The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2016–17**

Combinatorics **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) (a) How many solutions are there of the equation

$$y_1 + y_2 + \cdots + y_k = n,$$

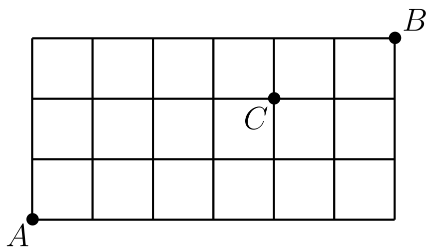
in which each y_i is a positive integer? Give a brief reason for your answer. ***(3 marks)***

(b) How many integer solutions are there to the equation

$$y_1 + y_2 + y_3 + y_4 = 37,$$

such that $y_1 > 1$, $y_2 > 2$, $y_3 > 3$ and $y_4 > 4$? ***(5 marks)***

(ii) Consider shortest routes from A to B along the lines of the grid below.



(a) How many such routes are there that pass through the point C ? ***(2 marks)***

(b) How many such routes are there that do not pass through the point C ? ***(2 marks)***

(c) Now consider an $m \times n$ grid with A bottom left at the point $(0, 0)$ and B top right at the point (m, n). Let C be the grid point (i, j). Consider shortest routes from A to B along the lines of this grid. How many such routes do not pass through C ? ***(3 marks)***

(iii) (a) Let $a, b, n \geq 1$ with $a + b \leq n$. By considering choosing $a + b + 1$ numbers from the set $\{0, 1, 2, \ldots, n\}$, and the possibilities for the number in position $a + 1$ when the chosen numbers are listed in increasing order, show that

$$\binom{n+1}{a+b+1} = \sum_{k=0}^{n} \binom{k}{a} \binom{n-k}{b}.$$

(6 marks)

(b) Hence, or otherwise, express

$$\sum_{j=0}^{n} \sum_{k=0}^{n} \binom{j}{a} \binom{k}{b} \binom{n-j-k}{c},$$

where $a + b + c \leq n$, as a single binomial coefficient. ***(4 marks)***

2 (i) Consider a rectangle m squares wide and n squares high.

(a) For which m and n can this be completely covered by non-overlapping dominoes (that is, by pieces which cover exactly two adjacent squares)? Justify your answer. ***(4 marks)***

(b) Now suppose that m and n are both even. Consider the shape resulting when two diagonally opposite corner squares are removed. Show that it is impossible to cover this completely by non-overlapping dominoes. ***(4 marks)***

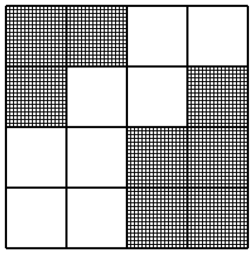
(ii) (a) Use the Pigeon-hole Principle to show that there are two powers of 17 whose difference is divisible by 123456789. ***(5 marks)***

(b) Show that, if $n + 1$ objects are placed in k boxes, then there must be at least one box that contains at least $\lfloor n/k \rfloor + 1$ objects. (Here $\lfloor x \rfloor$ denotes the integer part of x.) ***(3 marks)***

(iii) (a) State the Inclusion/Exclusion Principle. ***(3 marks)***

(b) Use the Inclusion/Exclusion Principle to find the number of permutations of the numbers $1, 2, \ldots, 10$ fixing at least one of 8, 9 or 10. ***(6 marks)***

3 (i) Calculate the rook polynomial of the (unshaded) board B:



(6 marks)

(ii) Let $m \leq n$. Show that the rook polynomial of a full $m \times n$ board is

$$\sum_{k=0}^{m} \binom{m}{k} \binom{n}{k} k!\, x^k.$$

(5 marks)

(iii) Which of the following polynomials can be the rook polynomial of a board? Give reasons for your answers, including examples of appropriate boards where relevant.

(a) $(1+x)(1+4x+2x^2)^2$, ***(2 marks)***

(b) $1+4x+7x^2+3x^3+x^4$. ***(2 marks)***

(iv) (a) Show that

$$(n-1,\ n-2,\ n-3,\ \ldots,\ 2,\ 1,\ 0)$$

are possible scores in a tournament of n players. ***(3 marks)***

(b) Now let n be odd with $n = 2m+1$. Show there exists a tournament of n players in which each player scores m. ***(3 marks)***

(c) Deduce that each of the following is a possible set of scores in a tournament of $2n$ players, where again $n = 2m+1$.

$$(4m+1, 4m, 4m-1, \ldots, 2m+1, m, m, \ldots m),$$
$$(3m+1, 3m+1, \ldots, 3m+1, 2m, 2m-1, 2m-2, \ldots, 2, 1, 0).$$

(4 marks)

4 (i) For what value of x can the following Latin rectangle be extended to a 6×6 Latin square?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & x & 6 & 2 \end{pmatrix}$$

Write down one such extension. ***(6 marks)***

(ii) (a) Define what it means for two $n \times n$ Latin squares $L = (l_{ij})$ and $M = (m_{ij})$ to be orthogonal. ***(2 marks)***

(b) Let p be a prime number. Define $p \times p$ matrices A_k for $k = 1, 2, \ldots, p-1$ by: $(A_k)_{i,j}$ is the element of $\{1, 2, \ldots, p\}$ congruent to $ki + j$ mod p. You may assume that A_k is a Latin square, for $k = 1, 2, \ldots, p-1$. Show that A_k and A_h are orthogonal, for $1 \leq k, h \leq p-1$ and $k \neq h$. ***(6 marks)***

(iii) Assume that a design exists consisting of v varieties and b blocks, with each block containing k varieties and with each pair of varieties in λ blocks. Show that each variety is in precisely r blocks, where

$$r = \frac{bk}{v} = \frac{\lambda(v-1)}{k-1}.$$

(7 marks)

(iv) We define four blocks:

$$\{1, 2, 3\}, \qquad \{1, 2, 4\}, \qquad \{1, 3, 4\}, \qquad \{2, 3, 4\}.$$

Write down the corresponding incidence matrix M and calculate $M^T M$. Deduce that these are the blocks of a design and list all the parameters of the design. ***(4 marks)***

End of Question Paper