Combinatorics Exam Solutions 2019-20

(1) Put $N = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, and consider subsets $U \subseteq N$.

(a) How many subsets are there in total? **(1 marks)**

(b) How many subsets U are there such that U contains at least two odd numbers? **(3 marks)**

(c) How many subsets U are there such that $|U|$ is divisible by 4? **(2 marks)**

(d) Say that $U \subseteq N$ is an *interval* if $|U| > 1$, and whenever $i < j < k$ with $i, k \in U$ we also have $j \in U$. How many intervals are there? **(3 marks)**

Solution: Part (a) is standard, the rest is similar to problems that have been seen.

(a) The total number of subsets is $2^{12} = 4096$. [1]

(b) Let N_0 be the subset of even numbers in N, and let N_1 be the subset of odd numbers, so $|N_0| = |N_1| = 6$. We are looking for subsets of the form $U = U_0 \cup U_1$, where $U_i \subseteq N_i$ and $|U_1| \geq 2$. The number of possibilities for U_0 is $2^6 = 64$. The number of possibilities for U_1 is

$$\binom{6}{2} + \binom{6}{3} + \cdots + \binom{6}{6} = 2^6 - \binom{6}{0} - \binom{6}{1} = 64 - 1 - 6 = 57.$$

Thus, the number of possibilities for U is $64 \times 57 = 3648$. [3]

(c) The number is

$$\binom{12}{0} + \binom{12}{4} + \binom{12}{8} + \binom{12}{12} = 1 + 495 + 495 + 1 = 992.[2]$$

(d) For any subset $\{i, k\} \subset N$ of size 2, we have an interval $\{i, i + 1, \ldots, k\}$. This gives a bijection between subsets of size 2 and intervals, so the number of intervals is $\binom{12}{2} = 66$. [3]

(2) Consider the equation $x_1 + x_2 + x_3 + x_4 = 18$, where the variables x_i are required to be integers.

(a) Find the number of solutions where $i \leq x_i$ for all i. **(2 marks)**

(b) Find the number of solutions where $i \leq x_i$ for all i and also $7 \leq x_3$. **(1 marks)**

(c) Find the number of solutions where $i \leq x_i$ for all i and also $7 \leq x_3$ and $6 \leq x_4$. **(1 marks)**

(d) Find the number of solutions where $i \leq x_i < 10 - i$ for all i. (You will need the Inclusion-Exclusion Principle for this, together with parts (b) and (c) and some similar calculations.) **(8 marks)**

Solution: Parts (a), (b) and (c) are very standard. The method used for (d) has also been seen. We first rewrite everything in terms of the variables $w_i = x_i - i$. The equation becomes $w_1 + w_2 + w_3 + w_4 = 18 - (1+2+3+4) = 8$.

(a) Let B denote the set of solutions for part (a). Here we merely require that $w_i \geq 0$ for all i, and by the standard method, the number of solutions is

$$|B| = \binom{8+3}{3} = \frac{11 \times 10 \times 9}{3 \times 2 \times 1} = 165.[2]$$

(b) Here we can write $x_3 = 7 + v_3$, and $x_i = i + w_i$ for $i \neq 3$. The equation is

$$(w_1 + 1) + (w_2 + 2) + (v_3 + 7) + (w_4 + 4) = 18,$$

or equivalently $w_1 + w_2 + v_3 + w_4 = 4$; the number of solutions is $\binom{4+3}{3} = 35$ [1].

(c) Here we can write $x_3 = 7 + v_3$ and $x_4 = 6 + v_4$ and $x_i = i + w_i$ for $i \neq 3, 4$. The equation is

$$(w_1 + 1) + (w_2 + 2) + (v_3 + 7) + (v_4 + 6) = 18,$$

or equivalently $w_1 + w_2 + v_3 + v_4 = 2$; the number of solutions is $\binom{2+3}{3} = 10$ [1].

(d) Now let $B_i \subseteq B$ be the subset of solutions where $x_i \geq 10 - i$, or equivalently $w_i \geq 10 - 2i$. The set of solutions for (d) is then $B^* = B \setminus (B_1 \cup B_2 \cup B_3 \cup B_4)$, and the IEP gives $|B^*| = \sum_I (-1)^{|I|} |B_I|$ [2]. (Here I runs over subsets of $\{1, 2, 3, 4\}$, and $B_I = \bigcap_{i \in I} B_i$.) In principle, this sum has 16 terms, but many of them are zero. Parts (a), (b) and (c) tell us that $|B_\emptyset| = |B| = 165$ and $|B_3| = 35$ and $|B_{34}| = 10$. Using the same method as in (b), we get

$$|B_1| = \binom{0+3}{3} = 1 \qquad |B_2| = \binom{2+3}{3} = 10 \qquad |B_3| = \binom{4+3}{3} = 35 \qquad |B_4| = \binom{6+3}{3} = 84.\text{[2]}$$

Using the same method as in (c), we get

$$|B_{24}| = \binom{0+3}{3} = 1 \qquad |B_{34}| = \binom{2+3}{3} = 10.\text{[2]}$$

The same method also shows that B_{12} is the set of nonnegative solutions for $(v_1+9)+(v_2+8)+(w_3+3)+(w_4+4) = 18$, or equivalently $v_1 + v_2 + w_3 + w_4 = -6$; this is clearly empty. In fact, we find that all the remaining sets B_I are empty. [1]This gives

$$\begin{aligned} |B^*| &= |B| - |B_1| - |B_2| - |B_3| - |B_4| + |B_{24}| + |B_{34}| \\ &= 165 - 1 - 10 - 35 - 84 + 1 + 10 = 46.\text{[1]} \end{aligned}$$

(3) Let P be the set of all prime numbers p such that $100 \leq p \leq 1000$. You can assume that $|P| = 143$.

(a) Can any of the primes in P be equal to $8 \pmod{12}$? Can any of them be equal to $9 \pmod{12}$? **(3 marks)**

(b) Show that there is a subset $Q \subseteq P$ such that $|Q| = 36$ and all the primes in Q are congruent to each other modulo 12. **(5 marks)**

Solution: Unseen, although other pigeonhole arguments for congruence have been seen.

For $0 \leq k < 12$ put $P_k = \{p \in P \mid p = k \pmod{12}\}$, so P is the disjoint union of the sets P_k. Part (a) asks about the sets P_8 and P_9. If $p \in P_8$ then $p = 8 + 12m$ for some m, so p is even, but the only even prime is 2, and $2 \notin P$ because $2 < 100$, so this is impossible. This means that $P_8 = \emptyset$ [2]. Similarly, if $p \in P_9$ then $p = 9 + 12m$ for some m, so p is divisible by 3. The only prime that is divisible by 3 is 3 itself, and $3 \notin P$ because $3 < 100$, so this is impossible. This means that $P_9 = \emptyset$ [1]. In the same way, we see that $P_0 = P_2 = P_3 = P_4 = P_6 = P_8 = P_9 = P_{10} = \emptyset$, so only the sets P_1, P_5, P_7 and P_{11} can be nonempty [1]. It follows that $|P_1| + |P_5| + |P_7| + |P_{11}| = |P| = 143$ [1]. If all of these sets had $|P_k| \leq 35$ then we would have $|P_1| + |P_5| + |P_7| + |P_{11}| \leq 4 \times 35 = 140$, which is false [2]. Thus, we can choose k with $|P_k| \geq 36$. We can then choose a subset $Q \subseteq P_k$ with $|Q| = 36$. All elements of Q are congruent to $k \pmod{12}$, so they are all congruent to each other modulo 12. [1]

(4) Recall that F_n denotes the $n \times n$ board with all squares white. Let B be a copy of F_5 with a single square blocked off.

(a) What is the relationship between $r_B(x)$, $r_{F_5}(x)$ and $r_{F_4}(x)$? **(2 marks)**

(b) Use this to calculate $r_B(x)$. **(4 marks)**

Solution: It is very standard to use the blocking and stripping relation forwards. The idea of using it backwards is unseen.

(a) The board B is obtained from F_5 by blocking one square, and the corresponding stripping operation converts F_5 to F_4, so we have the standard blocking and stripping relation $r_{F_5}(x) = r_B(x) + x\, r_{F_4}(x)$. [2]

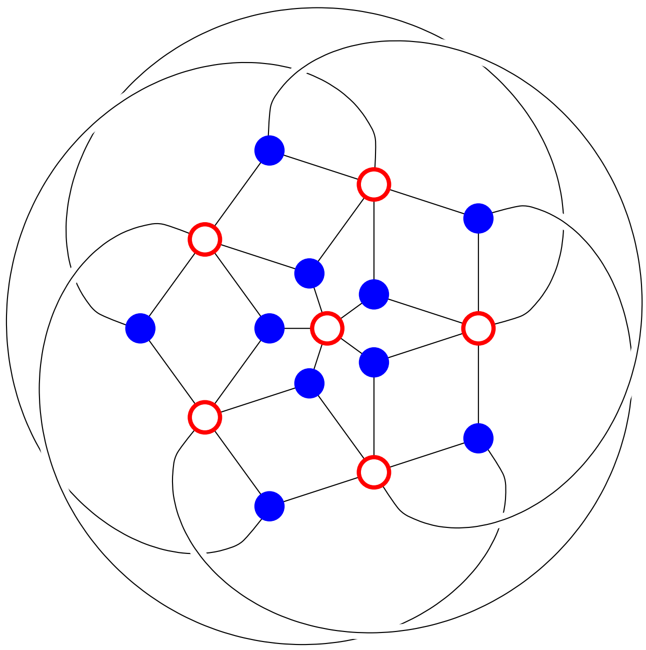
(b) This gives $r_B(x) = r_{F_5}(x) - x\, r_{F_4}(x)$. It is also standard that

$$r_{F_n}(x) = \sum_{k=0}^{n} \binom{n}{k}^2 k!\, x^k. \text{[1]}$$

Using this, we get

$$\begin{aligned} r_{F_5}(x) &= 1 + (5^2 \times 1)x + (10^2 \times 2)x^2 + (10^2 \times 6)x^3 + (5^2 \times 24)x^4 + (1^2 \times 120)x^5 \\ &= 1 + 25x + 200x^2 + 600x^3 + 600x^4 + 120x^5 \text{[1]} \\ r_{F_4}(x) &= 1 + (4^2 \times 1)x + (6^2 \times 2)x^2 + (4^2 \times 6)x^3 + (1^2 \times 24)x^4 \\ &= 1 + 16x + 72x^2 + 96x^3 + 24x^4 \text{[1]} \\ r_B(x) &= 1 + 24x + 184x^2 + 528x^3 + 504x^4 + 96x^5. \text{[1]} \end{aligned}$$

(5) Consider the following picture:



(Note that there are five curved lines, each one joining a vertex of the outer pentagon to the middle of the opposite edge.)

From this we can try to construct a block design. We have a block for each filled blue circle, and a variety for each unfilled red circle. A given variety lies in a given block iff there is a line joining the corresponding circles.

(a) Explain briefly why this does indeed give a block design, and find the corresponding parameters (v, b, r, k, λ). **(5 marks)**

(b) Write down the standard equations relating these parameters, and check that they are satisfied in this case. **(2 marks)**

Solution: This is unseen, but straightforward.

- v must be the number of varieties, or in other words the number of unfilled red circles, which is 6. [1]
- b must be the number of blocks, or in other words the number of filled blue circles, which is 10. [1]
- For this to be a block design, there must be a number r such that every variety lies in precisely r blocks, or in other words, every red circle is connected to precisely r blue circles. By inspection, every red circle is connected to precisely 5 blue circles, so $r = 5$. [1]

- For this to be a block design, there must be a number k such that every block contains precisely k varieties, or in other words, every blue circle is connected to precisely k red circles. By inspection, every blue circle is connected to precisely 3 blue circles, so $k = 3$. [1]
- For this to be a block design, there must be a number λ such that every pair of distinct varieties lies in precisely λ blocks. In other words, for every pair of red circles, there must be precisely λ blue circles that are connected to both of them. Close inspection shows that this is satisfied for $\lambda = 2$. [1]
- The standard equations are shown on the left below. On the right, we have filled in the values $(v, b, r, k, \lambda) = (6, 10, 5, 3, 2)$. It is clear that all the resulting equations are satisfied. [2]

$$
\begin{aligned}
bk &= rv & \qquad 10 \times 3 &= 6 \times 5 \\
bk(k-1) &= \lambda v(v-1) & \qquad 10 \times 3 \times 2 &= 2 \times 6 \times 5 \\
r(k-1) &= \lambda(v-1) & \qquad 5 \times 2 &= 2 \times 5.
\end{aligned}
$$

(6) Suppose we need to recruit people as follows:

(1) 1 rocket scientist

(2) 10 brain surgeons

(3) 100 hamburger chefs.

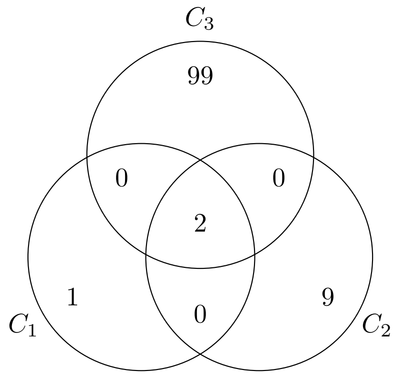
We have 111 candidates in total; let C_i be the set of candidates who are qualified for job i. It is given that

$$|C_1| = 3 \qquad |C_2| = 11 \qquad |C_3| = 101 \qquad |C_1 \cap C_2| = |C_1 \cap C_3| = |C_2 \cap C_3| = |C_1 \cap C_2 \cap C_3| = 2.$$

Can the job allocation problem be solved? Justify your answer. **(8 marks)**

Solution: Put $T = C_1 \cap C_2 \cap C_3$ (the set of multitalented people who can do all three jobs) and $C_i' = C_i \setminus T$. It is given that $|T| = 2$. We also have $|C_i'| = |C_i| - |T| = |C_i| - 2$, so $|C_1'| = 1$ and $|C_2'| = 9$ and $|C_3'| = 99$ [3].

Note that the sets $C_1 \cap C_2$, $C_1 \cap C_3$ and $C_2 \cap C_3$ all contain T and have size 2 so they are equal to T. In other words, anyone who can do two jobs can in fact do all three [2]. It follows that the sets C_1', C_2' and C_3' are disjoint [1]. We can thus allocate everyone in C_i' to do job i, allocate one multitalented person to be a brain surgeon, and allocate the other multitalented person to be a hamburger chef; this solves the allocation problem [2].



As an alternative, we can verify the Hall plausibility conditions. We need to show that

$$
\begin{array}{ccc}
|C_1| \geq 1 & |C_2| \geq 10 & |C_3| \geq 100 \\
|C_1 \cup C_2| \geq 11 & |C_1 \cup C_3| \geq 101 & |C_2 \cup C_3| \geq 110 \\
|C_1 \cup C_2 \cup C_3| \geq 111. & &
\end{array}
$$

The inequalities on the first row are immediate from the given data. For the second row, the IEP gives $|C_i \cup C_j| = |C_i| + |C_j| - |C_i \cap C_j|$. It is given that $|C_i \cap C_j| = 2$ for all $i \neq j$, so $|C_i \cup C_j| = |C_i| + |C_j| - 2$. This gives

$$|C_1 \cup C_2| = 12 \qquad |C_1 \cup C_3| = 102 \qquad |C_2 \cup C_3| = 110,$$

so the inequalities in the second row are satisfied. Similarly, we have

$$|C_1 \cup C_2 \cup C_3| = 3 + 11 + 101 - 2 - 2 - 2 + 2 = 111,$$

so the final inequality is also satisfied. It follows by the team version of Hall's Theorem that the allocation problem can be solved.