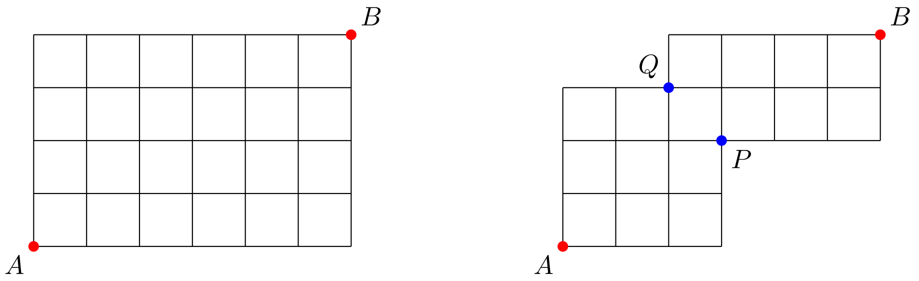
Combinatorics Exam Solutions 2021-22

(1) Consider the following diagram:



We are interested in paths through this grid from A to B (with each path consisting of steps of length one upwards or to the right, as usual).

(a) How many paths are there from A to B in the left hand diagram? **(3 marks)**

(b) How many paths are there from A to B in the right hand diagram?
[Hint: Consider whether such paths pass through P or Q or both or neither.] **(7 marks)**

Solution:

(a) **This is discussed at length in the notes.**

To get from A to B we must take $6 + 4 = 10$ steps, of which 4 must be vertical. The route can be specified completely by deciding which steps are vertical steps, so the number of possibilities is

$$\binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210.[3]$$

(b) **Unseen.**

It is easy to see that every path from A to B must pass through either P or Q, but not both [1]. For a path passing through P, we must first cross a 3×2 grid to go from A to P, then cross another 3×2 grid to go from P to B. By the method used in (a), the number of paths across a $p \times q$ rectangular grid is $\binom{p+q}{p}$ [1]. Thus, the number of routes via P is

$$\binom{3+2}{3} \times \binom{3+2}{3} = 10 \times 10 = 100.[2]$$

On the other hand, for a path passing through Q, we need to cross a 2×3 grid to go from A to Q, then a 4×1 grid to go from Q to B. The number of ways to do this is

$$\binom{2+3}{2} \times \binom{4+1}{4} = 10 \times 5 = 50.[2]$$

Thus, there are 150 routes in total. [1]

(2) Find the number of integer solutions for each of the following problems:

(a) $x_1 + \cdots + x_{10} = 21$ with $x_1, \ldots, x_{10} \geq 0$. **(3 marks)**

(b) $x_1 \times \cdots \times x_{10} = 21$ with $x_1, \ldots, x_{10} \geq 0$. **(3 marks)**

(c) $x_1 + \cdots + x_{10} = 3 \pmod{10}$ with $0 \leq x_1, \ldots, x_{10} < 10$. **(3 marks)**

(d) $x_1^2 + \cdots + x_{10}^2 = 3$ with $x_1, \ldots, x_{10} \in \mathbb{Z}$. **(3 marks)**

Solution:

(a) **Standard.** The number of variables is $k = 10$, the right hand side of the equation is $m = 21$ and the variables are nonnegative integers. By the standard method explained in the notes, the number of solutions is $\binom{m+k-1}{k-1} = \binom{30}{9} = 14307150$. [3]

(b) **Unseen. The most obvious solution is a little longer than that given here.**

A factor of 3 must appear in one of the variables x_i, and a factor of 7 must appear in one of the variables x_j (where j might be equal to i). There are 10 choices for i and 10 choices for j giving 100 solutions altogether. [3]

(c) **Essentially the same problem appeared on a problem sheet.**

Here we can choose $x_1, \dots, x_9$ arbitrarily and then set $x_{10} = (3 - (x_1 + \cdots + x_9)) \pmod{10}$. Thus, the number of solutions is 10^9. [3]

(d) **Unseen.** Here three of the variables must be ± 1 and the rest must be zero. There are $\binom{10}{3} = 120$ ways to choose which variables are nonzero, then $2^3 = 8$ ways to choose the $\pm$ signs, giving $2^3\binom{10}{3} = 960$ solutions altogether. [3]

(3) For a finite nonempty set $A \subset \mathbb{N}$, we define $\text{width}(A) = \max(A) - \min(A)$ (so $\{10, 13, 27\}$ has width 17, for example). Suppose we are given integers $n, w, s \geq 2$ with $s, w < n$. Give a formula for the number of sets $A \subseteq \{1, \dots, n\}$ such that $\text{width}(A) = w$ and $|A| = s$. **(6 marks)**

Solution: This is unseen, but problems with similar ingredients have been seen.

To choose A, we first choose the bottom element i and the top element $j = i + w$ [1]. These must both lie in $\{1, \dots, n\}$, so $i \in \{1, \dots, n - w\}$, so there are $n - w$ choices for this step [1]. We then need to choose $s - 2$ additional elements from the set $\{i + 1, \dots, j - 1\}$ [1], which has size $(j - 1) - (i + 1) + 1 = j - i - 1 = w - 1$ [1]. Thus the number of choices for this step is $\binom{w-1}{s-2}$ [1], and the overall number of choices is $(n - w)\binom{w-1}{s-2}$ [1].

(4) Suppose we have a row of 20 chairs, and we want to seat 5 people with at least two empty chairs between each person and the next person. How many ways are there to do this? **(4 marks)**

Solution: The corresponding problem with gaps of size at least one is discussed at length in the notes. The case of gaps of size at least two is unseen.

This is similar to the gappy set problem. We can seat 5 people arbitrarily in a row of 12 chairs, then insert two extra chairs to the right of each of the first 4 people. This gives every configuration of the required type precisely once, so the number of solutions is $\binom{12}{5} = 792$. [4]

(5) Let a_n be the number of ways of covering an $n \times 2$ board with disjoint dominos.

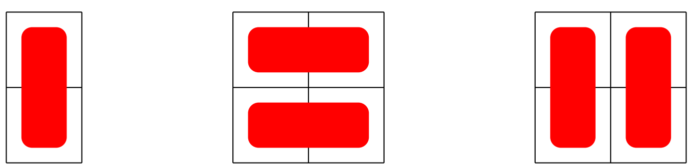
(a) Find a_1 and a_2. **(2 marks)**

(b) By considering how the top left square is covered, show that $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. **(4 marks)**

(c) Thus find a_6. **(3 marks)**

Solution: This is unseen.

(a) To cover a 1×2 board, we have no choice but to use a single vertical domino; so $a_1 = 1$ [1]. To cover a 2×2 board, we can either use two vertical dominoes or two horizontal dominoes; so $a_2 = 2$ [1].

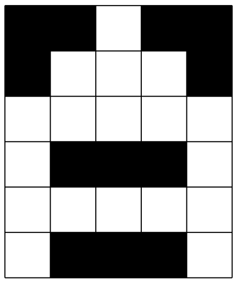


(b) Now suppose we have an $n \times 2$ board, with $n \geq 3$. We can cover the top left square with a vertical domino; this leaves a board of shape $(n-1) \times 2$, and there are a_{n-1} ways to cover that [2]. Alternatively, we can cover the top left square with a horizontal domino. Then the only way to cover the bottom left square is with another horizontal domino placed directly underneath. This leaves a board of shape $(n-2) \times 2$, and there are a_{n-2} ways to cover that [2]. Altogether this gives $a_{n-1} + a_{n-2}$ different ways to cover the original $n \times 2$ board, so $a_n = a_{n-1} + a_{n-2}$.

(c) Using the relation $a_n = a_{n-1} + a_{n-2}$ repeatedly, we get

$$\begin{aligned} a_3 &= a_2 + a_1 = 2 + 1 = 3 \\ a_4 &= a_3 + a_2 = 3 + 2 = 5 \\ a_5 &= a_4 + a_3 = 5 + 3 = 8 \\ a_6 &= a_5 + a_4 = 8 + 5 = 13. \text{[3]} \end{aligned}$$

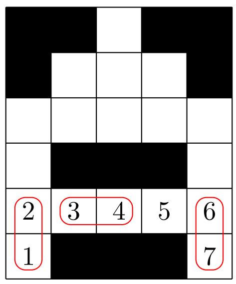
(6) Can the following board be covered by disjoint dominoes? Justify your answer carefully. **(4 marks)**



Solution: Similar examples have been seen.

The only way to cover the square marked 1 is to place a domino over squares 1 and 2 [1]. Having done that, the only way to cover 3 is to place a domino over 3 and 4 [1]. Also, the only way to cover 7 is to place a domino over 6 and 7 [1]. After placing these 3 dominoes, it is clearly impossible to cover 5. [1]

Up to 2 marks could be given for clear and correct statements about colouring and parity, although such considerations do not settle the question here.



(7)

(a) State the pigeonhole principle.**(3 marks)**

(b) Let P be the set of primes p such that $1000 \leq p \leq 2000$. It is given that $|P| = 135$. Show that there exist subsets $\{p_1, p_2\} \subseteq P$ and $\{p_3, p_4\} \subseteq P$ of size 2 such that $\{p_1, p_2\} \neq \{p_3, p_4\}$ but $p_1 + p_2 = p_3 + p_4$. **(5 marks)**

Solution:

(a) **Bookwork.** Any correct statement of the right general nature will be accepted. For example, if $f\colon A \to B$ is a map of finite sets with $|A| > |B|$ then there exist elements $a_1, a_2 \in A$ with $a_1 \neq a_2$ but $f(a_1) = f(a_2)$. [3]

(b) **Similar examples have been seen.**

Let A be the set of subsets of P of size two, so $|A| = \binom{135}{2} = 9045$ [2]. Put $B = \{2000, \ldots, 4000\}$, so $|B| = 2001 < |A|$ [1]. We can define $f\colon A \to B$ by $f(\{p_1, p_2\}) = p_1 + p_2 \in B$ [1]. As $|A| > |B|$, the pigeonhole principle tells us that there are distinct elements $\{p_1, p_2\}, \{p_3, p_4\} \in A$ with $f(\{p_1, p_2\}) = f(\{p_3, p_4\})$, or in other words $p_1 + p_2 = p_3 + p_4$, as required [1]. Explicitly, one such example is given by $(p_1, \ldots, p_4) = (1009, 1031, 1019, 1021)$.

(8)

(a) State the positive form of the inclusion exclusion principle, carefully defining all notation that you use. **(6 marks)**

(b) How many permutations of $\{1,\dots,8\}$ send at least one odd number to itself? **(6 marks)**

Solution:

(a) **Bookwork.** Let A and B be finite sets [1], and let $(B_a)_{a\in A}$ be a family of subsets of B [1]. For $I\subseteq A$ we put $B_I=\bigcap_{a\in I}B_a$ [1]. We also put $B'=\bigcup_{a\in A}B_a$ [1]. Then

$$|B'|=\sum_{I\neq\emptyset}(-1)^{|I|-1}|B_I|.\text{[2]}$$

(Other forms of notation will also be accepted, provided that they are explained clearly.)

(b) **This is an adaptation of the count of derangements which is discussed at length in the notes. Similar adaptations have appeared in earlier exam papers.**

Let P be the set of permutations of $\{1,\dots,8\}$, put $A=\{1,3,5,7\}$, and let P_i be the set of permutations $\sigma\in P$ with $\sigma(i)=i$, and put $P'=P_1\cup P_3\cup P_5\cup P_7$. We must find $|P'|$ [3]. For $I\subseteq A$ we note that the elements of P_I are essentially the same as the permutations of $\{1,\dots,8\}\setminus I$, so $|P_I|=(8-|I|)!$ [1]. For $k=1,\dots,4$ we note that there are $\binom{4}{k}$ possible choices of I with $|I|=k$ [1], and each of these has $|P_I|=(8-k)!$. Thus, the IEP gives

$$|P'|=\binom{4}{1}7!-\binom{4}{2}6!+\binom{4}{3}5!-\binom{4}{4}4!=4\times5040-6\times720+4\times120-1\times24=16296.\text{[1]}$$

(9) Find a board B with rook polynomial $r_B(x)=1+200x+9900x^2$. **(4 marks)**

Solution: Similar problems have been seen.

Because the coefficient of x is 200, we need a board with 200 white squares. Because the polynomial has degree 2, it must be possible to place two non-challenging rooks, but impossible to place 3 non-challenging rooks. This leads us to try taking B to be an empty board of shape 100×2. To place 2 rooks on this board, we have 100 choices for the rook on the first row, and then 99 choices for another rook on the second row, giving 9900 possibilities altogether. Thus, the rook polynomial is $1+200x+9900x^2$ as required. [4] (There is at least one other correct answer; full credit will be given for any answer that is correct.)

(10) Say that a tournament has property P if there are at least three players, and one of them wins every match, and another one loses every match, and all other players have the same score.

(a) Prove that if a tournament has property P, then the number of players is odd. **(4 marks)**

(b) Give an example of a tournament with 7 players that has property P. **(4 marks)**

Solution:

(a) **This is similar to the proof in the notes that if all scores are equal then the number of players is odd.**

Let n be the number of players (so $n\geq3$ by assumption). One player has a score of $n-1$, another has a score of 0, and the remaining $n-2$ players all have the same score, say s. As always, the total score of all players must be $\binom{n}{2}=n^2/2-n/2$. This gives $n-1+0+(n-2)s=n^2/2-n/2$, so $(n-2)s=n^2/2-3n/2+1=(n-1)(n-2)/2$. As $n\geq3$ it is valid to divide by $n-2$. This gives $s=(n-1)/2$, so $n=2s+1$, so n is odd. [4]

(b) **Similar problems have been seen.**

To construct the required tournament we can just take the odd modular tournament of size 5, add in an extra player p who beats all others, and then add in another player q who loses to all others. The result is as follows. [4]

	0	1	2	3	4	p	q
0		W	W	L	L	L	W
1	L		W	W	L	L	W
2	L	L		W	W	L	W
3	W	L	L		W	L	W
4	W	W	L	L		L	W
p	W	W	W	W	W		W
q	L	L	L	L	L	L	

(11) Let L be a 3×4 latin rectangle with entries in $\{1, \dots, 7\}$. Using an appropriate theorem from the notes, show that X can be extended to a 7×7 latin square. **(5 marks)**

Solution: Unseen.

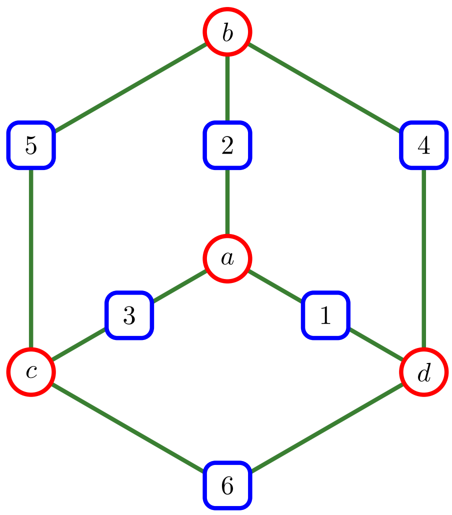
The standard extension theorem refers to a $p \times q$ latin rectangle with entries in $\{1, \dots, n\}$; in the present case we have $p = 3$ and $q = 4$ and $n = 7$ [1]. Let $m_L(k)$ be the number of occurrences of k in L; it is clear that $m_L(k) \geq 0$ [1]. The standard theorem is formulated in terms of the numbers $e_L(k) = m_L(k) + n - p - q$ [1], which is just the same as $m_L(k)$ in this case [1]. The theorem says that L can be extended provided that $e_L(k) \geq 0$ for all k, and this is clearly satisfied. [1]

(12)

(a) Define what is meant by a *block design*. **(6 marks)**

(b) State the three standard identities that hold between the parameters (v, b, r, k, λ) of a block design. **(3 marks)**

(c) Consider the following picture:



From this we can try to make a block design in two different ways.

- For D, we have blocks $\{a, b, c, d\}$ and varieties $\{1, \dots, 6\}$. A block lies in a variety iff they are connected by a line in the diagram.
- For D', we have blocks $\{1, \dots, 6\}$ and varieties $\{a, b, c, d\}$. A block lies in a variety iff they are connected by a line in the diagram.

One of these gives a block design, and the other does not.

(i) For the one that does give a block design, explain why, and find the parameters (v, b, r, k, λ). **(6 marks)**

(ii) For the one that does not give a block design, explain why not. **(3 marks)**

Solution:

(a) **Bookwork.** A block design with parameters (v, b, r, k, λ) consists of a finite set V of varieties, a finite set B of blocks, a collection of subsets $C_j \subseteq V$ for each $j \in B$, [2] and the corresponding sets $R_p = \{j \in B \mid p \in C_j\} \subseteq B$ [1]. These must satisfy

- $|V| = v$ and $|B| = b$
- For each $p \in V$ we have $|R_p| = r$
- For each $j \in B$ we have $|C_j| = k$
- For all $p, q \in V$ with $p \neq q$ we have $|R_p \cap R_q| = \lambda$. [3]

(b) **Bookwork.** The standard relations are $bk = vr$ [1]and $bk(k-1) = \lambda v(v-1)$ [1]and $r(k-1) = \lambda(v-1)$ [1].

(c) **Part (i) is similar to a question on last year's exam. Part (ii) is similar to a question on a problem sheet.**

(i) Definition D' gives a block design with parameters $(v, b, r, k, \lambda) = (4, 6, 3, 2, 1)$ [3]. Indeed, the set $\{a, b, c, d\}$ of varieties has size $v = 4$, and the set $B = \{1, 2, 3, 4, 5, 6\}$ of blocks has size $b = 6$. In the incidence graph each letter is linked to three numbers, so each set R_x has size $r = 3$. Each number is linked to two letters, so each set C_j has size $k = 2$. For any pair of distinct letters, there is precisely one number that is linked to both of them; thus, all sets of the form $R_x \cap R_y$ (with $x \neq y$) has size $\lambda = 1$. [3]

(ii) For D, we have a row set R_p for each number $p \in \{1, \ldots, 6\}$, consisting of the letters that are linked to it. In particular, we have $R_1 = \{a, d\}$ and $R_2 = \{a, b\}$ and $R_5 = \{b, c\}$, so $|R_1 \cap R_2| = 1$ and $|R_1 \cap R_5| = 0$. As the intersections $R_p \cap R_q$ do not all have the same size, this is not a block design. [3]