MAS334 COMBINATORICS — PROBLEM SHEET 1 — Solutions

Please hand in Exercises 1.1 and 1.3 by the end of Week 2.

Exercise 1.1. Use the Binomial Theorem to show that, for any positive integer n,

(A) $$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$$

(B) $$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0$$

(C) $$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = 2^{n-1}$$

(D) $$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots = 2^{n-1}$$

Solution: Recall the Binomial Theorem (Example 1.17):

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = (1+x)^n.$$

Equation (A) follows by taking $x = 1$, and Equation (B) follows by taking $x = -1$. If we add (A) and (B) and divide by 2 we get (C). If we subtract (A) and (B) and divide by 2 we get (D).
Feedback: In the homework, most students took the approach described above. Here are two interesting alternatives. First, we want to understand the sums

$$S = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \sum_{k\geq 0}\binom{n}{2k}$$

$$T = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = \sum_{k\geq 0}\binom{n}{2k+1}.$$

Pascal's relation tells us that $\binom{n}{2k+1} = \binom{n-1}{2k} + \binom{n-1}{2k+1}$, so

$$T = \sum_{k\geq 0}\binom{n-1}{2k} + \sum_{k\geq 0}\binom{n-1}{2k+1} = \sum_{\text{even } j\geq 0}\binom{n-1}{j} + \sum_{\text{odd } j\geq 0}\binom{n-1}{j} = \sum_{j\geq 0}\binom{n-1}{j} = 2^{n-1}.$$

By subtracting this from the relation $S + T = \sum_{j\geq 0}\binom{n}{j} = 2^n$, we also get the relation $S = 2^{n-1}$.

So far we have just given algebraic proofs. For a combinatorial approach:

- Let E be the collection of subsets $A \subseteq \{1, \ldots, n\}$ such that $|A|$ is even.
- Let O be the collection of subsets $B \subseteq \{1, \ldots, n\}$ such that $|B|$ is odd
- Let P be the collection of subsets $C \subseteq \{1, \ldots, n-1\}$ of any size.

It is easy to see that $|E| = S$ and $|O| = T$ and $|P| = 2^{n-1}$, so we want to show that $|E| = |O| = |P|$, which we can do by giving a one-to-one correspondence between these sets.

Given a set $C \subseteq \{1, \ldots, n-1\}$ we define

$$A = \begin{cases} C & \text{if } |C| \text{ is even} \\ C \cup \{n\} & \text{if } |C| \text{ is odd} \end{cases} \qquad B = \begin{cases} C & \text{if } |C| \text{ is odd} \\ C \cup \{n\} & \text{if } |C| \text{ is even.} \end{cases}$$

It is not hard to check that $A \in E$ and $B \in O$ and that in all cases we have

$$A \cap \{1, \ldots, n-1\} = B \cap \{1, \ldots, n-1\} = C.$$

Using this, we see that this procedure gives the required one-to-one correspondence between P, E and O.

Exercise 1.2.

(a) If you draw n (infinite) straight lines in the plane with no two parallel and no three meeting at a point, how many intersection points will there be altogether?

(b) Now n straight lines are drawn in the plane consisting precisely of x_1 parallel in one direction, x_2 parallel in another direction, ... and x_k parallel in another direction, and with no three meeting at a point. By considering the number of intersection points lost by having parallel lines, show that the number of intersection points will be

$$\frac{1}{2}\,(n^2 - x_1^2 - x_2^2 - \ldots - x_k^2).$$

(c) Draw 14 straight lines in the plane, with no three meeting at a point, so that there are 61 intersection points altogether.

Solution:

(a) Any two lines from the n give rise to a different intersection point, so there are $\binom{n}{2}$ points altogether.

(b) The x_i parallel lines will give no intersection points, resulting in these $\binom{x_i}{2}$ points 'being lost'. So now the number of intersection points is

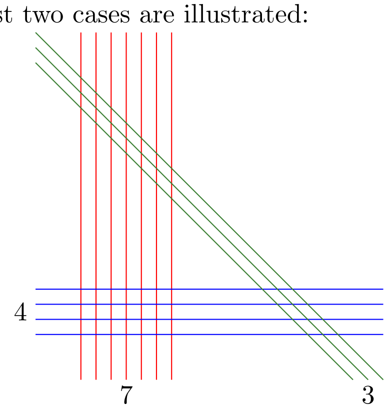
$$\binom{n}{2} - \binom{x_1}{2} - \ldots - \binom{x_k}{2} = \frac{1}{2}(n^2 - n - x_1^2 + x_1 - \ldots - x_k^2 + x_k) = \frac{1}{2}(n^2 - x_1^2 - \ldots - x_k^2)$$

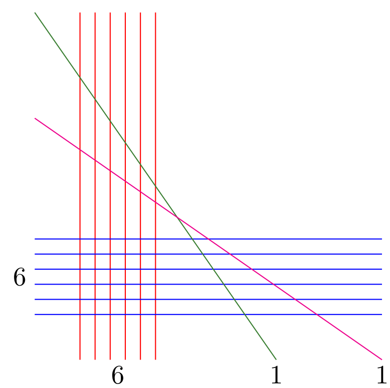
(the last equality being due to the fact that $n = x_1 + \ldots + x_k$).

(c) We need to find some $x_1, x_2, \ldots, x_k$ whose sum is 14 and whose squares sum to 74 (since $\frac{1}{2}(14^2 - x_1^2 - x_2^2 + \ldots - x_k^2) = 61$). Hence the possible values of the x_i's are

$$8, 2, 2, 1, 1 \text{ or } 7, 4, 3 \text{ or } 6, 6, 1, 1;$$

the last two cases are illustrated:





Exercise 1.3. By considering colouring k items out of n items using either red or blue to colour each item, show that

$$\sum_{j=0}^{k} \binom{n}{j}\binom{n-j}{k-j} = \binom{n}{0}\binom{n}{k} + \binom{n}{1}\binom{n-1}{k-1} + \binom{n}{2}\binom{n-2}{k-2} + \cdots + \binom{n}{k}\binom{n-k}{0} = 2^k\binom{n}{k}.$$

Solution: Here are two different ways in which we can choose a colouring of the required type.

- We can start by choosing k items to be coloured. There are $\binom{n}{k}$ ways to do this.
- For each element of that set, we can choose one of the two colours. There are 2^k ways to do this.

This shows that there are $2^k\binom{n}{k}$ possibilities altogether.

Alternatively:

- We can start by choosing how many items to colour red. This is a number j with $0 \le j \le k$.
- Then we can choose j of the n items and colour them red. There are $\binom{n}{j}$ ways to do this.
- We now have $n - j$ uncoloured balls, and we choose $k - j$ of them and color them blue, to give k coloured balls altogether. There are $\binom{n-j}{k-j}$ ways to do this.

From this approach we see that the total number of possibilities is $\sum_{j=0}^{k}\binom{n}{j}\binom{n-j}{k-j}$.
Feedback: In the homework, many students gave an algebraic proof, which is also valid, but not really in the spirit of this course. The argument is as follows: we have

$$\begin{aligned}\binom{n}{j}\binom{n-j}{k-j} &= \frac{n!}{j!(n-j)!}\frac{(n-j)!}{(k-j)!(n-k)!} = \frac{n!}{j!(k-j)!(n-k)!} \\ &= \frac{n!}{k!(n-k)!}\frac{k!}{j!(k-j)!} = \binom{n}{k}\binom{k}{j}.\end{aligned}$$

By taking the sum over j, we get

$$\sum_{j}\binom{n}{j}\binom{n-j}{k-j} = \binom{n}{k}\sum_{j}\binom{k}{j} = \binom{n}{k}2^k$$

as claimed.

The identity $\binom{n}{j}\binom{n-j}{k-j} = \binom{n}{k}\binom{k}{j}$ itself has a combinatorial interpretation, by counting the number of ways to colour j items red and $k-j$ items blue, in two different ways.

Exercise 1.4.

(i) Show by each of the following methods that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

(a) Consider the choice of n people from a group consisting of n men and n women.
(b) Use the expansion of $(1+x)^{2n}$.
(c) Count the number of routes in a suitable grid.

(ii) Given a collection of $2n$ people consisting of n men and n women, how many ways can a subset be chosen, the only restriction being that the number of women chosen equals the number of men chosen?

(iii) By considering the number of ways of choosing a subset of a set of n people and one person as a leader in the subset, or otherwise, show that

$$\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n\,2^{n-1}.$$

Solution:

i) Put

$$S = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \sum_{i=0}^{n}\binom{n}{i}^2.$$

We are asked to prove (in several ways) that $S = \binom{2n}{n}$. The first thing to note is that $\binom{n}{i} = \binom{n}{n-i}$, so $\binom{n}{i}^2$ can be rewritten as $\binom{n}{i}\binom{n}{n-i}$, so S is the same as

$$S' = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0} = \sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}.$$

In all three methods, the answer will naturally come out as S' rather than S.

a) There are $\binom{2n}{n}$ ways to choose n people from $2n$. On the other hand, we can proceed as follows. We first decide how many women to include, which will be a number i between 0 and n. We then choose i women (for which there are $\binom{n}{i}$ possibilities) and $n-i$ men (for which there are $\binom{n}{n-i}$ possibilities). This gives $\binom{n}{i}\binom{n}{n-i}$ possibilities involving i women, and thus $\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$ possibilities overall. This number is just S', and it must agree with the answer of $\binom{2n}{n}$ that we obtained more directly, so $S' = \binom{2n}{n}$.

b) $\binom{2n}{n}$ is the coefficient of x^n in the expansion of $(1+x)^{2n}$. However, we can also get $(1+x)^{2n}$ by squaring the polynomial

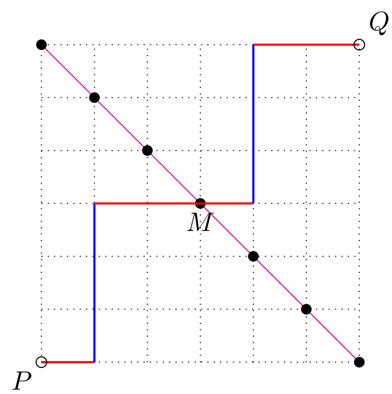
$$p = (1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

To get the coefficient of x^n in p^2, we multiply the coefficients of x^i and x^{n-i} in p, and take the sum over i. These coefficients are $\binom{n}{i}$ and $\binom{n}{n-i}$, so we get

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = S'$$

again.

c) Consider grid paths from P to Q in an $n \times n$ grid, as illustrated below for the case $n = 6$.



As we have discussed previously, the number of such paths is $\binom{2n}{n}$ (because we need to take $2n$ steps, and we need to choose which n of them are horizontal). However, we can choose such a path in a different way. We first choose the point where the path will cross the diagonal line: this will be a point $M = (i, n-i)$ for some $i \in [0, n]$. We then have to choose a grid path from P to M, and a path from M to Q. The first stage crosses a box of size $i \times (n-i)$, so the number of possibilities is $\binom{i+(n-i)}{i} = \binom{n}{i}$. The second stage crosses a box of size $(n-i) \times i$, so the number of possibilities is $\binom{(n-i)+i}{n-i} = \binom{n}{n-i}$. Thus, the nummber of paths passing through $(i, n-i)$ is $\binom{n}{i}\binom{n}{n-i}$, and by taking the sum over i we get $\binom{2n}{n} = S'$ again.

ii) We need to choose a team with the same number of men as women. We first choose the number of women, say i; then we choose i women, for which there are $\binom{n}{i}$ possibilities; then we choose i men, for which there are again $\binom{n}{i}$ possibilities. This gives $\sum_{i=0}^{n} \binom{n}{i}^2$ possibilities altogether, but we saw in part (i) that this is the same as $\binom{2n}{n}$.

Alternatively, we can choose any group of n people, without regard to gender; then we can tell the men to go away, and replace them by the men who were previously left out. The original group must contain i women and $n-i$ men, for some i; the new group will then contain i women and i men. It is not hard to see that this construction gives a one-to-one correspondence between groups of size n and gender-balanced groups, so the number of gender-balanced groups is $\binom{2n}{n}$.

iii) One approach is as follows. We first choose the leader: there are n possibilities for this. There are then $n-1$ remaining people, and we choose some subset of them to form the rest of the team. There are 2^{n-1} possibilities for the subset, and thus $n\,2^{n-1}$ possibilities for the original problem. Alternatively, we can first choose the team size, say i. This must be at least one, because the team must include a leader. We then choose the set of team members: there are $\binom{n}{i}$ possibilities. We then choose one of the team members to be the leader: there are i possibilities. This shows that there are $i\binom{n}{i}$ possibilities for a team of size i, and $\sum_{i=1}^{n} i\binom{n}{i}$ possibilities altogether. These two answers must be the same, so $\sum_{i=1}^{n} i\binom{n}{i} = n\,2^{n-1}$.

It is also possible to give an algebraic proof, as follows:

$$\begin{aligned}\sum_{i=1}^{n} i\binom{n}{i} &= \sum_{i=1}^{n} i\frac{n!}{i!(n-i)!} = \sum_{i=1}^{n} \frac{n!}{(i-1)!(n-i)!} \\ &= n\sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!((n-1)-(i-1))!} = n\sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} \\ &= n(1+1)^{n-1} = n2^{n-1}.\end{aligned}$$

Exercise 1.5. Consider the equation $w_1 + \cdots + w_{10} = 57$, with $w_i \geq i$ for all i. How many solutions are there?

Solution: We can write $w_i = i - 1 + x_i$ with $x_i > 0$. The equation becomes

$$x_1 + (1 + x_2) + (2 + x_3) + \cdots + (9 + x_{10}) = 57.$$

The sum of the extra terms on the left is

$$1 + 2 + \cdots + 9 = \binom{10}{2} = 45$$

(by Proposition 1.18). Thus, we can rearrange to get $x_1 + \cdots + x_{10} = 57 - 45 = 12$. According to Proposition 2.1, the number of solutions is $\binom{12-1}{10-1} = \binom{11}{9} = 55$.

Alternatively, we could write $w_i = i + y_i$ with $y_i \geq 0$. The equation becomes

$$(1 + y_1) + (2 + y_2) + (3 + y_3) + \cdots + (10 + x_{10}) = 57.$$

The sum of the extra terms on the left is

$$1 + 2 + \cdots + 10 = \binom{11}{2} = 55,$$

so we can rearrange to get $y_1 + \cdots + y_{10} = 2$. According to Proposition 2.4, the number of solutions is $\binom{2+10-1}{10-1} = \binom{11}{9} = 55$ again. Alternatively, from $\sum_i y_i = 2$ we see that either

(a) One of the variables y_i is 2, and the rest are zero; or
(b) Two of the variables are 1, and the rest are zero.

For case (a), there are 10 ways to choose which variable is 2. For case (b), there are $\binom{10}{2} = 45$ ways to choose which two variables are 1. The total number of possibilities is again 55.

Exercise 1.6. The monster size Toblerone bar weighs 4.5kg, is 80cm long and contains 12 triangles.

(a) Suppose we want to share the bar among four people, by breaking it into four pieces in the obvious way, but the pieces are allowed to be of different sizes. How many ways are there to do this?
(b) Suppose instead that we want to divide the bar into pieces, but we do not specify in advance how many pieces there should be (and we allow the degenerate case where there is just one piece). How many ways are there to do this?

Solution: There are 11 gaps between the 12 pieces. For part (a), we break the bar at three of the gaps. There are $\binom{11}{3} = 165$ ways to do this. For part (b), we choose an arbitrary set of gaps, and break the bar at those points. There are $2^{11} = 2048$ possible sets of gaps.

We could also do part (a) as follows: if person i gets x_i triangles, then we have $x_1 + x_2 + x_3 + x_4 = 12$ with $x_i > 0$. By Proposition 2.1, the number of solutions is $\binom{12-1}{4-1} = \binom{11}{3} = 165$ again. But this is really the same argument as before, because the proof of Proposition 2.1 effectively involves choosing three gaps. Similarly, for part (b), the number of pieces can be anywhere between 1 and 12. The number of ways of making k pieces is $\binom{12-1}{k-1} = \binom{11}{k-1}$. Thus, the total number of possibilities is

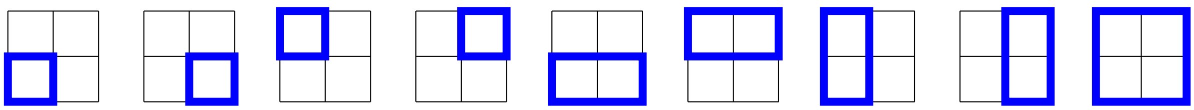
$$\binom{11}{0} + \binom{11}{1} + \cdots + \binom{11}{11}.$$

By the binomial expansion, this is just $(1+1)^{11} = 2^{11} = 2048$ as before.

Exercise 1.7. Consider the equation $u_1 + \cdots + u_{10} = 5 \pmod{10}$, with $0 \leq u_i < 10$. How many solutions are there? (This is easier than Proposition 2.1.)

Solution: Here we can choose $u_1, \ldots, u_9 \in \{0, \ldots, 9\}$ arbitrarily, and then put $u_{10} = 5 - \sum_{i=1}^{9} u_i \pmod{10}$. There are 10 choices for each of $u_1, \ldots, u_9$, making 10^9 possibilities altogether.

Exercise 1.8. The following picture shows that in a 2×2 square, there are 9 different rectangles.



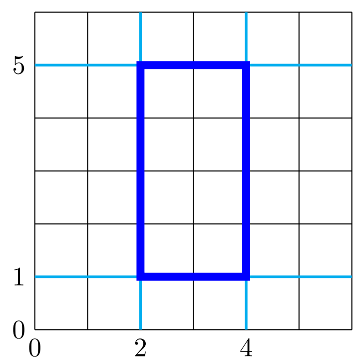
How many rectangles are there in an $n \times n$ square?

(It is easiest to see this by thinking about the sides of the rectangles. If you have trouble spotting a solution, you should begin by considering small examples. Work out what the answer is in the $n \times n$ case, where $n = 1, 2, 3, 4, 5$, by directly counting. Make sure you do this carefully and accurately, as mistakes here will lead to you failing to spot a pattern. Now look for a pattern and *guess* the general answer based on this. Then look for an argument that justifies your guess.)

Solution:

Interactive demo

A rectangle is determined by its sides. These are obtained by choosing any two of the $n+1$ horizontal lines and any two of the $n+1$ vertical lines. For example, the rectangle shown below is determined by horizontal lines 1 and 5, together with vertical lines 2 and 4. So the number of rectangles is $\binom{n+1}{2}^2$.



If you did not see this approach immediately, you could have proceeded as follows. For $n = 1, \ldots, 5$ you should obtain the numbers $1, 9, 36, 100, 225$ by careful enumeration. Now look for a pattern. The first thing you notice is that these numbers are all squares: $1^2, 3^2, 6^2, 10^2, 15^2$. Now you need to see a pattern in the numbers $1, 3, 6, 10, 15$. These you should spot in Pascal's triangle, as $\binom{n+1}{2}$ for $n = 1, 2, 3, 4, 5$. So at this point you *guess* that the general answer is $\binom{n+1}{2}^2$. You still need to find an argument to justify this guess in the general case. But now you know that you need to look in the problem for a choice of two things from $n+1$ things, twice over. Now you should be able to find the two line argument above.

Exercise 1.9. Fix an integer $n > 0$. At some point you have probably seen the identity

$$\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2 = \frac{n^2(n+1)^2}{4}.$$

The usual proof (which is not hard) is by induction. Here we will instead give a bijective proof (due to Benjamin and Orrison).

(a) Put $S = \{(h, i, j, k) \mid 1 \leq h, i, j \leq k \leq n\}$. Show that $|S| = \sum_{k=1}^{n} k^3$.

(b) Put $T_0 = \{(a, b) \mid 1 \leq a \leq b \leq n\}$ and $T = \{(a, b, c, d) \mid 1 \leq a \leq b \leq n,\ 1 \leq c \leq d \leq n\}$. Show that $|T_0| = \binom{n+1}{2}$ and $|T| = \binom{n+1}{2}^2$.

(c) Define $f(h,i,j,k)$ (for $(h,i,j,k) \in S$) and $g(a,b,c,d)$ (for $(a,b,c,d) \in T$) as follows:

$$f(h,i,j,k) = \begin{cases} (h,i,j,k) & \text{if } h \leq i \\ (j,k,i,h-1) & \text{if } h > i \end{cases} \qquad g(a,b,c,d) = \begin{cases} (a,b,c,d) & \text{if } b \leq d \\ (d+1,c,a,b) & \text{if } b > d. \end{cases}$$

Show that $f(h,i,j,k) \in T$ and $g(a,b,c,d) \in S$.

(d) Show that the maps $f\colon S \to T$ and $g\colon T \to S$ are inverse to each other.

(e) Deduce that $\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2$.

Solution:

(a) To choose an element $(h,i,j,k) \in S$, we first choose $k \in \{1,\dots,n\}$, then we choose $h,i,j \in \{1,\dots,k\}$. There are k possible choices for each of h, i and j, so there are k^3 choices for the triple (h,i,j). This makes it clear that $|S| = \sum_{k=1}^{n} k^3$.

(b) For each $(a,b) \in T_0$, we have a subset $\{a, b+1\} \subseteq \{1,\dots,n+1\}$. This gives a bijection between T_0 and the set of subsets of size 2 in $\{1,\dots,n+1\}$, so $|T_0| = \binom{n+1}{2}$. To give an element $(a,b,c,d) \in T$ is the same as to give an ordered pair of elements $(a,b) \in T_0$ and $(c,d) \in T_0$, so $|T| = |T_0|^2 = \binom{n+1}{2}^2$.

(c) Consider an element $(h,i,j,k) \in S$, so h,i,j,k all lie in $\{1,\dots,n\}$. Suppose that $h \leq i$, so $f(h,i,j,k) = (h,i,j,k)$. Here $h \leq i$ by assumption, and $j \leq k$ by the definition of S, so $(h,i,j,k) \in T$. Suppose instead that $h > i$, so $f(h,i,j,k) = (j,k,i,h-1)$. Here $j \leq k$ by the definition of S, and $i \leq h-1$ by the assumption that $h > i$, so $(j,k,i,h-1) \in T$. We therefore have $f(h,i,j,k) \in T$ in all cases.

Now consider an element $(a,b,c,d) \in T$, so that $a \leq b$ and $c \leq d$. Suppose that $b \leq d$ (so $g(a,b,c,d) = (a,b,c,d)$). Together with the inequalities $a \leq b$ and $c \leq d$ this implies that $a,b,c \leq d$ so $(a,b,c,d) \in S$. Suppose instead that $b > d$, so $d+1 \leq b$ and $g(a,b,c,d) = (d+1,c,a,b)$. Together with the inequalities $a \leq b$ and $c \leq d$ this implies that $d+1,c,a \leq b$ so $(d+1,c,a,b) \in S$. We therefore have $g(a,b,c,d) \in S$ in all cases.

(d) Consider again an element $(h,i,j,k) \in S$. Suppose that $h \leq i$, so $f(h,i,j,k) = (h,i,j,k)$. Here $i \leq k$ by the definition of S, so the first clause in the definition of g is applicable and we get $g(f(h,i,j,k)) = g(h,i,j,k) = (h,i,j,k)$. Suppose instead that $h > i$, so $f(h,i,j,k) = (j,k,i,h-1)$. From the definition of S we have $k \geq h$ and so $k > h-1$. This means that the second clause in the definition of g is applicable, giving $g(j,k,i,h-1) = (h-1+1,i,j,k) = (h,i,j,k)$, or in other words $g(f(h,i,j,k)) = (h,i,j,k)$. This shows that $g \circ f = \mathrm{id}\colon S \to S$, and a very similar proof shows that $f \circ g = \mathrm{id}\colon T \to T$, so f and g are inverse to each other.

(e) As we have a bijection between S and T, we see that $|S| = |T|$. Using (a) and (b), this becomes $\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2$.