MAS334 COMBINATORICS — AUTUMN SEMESTER 2017-2018
EXAM SOLUTIONS AND MARK SCHEME

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Solution to Question 1

(ia) **(bookwork)** For $n \neq 0$ and all k,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(2 Marks)

(ib) **(bookwork)** For $n \in \mathbb{N}$,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \qquad \textbf{(2 Marks)}$$

(ic) **(unseen, similar to seen problem)**
Multiplying both sides of the equation appearing in the Binomial Theorem by $1+x$, we have

$$\begin{aligned}(1+x)^{n+1} &= (1+x)\sum_{k=0}^{n} \binom{n}{k} x^k \qquad \textbf{(1 Mark)} \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k + \sum_{k=0}^{n} \binom{n}{k} x^{k+1} \\ &= \sum_{k=0}^{n+1} \left(\binom{n}{k} + \binom{n}{k-1} \right) x^k \qquad \textbf{(2 Marks)}\end{aligned}$$

Thus

$$\sum_{k=0}^{n+1} \binom{n+1}{k} x^k = \sum_{k=0}^{n+1} \left(\binom{n}{k} + \binom{n}{k-1} \right) x^k,$$

and equating coefficients gives Pascal's Identity. **(1 Mark)**

(iia) **(standard example)** By a standard procedure, with a horizontal grid line for each variable and measuring progress to the right along the horizontal grid lines, these solutions are in bijection with shortest routes from bottom left to top right in a 19 by $4-1=3$ grid. **(2 Marks)**

Since the 3 up steps can be taken at any stage, there are $\binom{22}{3} = 1540$ such solutions. **(1 Mark)** (iib) **(unseen, standard type of example)** Let property 1 be $x_1 > 2$, property 2 be $x_2 > 3$ and property 3 be $x_3 > 4$. We want the number of solutions with at least one of the properties. **(2 Marks)**

Using usual inclusion/exclusion notation, this is given by

$$N(1) + N(2) + N(3) - N(1,2) - N(1,3) - N(2,3) + N(1,2,3).$$

(1 Mark)

Write $x_1 = x_1' + 3$. Then $N(1)$ is the number of non-negative integer solutions to

$$x_1' + x_2 + x_3 + x_4 = 16,$$

and this is $\binom{19}{3}$, by part (a). **(1 Mark)**

Similarly,

$$N(2) = \binom{18}{3}, N(3) = \binom{17}{3}, N(1,2) = \binom{15}{3},$$
$$N(1,3) = \binom{14}{3}, N(2,3) = \binom{13}{3}, N(1,2,3) = \binom{10}{3}.$$ **(3 Marks)**

So we get

$$\binom{19}{3} + \binom{18}{3} + \binom{17}{3} - \binom{15}{3} - \binom{14}{3} - \binom{13}{3} + \binom{10}{3} = 1480.$$

(1 Mark)

[**Alternative:** Observe that we want the total minus solutions with $x_1 \leq 2$ and $x_2 \leq 3$ and $x_3 \leq 4$. With these conditions there are 3 possible values for x_1, 4 for x_2 and 5 for x_3. All combinations of these are possible and x_4 is then uniquely determined, so we get $1540 - 3 \cdot 4 \cdot 5 = 1540 - 60 = 1480$.]

(iii) **(unseen)** Let $t_n = \sum_{i=0}^{n} (i+1)\binom{n}{i}$. It's enough to show that t_n satisfies the same initial condition and recurrence relation as s_n. **(1 Mark)**

First, we see that $t_0 = 1$. **(1 Mark)**

Then

$$t_n = \sum_{i=0}^{n} (i+1)\binom{n}{i} = \sum_{i=0}^{n} (i+1)\left(\binom{n-1}{i} + \binom{n-1}{i-1}\right)$$

by Pascal's Identity

$$= t_{n-1} + \sum_{i=1}^{n} (i+1)\binom{n-1}{i-1}$$
$$= t_{n-1} + \sum_{j=0}^{n-1} (j+2)\binom{n-1}{j}$$
$$= t_{n-1} + \sum_{j=0}^{n-1} (j+1)\binom{n-1}{j} + \sum_{j=0}^{n-1} \binom{n-1}{j}$$
$$= 2t_{n-1} + 2^{n-1} \qquad \text{by the Binomial Theorem.}$$ **(4 Marks)**

Solution to Question 2

(ia) **(unseen, standard)** Either n or $n-3$ is even, so we can fill the board row-wise or column-wise by dominoes. **(2 Marks)**

(ib) **(unseen, standard)** If n is odd, each row has an even number of squares ($n-5$ for the top row, $n-3$ for the rest), so we can fill with dominoes row-wise. **(1 Mark)**

If n is even, consider colouring alternating squares black and white. The full $(n-3) \times n$ board would have equal numbers of each, since the columns have an even number of squares. The two removed corner squares are of the same colour, so the resulting shape has unequal numbers of black and white squares. It is thus impossible to cover it completely with dominoes, since a domino covers one square of each colour. **(3 Marks)**

(iia) **(bookwork)** If more than n items are placed in n pigeon-holes, then some pigeon-hole will contain more than one item. **(2 Marks)**

(iib) **(unseen)** The minimum possible sum of 5 elements from $\{1, 2, \ldots, 100\}$ is $1 + 2 + 3 + 4 + 5 = 15$. The maximum such sum is $100 + 99 + 98 + 97 + 96 = 490$. **(1 Mark)**

Label 476 pigeon-holes with the possible sums $15, 16, \ldots, 490$. **(1 Mark)**

There are $\binom{12}{5} = 792$ subsets of X with 5 elements. **(1 Mark)**

Place each of these in the pigeon-hole according to the sum of its elements. Since there are more items than pigeon-holes, by the PHP, two such subsets are in the same pigeon-hole and have the same sum. **(1 Mark)**

(iiia) **(unseen, easy)** No. We can only pick 7 from A_3, meaning we have to pick 6 from A_4, but only A_4 contains a 3. **(1 Mark)**

(iiib) **(bookwork)** Any r of the sets have at least r elements between them. **(2 Marks)**

(iva) **(unseen)** The number of permutations of $\{1, 2, \ldots, n\}$ is $n!$ and each permutation fixes some number of integers k, with $0 \leq k \leq n$. **(1 Mark)**

The number of permutations fixing a given k integers (and none of the rest) is equal to the number of derangements of the rest, d_{n-k}. **(1 Mark)**

So the number of permutations fixing exactly k integers is equal to the number of ways of choosing the k integers times the number of derangements of the rest, $\binom{n}{k} d_{n-k}$. **(1 Mark)**

Thus $\sum_{k=0}^{n} \binom{n}{k} d_{n-k} = n!$. **(1 Mark)**

(ivb) **(unseen)** There are $n-1$ possibilities for the image of 1 under a derangement, and the same number of derangements sending 1 to each of these $n-1$ possibilities. **(1 Mark)**

Therefore $d_n = (n-1)d'_n$, where d'_n is the number of derangements sending 1 to 2. **(1 Mark)**

A derangement sending 1 to 2 either sends 2 to 1 or 2 to some other number. **(1 Mark)**

In the first case, the number of such is equal to the number of derangements of $\{3, 4, \ldots, n\}$, that is, d_{n-2}. **(1 Mark)**

In the second case, the number of such is equal to the number of derangements of $\{1, 3, 4, \ldots, n\}$, that is, d_{n-1}. **(1 Mark)**
So $d'_n = d_{n-2} + d_{n-1}$ and $d_n = (n-1)(d_{n-2} + d_{n-1})$. **(1 Mark)**

Solution to Question 3

(i) **(unseen, standard)**
There are many ways of calculating, using Theorem 43 (select a square) and Theorem 46 (disjoint boards) from the course. The obvious one for this example is to select the third square on the third row down, to obtain disjoint boards at the next stage.
We get:

$$r_B(x) = r_C(x) + x r_D(x),$$

where C consists of two disjoint boards, a full 2×1 and a full 2×3 board and D consists of disjoint full 1×1 and 2×2 boards.
(3 Marks)
So

$$\begin{aligned} r_B(x) &= (1+2x)(1+6x+6x^2) + x(1+x)(1+4x+2x^2) \\ &= 1 + 9x + 23x^2 + 18x^3 + 2x^4 . \qquad \textbf{(3 Marks)} \end{aligned}$$

[**Alternative:** Direct counting by hand will be given full marks if the correct answer is obtained (even without justification); partial marks may be obtained for a partially correct answer depending on the explanation given for the counting procedure adopted.]

(iia) **(unseen)** We apply the "select a square" theorem, choosing the top right square s. **(1 Mark)**
Deleting s produces a board consisting of n disjoint 1×1 boards, and so having rook polynomial $(1+x)^n$. **(1 Mark)**
Deleting the row and column of s produces a board consisting of $n-2$ disjoint 1×1 boards, and so having rook polynomial $(1+x)^{n-2}$. **(1 Mark)**
Thus the rook polynomial of B_n is

$$(1+x)^n + x(1+x)^{n-2}.$$

(1 Mark)
The number of ways of placing k non-challenging rooks on B_n is the coefficient of x^k, which is $\binom{n}{k} + \binom{n-2}{k-1}$, as required. **(2 Marks)**

(iib) **(unseen)** Now apply the "select a square" theorem, choosing the bottom left square s. **(1 Mark)**
Deleting s produces B_n. **(1 Mark)**
Deleting the row and column of s produces a board consisting of $n-1$ disjoint 1×1 boards, and so having rook polynomial $(1+x)^{n-1}$. **(1 Mark)**

Thus

$$\begin{aligned} r_{C_n}(x) &= r_{B_n}(x) + x(1+x)^{n-1} \\ &= (1+x)^n + x(1+x)^{n-2} + x(1+x)^{n-1} \\ &= \sum_{k=0}^{n} \left(\binom{n}{k} + \binom{n-1}{k-1} + \binom{n-2}{k-1} \right) x^k. \end{aligned}$$

(3 Marks)

(iiia) **(unseen)** The total of the scores is $28 = \binom{8}{2}$. **(1 Mark)**

The sum of any r is at least the sum of the smallest r, shown in the table and this is at least $\binom{r}{2}$.

r	sum of r smallest scores	$\binom{r}{2}$
1	2	0
2	4	1
3	6	3
4	10	6
5	14	10
6	18	15
7	22	21

(2 Marks)

So, by Landau's Theorem, they are the scores of a tournament. **(1 Mark)**

(iiib) **(unseen)** Consider putting together two sub-tournaments of 8 players, each as in part (a), such that the players of one sub-tournament beat all the players in the other. So the bottom sub-tournament players retain their scores as in (a). And each of the top sub-tournament players adds 8 to their score. This gives the required tournament. **(3 Marks)**

Solution to Question 4

(i) **(bookwork)** Let $L(i)$ denote the number of times i appears in the given $p \times q$ Latin rectangle. This is extendable to an $n \times n$ Latin square if and only if $L(i) \geq p + q - n$ for each $i \in \{1, \ldots, n\}$. **(2 Marks)**

(ii) **(standard problem)** We want to extend a $p \times q$ Latin rectangle to an $n \times n$ Latin square, where $p = 4$, $q = 4$ and $n = 6$. This is possible iff $L(i) \geq 4 + 4 - 6 = 2$ for $1 \leq i \leq 6$, where $L(i)$ denotes the number of occurrences of i in the given rectangle. This happens iff $x = 2$. **(2 Marks)**

One extension is

$$\begin{pmatrix} 1 & 4 & 2 & 3 & 6 & 5 \\ 4 & 1 & 6 & 5 & 2 & 3 \\ 6 & 3 & 5 & 4 & 1 & 2 \\ 2 & 5 & 4 & 1 & 3 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \\ 5 & 6 & 3 & 2 & 4 & 1 \end{pmatrix}$$

(**4 Marks**)

[Marking: 2 marks for any correct extension to 4×6 or 6×4; 2 marks for any correct extension from there to 6×6. Partial marks available for "partially correct" extensions.]

(iii) **(bookwork)** Let L_1, L_2, ..., L_q be mutually orthogonal $n \times n$ Latin squares. We need to show that $q \leq n-1$. **(1 Mark)**
If the first row of L_1 is $(a_1, a_2, \ldots, a_n)$, then replace a_i by i throughout L_1 to give L_1'. **(1 Mark)**
Now L_1' is still a Latin square and it is straightforward to see that it is still orthogonal to all the rest. **(1 Mark)**
Repeat this process for L_2, ..., L_q. **(1 Mark)**
Then we have q mutually orthogonal Latin squares L_1', L_2', ..., L_q', all with first row $(1, 2, \ldots, n)$. **(1 Mark)**
In each of these q Latin squares, consider the $(2,1)$ entry. Because the squares are Latin none of these is a 1, so these entries are all in $\{2, \ldots, n\}$. **(1 Mark)**
Also, they are all different, because a repeat of x, say, in this position in L_i' and L_j' would mean that the pair (x, x) occurs twice among the $\left((L_i')_{ab}, (L_j')_{ab}\right)$, corresponding to position $(1, x)$ and to position $(2,1)$, contradicting the orthogonality of L_i' and L_j'. **(1 Mark)**
So the q entries in the $(2,1)$ positions of the squares are different elements of $\{2, \ldots, n\}$ and so $q \leq n-1$ as required. **(1 Mark)**

(iv) **(bookwork)** The number of blocks of the design each variety appears in is r. **(1 Mark)**
The number of varieties per block is k. **(1 Mark)**
The number of blocks of the design each pair of varieties appears in is λ. **(1 Mark)**

(v) **(unseen)** We number of varieties, v, is $2^4 - 1 = 15$. **(1 Mark)**
The number of varieties per block, k, is 3. **(1 Mark)**
The number of blocks, b, is $\frac{15 \cdot 14}{6}$, since there are 15 choices for $\mathbf{x}$, leaving 14 for $\mathbf{y}$, with $\mathbf{z}$ determined and different permutations of $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ give the same block. **(1 Mark)**
Each pair of vectors $\mathbf{x}$, $\mathbf{y}$ uniquely determines the third $\mathbf{z}$ and thus a block, so each pair is in one block. Thus this is a design, with $\lambda = 1$. **(1 Mark)**
Then $r = \frac{bk}{v} = \frac{35 \cdot 3}{15} = 7$. **(1 Mark)**
Thus this is a design with parameters $(v = 15; b = 35; r = 7; k = 3; \lambda = 1)$. **(1 Mark)**