The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2015–2016

MAS435 Algebraic Topology — 2 hours 30 minutes

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) (a) What does it mean to say that a topological space is *compact*? What does it mean to say that a topological space is *Hausdorff*? ***(4 marks)***

(b) Prove that any continuous bijection from a compact space to a Hausdorff space is a homeomorphism. ***(6 marks)***

(c) Show that the quotient space $\overline{B}^2/S^1$ (the closed disc mod the boundary circle) is homeomorphic to S^2. ***(3 marks)***

(ii) (a) What is a path from a to b in a topological space X? ***(2 marks)***

(b) Define the fundamental group $\pi_1(X, x_0)$. [You should define the underlying set and the group multiplication, and check the multiplication is well defined. You need not check the group axioms are satisfied.] ***(6 marks)***

(c) Show that for any two topological spaces X, Y, with basepoints $x_0 \in X, y_0 \in Y$, the fundamental group $\pi_1(X \times Y; (x_0, y_0))$ is isomorphic to the product of $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$. ***(4 marks)***

2 (i) (a) What is a *covering map*? ***(4 marks)***

(b) State the Path Lifting Lemma for a covering map $p : Y \longrightarrow X$, and explain how it can be used to define a function

$$\ell : \pi_1(X, x_0) \longrightarrow p^{-1}(x_0),$$

where $x_0 \in X$. State conditions under which ℓ is a bijection. ***(8 marks)***

(ii) Let

$$SU(2) = \{A \in M_2(\mathbb{C}) \mid A\overline{A}^T = I, \det(A) = 1\}$$

be the 2×2 special unitary group.

(a) Show that

$$SU(2) = \{\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \mid |\alpha|^2 + |\beta|^2 = 1\}.$$

Use this to show that $\pi_1(SU(2), I)$ is the trivial group, where I is the identity matrix. ***(8 marks)***

(b) Consider the subgroup $Z = \{I, -I\}$ of $SU(2)$ and let $PSU(2) = SU(2)/Z$. Determine the fundamental group of $PSU(2)$ using the image of I as basepoint. ***(5 marks)***

3 (i) (a) What is a *chain complex* of abelian groups? What does it mean to say that two chain maps $\theta, \phi : C_\bullet \longrightarrow D_\bullet$ are *chain homotopic*. ***(5 marks)***

(b) Show that if θ and ϕ are chain homotopic, they induce the same map in homology. ***(4 marks)***

(ii) (a) Show that if K is a simplicial complex, P is a new vertex and $c_P K$ is the P-cone on K then $H_i(c_P K) = 0$ for $i > 0$. Explain the relevance of this to the homology of the standard abstract simplex Δ^n. ***(8 marks)***

(b) If K consists of all subsets of $\{0, 1, 2, 3, 4\}$ with ≤ 3 elements (i.e., the 2-skeleton of Δ^4), calculate $H_*(K)$. ***(8 marks)***

4 (i) (a) Suppose that $C_\bullet$ is a chain complex with only finitely many terms non-zero and all terms finite dimensional vector spaces. What is the *Lefschetz number* $\Lambda(\theta)$ of a chain map $\theta : C_\bullet \longrightarrow C_\bullet$? ***(2 marks)***

(b) Show that $\Lambda(\theta) = \Lambda(\theta_*)$ where $\theta_* : H_*(C_\bullet) \longrightarrow H_*(C_\bullet)$ is the induced map in homology. ***(6 marks)***

(c) State the Lefschetz Fixed Point Theorem. ***(2 marks)***

(ii) Consider maps $f : T \longrightarrow T$, where T denotes the 2-torus.

(a) Calculate the homology of T. ***(6 marks)***

(b) Is there a map $f : T \longrightarrow T$ homotopic to the identity which has no fixed points? Justify your answer. ***(2 marks)***

(c) Suppose that f induces the identity map on $H_0(T)$ and $H_2(T)$ and that $f_*(x) = -x$ for $x \in H_1(T)$. Calculate the Lefschetz number of f and deduce that f must have a fixed point. Describe such a map f which has exactly 4 fixed points. ***(3 marks)***

(d) Suppose that a group G of order 2 acts on T with 4 fixed points, and that T/G is an orientable surface. Find the genus of T/G. ***(4 marks)***

5 Are the following true or false. Justify your answers.

(i) Any self-map of a contractible space has a fixed point. ***(5 marks)***

(ii) $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^3$. ***(5 marks)***

(iii) The Klein bottle admits the structure of a topological group. ***(5 marks)***

(iv) The Euler characteristic distinguishes the homotopy types of connected one dimensional simplicial complexes. ***(5 marks)***

(v) If K and L are simplicial complexes, any continuous function $f : |K| \longrightarrow |L|$ is homotopic to a map $|s|$ for a simplicial map $s : K \longrightarrow L$. ***(5 marks)***

End of Question Paper