma\}], [\pi_Y \circ \{\omega, \sigma\}]) = ([\omega],[\sigma])$$
$$\varphi\theta([\{\omega,\sigma\}]) = \varphi([\pi_X \circ \{\omega,\sigma\}], [\pi_Y \circ \{\omega,\sigma\}]) = \varphi([\omega],[\sigma]) = [\{\omega,\sigma\}]$$ 2

(i) Bookwork
(ii) Seen for S^3 & the antipodal map. SU(2) variant unseen.

A2 (i)(a) A covering map is a continuous function $p : \tilde{X} \to X$ so that for every $x \in X$ there is a neighbourhood $U \ni x$, so that $p^{-1}(U) = \coprod_\alpha \tilde{U}_\alpha$ & $p|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \xrightarrow{\cong} U$ is a homeomorphism. 2 2

(b) Path lifting lemma: Given a path ω from x_0 to x, in X & a point $\tilde{x}_0 \in Y$ over x_0, there is a unique path $\tilde{\omega}$ from $\tilde{x}_0$ in Y with $p \circ \tilde{\omega} = \omega$ 3

We define a map
$\ell : \pi_1(X, x_0) \longrightarrow p^{-1}(x_0)$
as follows.
If ω is a loop based at x_0, we may lift ω to a path $\tilde{\omega}$ from $\tilde{x}_0$ & then $\ell([\omega]) = \tilde{\omega}(1)$ 2
The map is surjective if Y is path connected
& injective if $\pi_1(Y, \tilde{x}_0) = 1$ 3

(ii)(a) Let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU(2)$, so $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix} = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & \alpha\overline{\gamma} + \beta\overline{\delta} \\ \overline{\alpha}\gamma + \beta\overline{\delta} & |\gamma|^2 + |\delta|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ If $\beta = 0$ then $\gamma = 0, |\alpha|^2 = 1, \delta = \overline{\alpha}$
Otherwise $\delta = -\overline{\alpha}\gamma/\beta$ & $\gamma = (|\alpha|^2 + |\beta|^2)\gamma = \overline{\beta}$, $\alpha = \overline{\delta}$ & $A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$ as required

$\therefore SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \;\middle|\; |\alpha|^2 + |\beta|^2 = 1 \right\} \cong S^3$ 2

Hence $\pi_1(SU(2), I) = \pi_1(S^3; x_0) = 1$

[By van Kampen's theorem, since $S^3 \underset{2}{=} (S^3 \setminus \{N\}) \cup (S^3 \setminus \{S\})$ and $S^3 \setminus \{N\}, S^3 \setminus \{S\} \overset{2}{\simeq} *$
& $(S^3 \setminus \{N\}) \cap (S^3 \setminus \{S\}) \underset{2}{=} S^3 \setminus \{N, S\}$ is path connected]

(b) Now $SU(2) \to SU(2)/Z$ is a covering map & so we have 2

$\ell : \pi_1(SU(2)/Z, \bar{I}) \to p^{-1}(I) = Z$. Since $SU(2)$ is 1-connected 2 ℓ is a bijection so

$\pi_1(SU(2)/Z)$ is of order 2 & hence $\cong C_2$. 1

(i)(a), (b), (ii)(a) bookwork

(ii) (b) unseen. Method for $(\Delta^4)^{(3)} \cong S^3$ seen.

A3: (i) A chain complex is a sequence of abelian groups & group homomorphisms

$$\xrightarrow{d} C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d}$$

2

so that $d^2 = 0$. Two chain maps $\theta, \varphi : C \to D$ are chain homotopic if $\exists\ h : C_n \to D_{n+1}$ so that

$$dh + hd = \theta - \varphi$$

3

Suppose $\theta \simeq \varphi$ & $\alpha \in H_n C$.
Then there is a cycle $z \in C_n$ so $\alpha = [z]$ & $\theta_*[z] = [\theta z] = [(dh+hd+\varphi)z]$

$$= [\cancel{dh}z] + [h\cancel{dz}] + [\varphi z] = \varphi_*[z]$$

$\to 0$ $\to 0$ $dz = 0$

4

Thus $\theta_* = \varphi_*$ as required.

(b) We define $h : C_n(C_pK) \to C_{n+1}(C_pK)$ $(\sigma \in K)$ h is a chain homotopy $id \simeq const_p$. 3

$$\sigma \longmapsto p\sigma$$
$$p\sigma \longmapsto 0$$

Then for $n \geq 1$

$$(hd+dh)(\sigma) = p\,d\sigma + d\,p\sigma = p\,d\sigma + \sigma - p\,d\sigma = \sigma$$
$$(hd+dh)(p\sigma) = hd\,p\sigma + 0 = h(\sigma - p\,d\sigma) = p\sigma$$

2

For $n = 0$

$$(hd+dh)(\langle v\rangle) = 0 + d\langle p, v\rangle = \langle v\rangle - \langle p\rangle$$
$$(hd+dh)(\langle p\rangle) = 0$$

1

We conclude $c_pK \simeq *$. [or cut corners to just deal with $n \geq 1$)

Since $\Delta^n = c_n \Delta^{n-1}$ this says $H_i(\Delta^n) = 0$ for $i \geq 1$. 2

(c) We note $K \subseteq \Delta^4$
& hence $C_6K \subseteq C_6\Delta^4$. 2

$$\begin{tikzcd}
0 \arrow[r] & C_4K \arrow[r] \arrow[d] & C_3K \arrow[r] \arrow[d] & C_2K \arrow[r] \arrow[d, "="] & C_1K \arrow[r] \arrow[d, "="] & C_0K \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & C_4\Delta^4 \arrow[r] & C_3\Delta^4 \arrow[r] & C_3\Delta^4 \arrow[r] & C_1\Delta^4 \arrow[r] & C_0\Delta^4 \arrow[r] & 0
\end{tikzcd}$$

$C_4K = 0$, $C_3K = 0$

$$H_i(K) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}^3 & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

2

For $i \leq 1$, $H_i K = H_i \Delta^4$
For $i \geq 3$, $H_i K = 0$

$$H_2 K = \frac{Z_2 K}{B_2 K} = \frac{Z_2 \Delta^4}{0} \cong B_2 \Delta^4.$$

2.

$$\begin{array}{c|cccccc} & 01 & 02 & 03 & 12 & 13 & 23 \\ \hline d\langle 012\rangle & 1 & -1 & & 1 & & \\ d\langle 013\rangle & 1 & & -1 & & 1 & \\ d\langle 023\rangle & & 1 & -1 & & & 1 \\ d\langle 123\rangle & & & & 1 & -1 & 1 \end{array}$$

2 $B_2\Delta^4$ has rank 3
(since $d(\langle 012\rangle - \langle 013\rangle + \langle 023\rangle - \langle 123\rangle) = 0$)

(i) & (ii)(a) bookwork.

(ii) (b), (c) unseen (ii) (d) similar seen

A4 (i) (a)
$$\Lambda(f) = \sum_{i=} (-1)^i \operatorname{tr}(\theta_n : C_n \to C_n)$$

2

(b) We have $C_n \supseteq Z_n C \supseteq B_n C$
& we may choose a basis of $B_n C$, $b_n^1, \dots, b_n^{r_n}$
extend to a basis of $Z_n C$, $z_n^1, \dots, z_n^{s_n}$
extend to a basis of C_n, $c_n^1, \dots, z_n^{h_n}$

1

Furthermore we may choose the bases for decreasing n, starting with the top one.
Thus we may assume
$$b_n^i = d(c_{n+1}^i)$$
If θ_n has matrix
$$\Theta_n = \left(\begin{array}{c|c|c} B_nC & Z_n'C & C_n' \\ P_n & ? & ? \\ \hline ? & Q_n & ? \\ \hline ? & ? & R_n \end{array}\right)$$

2

Thus
$$\operatorname{tr}(\Theta_n) = \operatorname{tr} P_n + \operatorname{tr} Q_n + \operatorname{tr} R_n.$$
& since Θ_n is a chain map
$$R_{n+1} = P_n$$

1

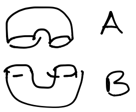
Hence $\Lambda(\theta) = \sum_i (-1)^i \operatorname{tr}(\Theta_i) = \sum (-1)^i [\operatorname{tr}(P_i) + \operatorname{tr}(Q_i) + \operatorname{tr}(R_i)]$
$$= \sum (-1)^i \operatorname{tr}(Q_i) = \Lambda(\theta_*).$$

2

(c) Lefschetz Fixed Point Theorem: If X is triangulable & $f : X \to X$ has $\Lambda(f_*) \neq 0$ then f has a fixed point

2

(ii) $T =$



$A \simeq S^1$
$A \cap B = S^1 \sqcup S^1$
$B \simeq S^1$

$$\begin{tikzcd} & T & \\ A \arrow[ur, "i_A"] & & B \arrow[ul, "i_B"'] \\ & A \cap B \arrow[ul, "j_A"] \arrow[ur, "j_B"'] & \end{tikzcd}$$

$$\begin{tikzcd} \cap \arrow[r] & \oplus \arrow[r] & \cup \\ 0 \arrow[r] & 0 \arrow[r] & H_2(T) \arrow[ddll] \\ \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "\alpha"] & \mathbb{Z}\oplus\mathbb{Z} & \\ H_1(S^1 \sqcup S^1) \arrow[u, equal] \arrow[r] & H_1(S^1) \oplus H_1(S^1) \arrow[u, equal] \arrow[r] & H_1(T) \arrow[dll] \\ H_0(S^1 \sqcup S^1) \arrow[d, equal] \arrow[r] & H_0(S^1) \oplus H_0(S^1) \arrow[d, equal] \arrow[r] & H_0(T) \arrow[d, equal] \arrow[r] & 0 \\ \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "\beta"] & \mathbb{Z}\oplus\mathbb{Z} & \mathbb{Z} \end{tikzcd}$$

2

2

By algebra or explicit reps, $\ker(\beta) \cong \mathbb{Z}$.

Hence $H_1(T) \cong \mathbb{Z} \oplus \operatorname{cok}(\alpha)$
$H_2(T) \cong \ker(\alpha)$.

We argue $\alpha = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ so $H_2(T) \cong \mathbb{Z}$
$H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$

2

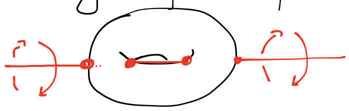
Up to sign this is clear from the homotopy equivalences including S^1s. Using a choice of generator in $H_1(A)$ to determine the others gives the signs 1

(b) Clearly we may rotate $T = S^1 \times S^1$ $(e^{i\theta}, e^{i\varphi}) \longmapsto (e^{i\theta+\theta_0}, e^{i\varphi})$ & thus is homotopic to 1 but has no fixed point 2

(c) $\Lambda(f) = tr(f_* : H_2(T) \to H_2(T)) - tr(H_1(T) \to H_1(T)) + tr(H_0(T) \to H_0(T))$

$$= 1 - (-2) + 1 = 4.$$ 2

Hence f has a fixed point by the Lefschetz fixed point theorem



For example rotation by a half turn around as shown. 1

(d) We have n_0, n_1, n_2 vertices, edges & faces &

$n_0^T = 2n_0^{T/G} - 4$
$n_1^T = 2n_1^{T/G}$
$n_2^T = 2n_2^{T/G}$ 2

$$0 = \chi(T) = n_0^T - n_1^T + n_2^T$$
$$= 2n_0^{T/G} - 2n_1^{T/G} + 2n_2^{T/G} - 4$$
$$= 2\chi(T/G) - 4$$ 1

$\therefore$ $\chi(T/G) = 2$ & T/G is of genus 0. 1

(i), (ii), (iii), (v) seen
(iv) "unseen"

2 for conclusion
3 for method

A5

(a) False ≤ 2 if conclusion wrong

$\mathbb{R} \simeq *$ but $x \longmapsto x+1$ is a self-map without a fixed point

(b) False

If $\mathbb{R}^2 \cong \mathbb{R}^3$ then $S^1 \simeq \mathbb{R}^2 - \{P\} \cong \mathbb{R}^3 - \{Q\} \simeq S^2$

However $\pi_1(S^1) \cong \mathbb{Z}$, $\pi_1(S^2) = 1$

(c) False.

$\pi_1(K)$ is nonabelian $\left(\langle A, B \rangle \quad \begin{array}{l} A(x,y) = (x+1, y) \\ B(x,y) = (-x, y+1) \end{array} \right)$ but $\pi_1(\text{Top}^l\text{gp})$ is abelian.

(d) True.

$X \simeq X/e$ where e is any edge with distinct ends. Since X is connected any two vertices are connected via an edge path. Hence $X \simeq X''$ where X' has one vertex, $X =$  $_n$ & $\chi(X) = 1-n$.

(e) False

There are finitely many simplicial maps $s: K \to L$ (determined by the map of vertices) -
However there are examples where there are infinitely many homotopy classes

(eg $|K| = |L| = S^1$ & $f_n(z) = z^n$; distinguished by their effect in π_1)