The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2020–21**

Combinatorics

This is an open book exam.
Answer all questions.
You can work on the exam during the 24 hour period starting at 10am (GMT), and you must submit your work within three hours of accessing the exam paper or by the end of the 24 hour period (whichever is earlier).
Late submission will not be considered without extenuating circumstances.
Unless it is explicitly stated otherwise, it is intended that calculations are performed by hand (possibly with the aid of a calculator). To gain full marks, you will need to show your working.
By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged, and that no unfair means have been used.

1 Put $N = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, and consider subsets $U \subseteq N$.

(a) How many subsets are there in total? ***(1 mark)***

(b) How many subsets U are there such that U contains at least two odd numbers? ***(3 marks)***

(c) How many subsets U are there such that $|U|$ is divisible by 4? ***(2 marks)***

(d) Say that $U \subseteq N$ is an *interval* if $|U| > 1$, and whenever $i < j < k$ with $i, k \in U$ we also have $j \in U$. How many intervals are there? ***(3 marks)***

2 Consider the equation $x_1 + x_2 + x_3 + x_4 = 18$, where the variables x_i are required to be integers.

(a) Find the number of solutions where $i \leq x_i$ for all i. ***(2 marks)***

(b) Find the number of solutions where $i \leq x_i$ for all i and also $7 \leq x_3$. ***(1 mark)***

(c) Find the number of solutions where $i \leq x_i$ for all i and also $7 \leq x_3$ and $6 \leq x_4$. ***(1 mark)***

(d) Find the number of solutions where $i \leq x_i < 10 - i$ for all i. (You will need the Inclusion-Exclusion Principle for this, together with parts (b) and (c) and some similar calculations.) ***(8 marks)***

3 Let P be the set of all prime numbers p such that $100 \leq p \leq 1000$. You can assume that $|P| = 143$.

(a) Can any of the primes in P be equal to $8 \pmod{12}$? Can any of them be equal to $9 \pmod{12}$? ***(3 marks)***

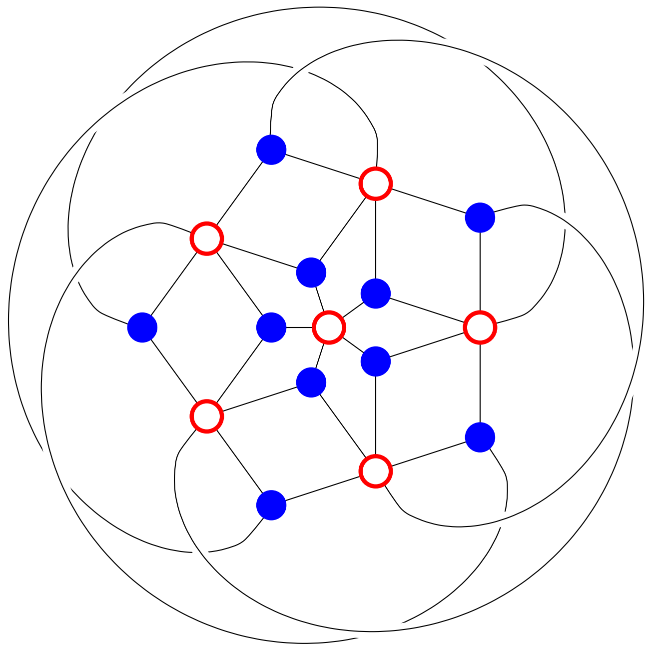
(b) Show that there is a subset $Q \subseteq P$ such that $|Q| = 36$ and all the primes in Q are congruent to each other modulo 12. ***(5 marks)***

4 Recall that F_n denotes the $n \times n$ board with all squares white. Let B be a copy of F_5 with a single square blocked off.

(a) What is the relationship between $r_B(x)$, $r_{F_5}(x)$ and $r_{F_4}(x)$? ***(2 marks)***

(b) Use this to calculate $r_B(x)$. ***(4 marks)***

5 Consider the following picture:



(Note that there are five curved lines, each one joining a vertex of the outer pentagon to the middle of the opposite edge.)

From this we can try to construct a block design. We have a block for each filled blue circle, and a variety for each unfilled red circle. A given variety lies in a given block iff there is a line joining the corresponding circles.

(a) Explain briefly why this does indeed give a block design, and find the corresponding parameters (v, b, r, k, λ). ***(5 marks)***

(b) Write down the standard equations relating these parameters, and check that they are satisfied in this case. ***(2 marks)***

6 Suppose we need to recruit people as follows:

(1) 1 rocket scientist

(2) 10 brain surgeons

(3) 100 hamburger chefs.

We have 111 candidates in total; let C_i be the set of candidates who are qualified for job i. It is given that

$$|C_1| = 3 \qquad |C_2| = 11 \qquad |C_3| = 101 \qquad |C_1 \cap C_2| = |C_1 \cap C_3| = |C_2 \cap C_3| = |C_1 \cap C_2 \cap C_3| = 2.$$

Can the job allocation problem be solved? Justify your answer. ***(8 marks)***

End of Question Paper