The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Autumn Semester 2019–20**

Combinatorics — **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 Put $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and consider subsets $U \subseteq N$.

(a) How many subsets are there in total? ***(1 mark)***

(b) How many subsets U are there such that U contains at least one odd number? ***(2 marks)***

(c) How many subsets U are there such that $|U|$ is odd? ***(2 marks)***

(d) How many subsets U are there such that $U \neq \emptyset$ and $\max(U)$ is even? ***(3 marks)***

2 State and prove Pascal's relation for binomial coefficients. ***(5 marks)***

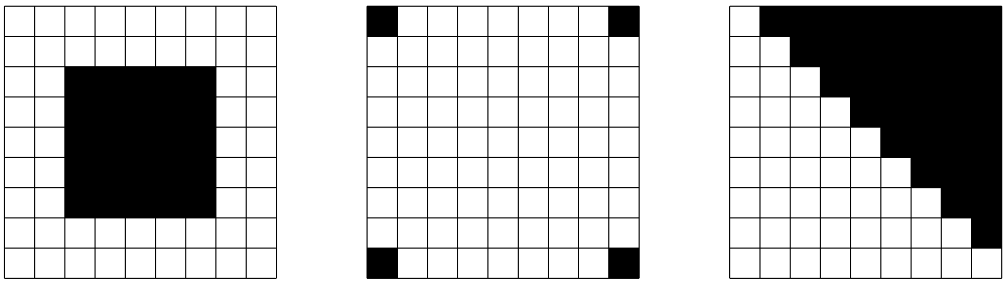
3 Consider the equation $x_1 + x_2 + \cdots + x_{10} = 16$.

(a) How many solutions are there with $1 \leq x_i$ for all i? ***(2 marks)***

(b) How many solutions are there with $1 \leq x_i \leq 2$ for all i? ***(2 marks)***

(c) How many solutions are there with x_i odd and positive for all i? ***(3 marks)***

4 Consider $n \times n$ boards as illustrated below for the case $n = 9$: in A_n the middle $(n-4)\times(n-4)$ square is black, in B_n the four corners are black, and in C_n everything above and to the right of the diagonal is black. We will investigate whether they can be covered by disjoint dominos. Some answers will depend on n, and some will not. We will always assume that $n \geq 5$.



A_9 B_9 C_9

(a) Can A_n be covered by disjoint dominos? ***(2 marks)***

(b) Can B_n be covered by disjoint dominos? ***(4 marks)***

(c) Suppose we colour C_n with alternating white and grey squares in the usual chessboard pattern, with the bottom left square being grey. By considering diagonal stripes, find the number of white squares and grey squares (this will depend on whether n is odd or even). ***(6 marks)***

(d) Can C_n be covered by disjoint dominos? ***(1 mark)***

5 (a) State the inclusion-exclusion principle. ***(3 marks)***

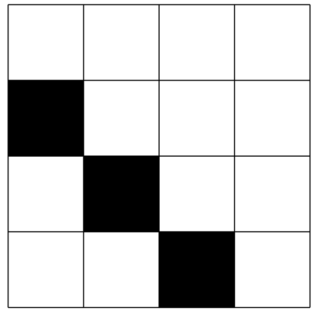
(b) Let B be a finite set with $|B| = n$. Suppose we have subsets $B_i \subset B$ for $i = 1, \ldots, m$, such that $|B_{i_1} \cap \cdots \cap B_{i_r}| = n/3^r$ for all $i_1, \ldots, i_r$ with $1 \leq r \leq m$ and $i_1 < \cdots < i_r$. Give a fully simplified formula for $|B_1 \cup \cdots \cup B_m|$. ***(5 marks)***

6 (a) State the Pigeonhole Principle. ***(2 marks)***

(b) Suppose we have a square with sides of length r. What is the maximum possible distance between two points in the square? ***(1 mark)***

(c) Suppose we have marked 10 points in a 3×3 square. Use the Pigeonhole Principle to show that there are two marked points such that the distance between them is less than or equal to $\sqrt{2}$. ***(4 marks)***

7 Consider the following board B:



(a) Calculate the rook polynomial of the complementary board $\overline{B}$. ***(2 marks)***

(b) Use (a) to calculate the number of ways of placing 4 non-challenging rooks on B. ***(4 marks)***

(c) Using (b), draw a board B' such that there are precisely 121 ways to place 8 non-challenging rooks on B'. ***(2 marks)***

8 (a) State Landau's theorem on scores in tournaments. ***(4 marks)***

(b) Suppose we have a tournament with $n > 0$ players, in which every player has the same score. Show that n must be odd. ***(3 marks)***

(c) Give an example of a tournament of 5 players in which every player has the same score. ***(3 marks)***

(d) Use Landau's criterion to show that there is a tournament of 6 players in which every player scores 2 or 3. ***(4 marks)***

(e) Find an explicit example of a tournament as in (d). ***(4 marks)***

9 Find numbers $a, \ldots, q$ such that the following matrix becomes a latin square: ***(6 marks)***

$$\begin{bmatrix} 0 & 1 & \mathbf{a} & \mathbf{f} & \mathbf{g} \\ \mathbf{d} & \mathbf{b} & 4 & 0 & \mathbf{c} \\ \mathbf{l} & \mathbf{m} & \mathbf{n} & \mathbf{h} & \mathbf{k} \\ \mathbf{i} & 3 & 2 & \mathbf{e} & \mathbf{j} \\ \mathbf{o} & \mathbf{p} & \mathbf{q} & 1 & 3 \end{bmatrix}$$

Explain your reasoning for at least three of the entries.

10 Recall that if there is a block design with parameters (v, b, r, k, λ) then the following equations are satisfied:

$$(A) \quad bk = vr \qquad\qquad (B) \quad r(k-1) = \lambda(v-1)$$

(a) Prove equation (A). ***(4 marks)***

(b) Suppose that $k = 2\lambda + 1$ and $b = 2k + 1$. Prove that $v = b$ and $r = k$. [Hint: write $r = k + s$, rewrite everything in terms of v, λ and s, then use (B) to find v, then use (A).] ***(8 marks)***

(c) Suppose that $b = v = 7$ and the first six column sets are as follows:

$$C_1 = \{2,4,6\} \qquad C_2 = \{1,4,5\} \qquad C_3 = \{3,4,7\}$$
$$C_4 = \{1,2,3\} \qquad C_5 = \{2,5,7\} \qquad C_6 = \{1,6,7\}$$

Find the parameters r, k and λ, the corresponding row sets R_i, and the last column set C_7. ***(8 marks)***

End of Question Paper