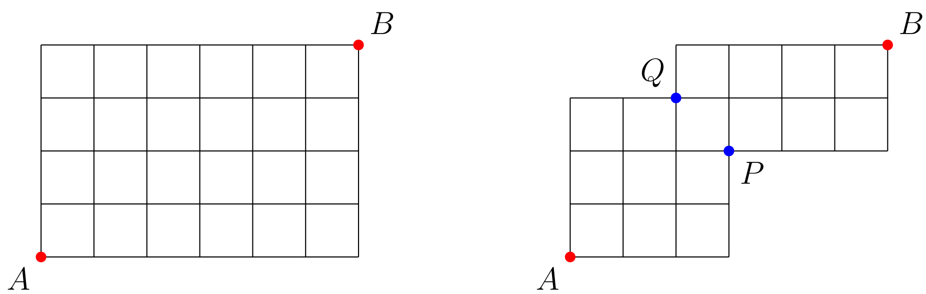
The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2021–22**

Combinatorics **3 hours**

Attempt all the questions. Give justification for all numerical answers. The allocation of marks is shown in brackets.

1 Consider the following diagram:



We are interested in paths through this grid from A to B (with each path consisting of steps of length one upwards or to the right, as usual).

(a) How many paths are there from A to B in the left hand diagram? ***(3 marks)***

(b) How many paths are there from A to B in the right hand diagram?
[Hint: Consider whether such paths pass through P or Q or both or neither.] ***(7 marks)***

2 Find the number of integer solutions for each of the following problems:

(a) $x_1 + \cdots + x_{10} = 21$ with $x_1, \ldots, x_{10} \geq 0$. ***(3 marks)***

(b) $x_1 \times \cdots \times x_{10} = 21$ with $x_1, \ldots, x_{10} \geq 0$. ***(3 marks)***

(c) $x_1 + \cdots + x_{10} = 3 \pmod{10}$ with $0 \leq x_1, \ldots, x_{10} < 10$. ***(3 marks)***

(d) $x_1^2 + \cdots + x_{10}^2 = 3$ with $x_1, \ldots, x_{10} \in \mathbb{Z}$. ***(3 marks)***

3 For a finite nonempty set $A \subset \mathbb{N}$, we define $\text{width}(A) = \max(A) - \min(A)$ (so $\{10, 13, 27\}$ has width 17, for example). Suppose we are given integers $n, w, s \geq 2$ with $s, w < n$. Give a formula for the number of sets $A \subseteq \{1, \ldots, n\}$ such that $\text{width}(A) = w$ and $|A| = s$. ***(6 marks)***

4 Suppose we have a row of 20 chairs, and we want to seat 5 people with at least two empty chairs between each person and the next person. How many ways are there to do this? ***(4 marks)***

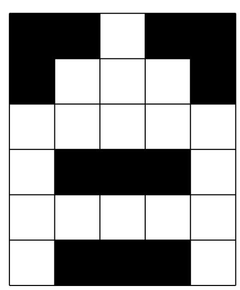
5 Let a_n be the number of ways of covering an $n \times 2$ board with disjoint dominos.

(a) Find a_1 and a_2. ***(2 marks)***

(b) By considering how the top left square is covered, show that $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. ***(4 marks)***

(c) Thus find a_6. ***(3 marks)***

6 Can the following board be covered by disjoint dominoes? Justify your answer carefully. ***(4 marks)***



7

(a) State the pigeonhole principle. ***(3 marks)***

(b) Let P be the set of primes p such that $1000 \leq p \leq 2000$. It is given that $|P| = 135$. Show that there exist subsets $\{p_1, p_2\} \subseteq P$ and $\{p_3, p_4\} \subseteq P$ of size 2 such that $\{p_1, p_2\} \neq \{p_3, p_4\}$ but $p_1 + p_2 = p_3 + p_4$. ***(5 marks)***

8

(a) State the positive form of the inclusion exclusion principle, carefully defining all notation that you use. ***(6 marks)***

(b) How many permutations of $\{1, \dots, 8\}$ send at least one odd number to itself? ***(6 marks)***

9 Find a board B with rook polynomial $r_B(x) = 1 + 200x + 9900x^2$. ***(4 marks)***

10 Say that a tournament has property P if there are at least three players, and one of them wins every match, and another one loses every match, and all other players have the same score.

(a) Prove that if a tournament has property P, then the number of players is odd. ***(4 marks)***

(b) Give an example of a tournament with 7 players that has property P. ***(4 marks)***

11 Let L be a 3×4 latin rectangle with entries in $\{1, \dots, 7\}$. Using an appropriate theorem from the notes, show that X can be extended to a 7×7 latin square. ***(5 marks)***

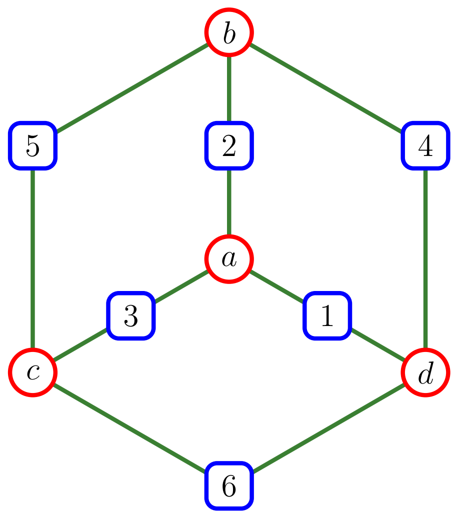
Turn over for Q12

12

(a) Define what is meant by a *block design*. ***(6 marks)***

(b) State the three standard identities that hold between the parameters (v, b, r, k, λ) of a block design. ***(3 marks)***

(c) Consider the following picture:



From this we can try to make a block design in two different ways.

- For D, we have blocks $\{a, b, c, d\}$ and varieties $\{1, \ldots, 6\}$. A block lies in a variety iff they are connected by a line in the diagram.
- For D', we have blocks $\{1, \ldots, 6\}$ and varieties $\{a, b, c, d\}$. A block lies in a variety iff they are connected by a line in the diagram.

One of these gives a block design, and the other does not.

(i) For the one that does give a block design, explain why, and find the parameters (v, b, r, k, λ). ***(6 marks)***

(ii) For the one that does not give a block design, explain why not. ***(3 marks)***

End of Question Paper