EXAMINABLE CONTENT FOR MAS334 (COMBINATORICS)

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This is the long version of this document, listing all definitions and results in the notes, whether or not they are starred. There is also a shorter version that lists only the starred points.

For starred definitions, you may be required to state the definition in the exam. For singly starred results, you may be required to state the result in the exam. For doubly starred results, you may be asked to reproduce the proof in the exam. You may also be asked to prove something where the required argument is a step in one of these specified proofs.

Although you will not be examined directly on the unstarred points, understanding those points may nonetheless be helpful for solving problems posed in the exam.

- Notation for intervals (Definition 1.1)
- Fencepost error (Remark 1.2)
- * Binary sequences (Definition 1.3)
- Binary sequences (Example 1.4)
- * Number of binary sequences (Proposition 1.5)
- * Power set (Definition 1.6)
- Power set (Example 1.7)
- * Size of power set (Proposition 1.8)
- * The set F_kA of lists with distinct entries (Definition 1.9)
- Lists with distinct entries (Example 1.10)
- ** The size of F_kA (Proposition 1.11)
- * Binomial coefficients (Definition 1.12)
- ** Number of subsets of given size (Corollary 1.13)
- Number of sequences with a given number of ones (Corollary 1.14)
- Number of ways to award medals (Problem 1.15)
- Number of ways to draw the National Lottery (Problem 1.16)
- Binomial expansion (Example 1.17)
- Sum of an arithmetic sequence (Proposition 1.18)
- ** Pascal's identity (Proposition 1.19)
- Symmetry of binomial coefficients (Proposition 1.20)
- * Gappy sets (Definition 1.21)
- Gappy sets (Example 1.22)
- ** Number of gappy sets (Proposition 1.23)
- Number of ways to seat patients (Problem 1.24)
- Adjacent numbers in the National Lottery (Problem 1.25)
- * Number of strictly positive solutions (Proposition 2.1)
- Number of strictly positive solutions (Example 2.2)
- Number of strictly positive solutions (Example 2.3)
- * Number of nonnegative solutions (Proposition 2.4)
- Number of nonnegative solutions (Example 2.5)
- Gappy sets and number of solutions (Remark 2.6)
- ** Number of grid routes (Problem 2.7)
- Number of grid routes and number of solutions (Remark 2.8)
- Extended Pascal identity (Proposition 2.9)
- Sum of numbers in a triangle (Proposition 2.10)
- Number of subtriangles (Example 2.11)
- Fibonacci numbers (Definition 2.12)

- Fibonacci numbers and binomial coefficients (Proposition 2.13)
- Cutting a circle with lines (Proposition 2.14)
- Cutting a circle with lines (Example 2.15)
- Integer equation with parity condition (Example 3.1)
- Integer equation with congruence condition (Example 3.2)
- Covering a full board with dominos (Problem 3.3)
- Covering a rectangle with tiles (Problem 3.4)
- Königsburg bridges (Problem 3.5)
- General theory of Euler circuits (Remark 3.6)
- Long lists must repeat (Remark 4.1)

* Pigeonhole principle (Proposition 4.2)

- People with the same number of hairs (Problem 4.3)
- People shaking the same number of hands (Proposition 4.4)
- Disjoint subsets with the same sum (Problem 4.5)
- Pedantry about the empty set (Remark 4.6)
- Consecutive subsequence with sum divisible by n (Problem 4.7)
- Money in a piggybank (Problem 4.8)
- Club with two sports (Problem 5.1)
- Club with three sports (Problem 5.2)

* Inclusion-Exclusion Principle (Theorem 5.3)

- Signed sum over subsets (Lemma 5.4)
- The set $A\langle b\rangle$ (Definition 5.5)
- $b \in B_I$ iff $I \subseteq A\langle b\rangle$ (Lemma 5.6)
- $A\langle b\rangle$ is empty iff $b \in B^*$ (Lemma 5.7)

* Derangements (Definition 5.8)

- Derangements of $\{1,2,3,4\}$ (Example 5.9)
- Derangements of hats (Example 5.10)

* Proportion of derangements (Proposition 5.11)
* Number of coprime numbers (Problem 5.12)
* Matching problems (Definition 6.1)

- Job allocation as a matching problem (Example 6.2)
- Romantic matching problem (Example 6.3)
- Club officers matching problem (Example 6.4)

* Row and column sets (Remark 6.5)

- A specific job allocation problem (Example 6.6)
- Partial matching pictures (Example 6.7)
- Rook counting example (Problem 7.1)

* Rook polynomial and coefficients (Definition 7.2)

- Coefficients of x^0 and x^1 (Remark 7.3)
- Rook polynomial symmetry (Remark 7.4)

* Linear board (Example 7.5)
* Diagonal board (Example 7.6)

- Full 3×3 board (Example 7.7)
- Rook polynomial example (Example 7.8)

* Full board notation (Definition 7.9)
* Rook placements and permutations (Proposition 7.10)

- Full board rook polynomial (Proposition 7.11)
- Full 3×3 board (Example 7.12)
- Rook placements and derangements (Problem 7.13)
- Job allocation as a rook problem (Problem 7.14)
- The game of Snap (Problem 7.15)
- The ménage problem (Problem 7.16)

** Blocking and stripping relation (Theorem 8.1)

- Blocking and stripping (Remark 8.2)
- Blocking and stripping (Example 8.3)
- Blocking and stripping degenerate examples (Example 8.4)

* Fully disjoint splitting (Definition 8.5)
- Fully disjoint splitting (Example 8.6)

** Factoring rook polynomials (Theorem 8.7)
- Factoring rook polynomials (Example 8.8)
- Extending a table as a rook problem (Problem 8.9)
- Extended rook polynomial calculation (Problem 8.10)
- Tabular method (Example 9.1)
- Complementary board (Definition 10.1)
- Complementary board (Example 10.2)

* Last rook coefficient of complementary board (Theorem 10.3)
- Rook polynomial of complementary board (Remark 10.7)
- The set S_X (Definition 10.5)
- The size of S_X (Lemma 10.6)
- Reversing roles of B and $\overline{B}$ (Remark 10.7)
- Complementary board degenerate examples (Example 10.8)
- Complementary boards and derangements (Example 10.9)
- Staircase board (Definition 10.10)
- Rook coefficients of a stairircase (Proposition 10.11)
- Finishing the ménage problem (Example 10.12)
- Finishing the Snap problem (Example 10.13)
- Solvability of matching problems (Definition 11.1)

* Types of plausibility (Definition 11.2)
- The empty set is plausible (Remark 11.3)
- Plausibility examples (Example 11.4)

* Solvable matching problems are plausible (Lemma 11.5)
- The name of Hall's Marriage Theorem (Remark 11.7)
- The completion problem for a partial matching (Construction 11.8)
- Zigzags in incomplete matchings (Definition 11.9)
- Shortest zigzags (Remark 11.10)
- Zigzags in matching problems (Example 11.11)
- More types of zigzags (Definition 11.12)
- The set of reachable jobs (Definition 11.13)
- Reachable jobs are end-reachable (Remark 11.14)
- Most reachable jobs have been assigned (Remark 11.15)
- Job allocation with enthusiasts (Theorem 12.1)
- Reformulation of the enthusiast plausibility condition (Lemma 12.2)
- The enthusiast plausibility condition is necessary (Remark 12.3)
- Problems with $|R_a| \leq d \leq |C_b|$ are solvable (Proposition 12.4)
- Job allocation with talented people (Corollary 12.5)

* Transversals (Definition 12.6)
- Transversals (Example 12.7)

* Hall's Theorem for transversals (Proposition 12.8)
- A list with no transversal (Example 12.9)

* Team allocation problems (Definition 12.10)

* Plausibility for team allocation (Definition 12.11)
- Solvable team allocation problems are plausible (Remark 12.12)

** Hall's Theorem for team allocation (Proposition 12.13)
- Tournaments (Definition 13.1)
- Interpretation of the formal definition (Remark 13.2)
- The rugby world cup (Example 13.3)

- Describing a tournament with medal collections (Remark 13.4)
* Consistent tournament (Example 13.5)
* Winning line (Definition 13.6)
* Winning line (Example 13.8)

** Every tournament has a winning line (Proposition 13.9)
* Score sequence (Definition 13.10)
- Score sequence (Example 13.11)
* Odd modular tournaments (Example 13.12)
- Sums of scores (Definition 13.13)
* Realisable score sequence (Definition 13.14)
- The total score must be $\binom{n}{2}$ (Lemma 13.15)
- There cannot be two zeros (Lemma 13.16)
- A binomial coefficient identity (Lemma 13.17)
- Equivalence of plausibility conditions (Proposition 13.18)
* Plausibility for score sequences (Definition 13.19)

** Every realisable sequence is plausible (Lemma 13.20)
- Plausible score sequences (Example 13.21)
* Landau's Theorem: every plausible sequence is realisable (Theorem 13.22)
- Constructing a tournament (Example 13.23)
- An implausible score sequence (Example 13.24)
- Sum of tournaments (Definition 13.25)
- Sum of tournaments (Remark 13.26)
- Sum of odd modular tournaments (Example 13.27)
- Product of tournaments (Definition 13.28)
- Product of tournaments interpretation (Remark 13.29)
- Scores for a product of tournaments (Lemma 13.30)
- Scores for a product of tournaments (Example 13.31)
* Latin rectangles (Definition 14.1)
- Maximum size of latin rectangles (Remark 14.2)
- Latin square (Definition 14.3)
- A latin rectangle (Example 14.4)
- A latin square (Example 14.5)
- Latin squares from finite groups (Example 14.6)
- Latin squares from modular addition (Example 14.7)
* Wide latin rectangles can be squared (Theorem 14.8)
- Interpretation with a job experience scheme (Remark 14.9)
* Multiplicity and excess (Definition 14.10)
- Multiplicity and excess (Example 14.11)
- Reformulations of multiplicity (Remark 14.12)
- Multiplicity in the wide or tall case (Lemma 14.13)
- Latin rectangle extension example (Example 14.14)
- Tall latin rectangles can be squared (Corollary 14.15)
- A latin rectangle that cannot be extended (Example 14.16)
- A latin rectangle that cannot be extended (Example 14.17)
* Plausibility for latin rectangles (Definition 14.18)
* Extendable rectangles are plausible (Proposition 14.19)
- The converse is not yet proved (Example 14.20)
- Plausible latin rectangles can be extended once (Lemma 14.21)
* Plausible latin rectangles can be fully extended (Theorem 14.22)
- Full extension example (Example 14.23)
- Reduced latin squares (Definition 14.24)
- Reduced latin squares of size one (Example 14.25)
- Reduced latin squares of size two (Example 14.26)

- Reduced latin squares of size three (Example 14.27)
- Numbers of reduced and unreduced latin squares (Proposition 14.28)
- Number of reduced latin squares of size four (Proposition 14.29)
- The numbers grow rapidly (Remark 14.30)
- The starring operation (Example 14.31)
- Orthogonality for latin squares (Definition 14.32)
- Solving for an orthogonal pair (Example 14.33)
- Upper bound on the number of orthogonal squares (Theorem 14.34)

* $\mathbb{Z}/p$ is a field (Proposition 14.35)

- Division modulo a prime (Remark 14.36)
- Orthogonal families from linear algebra (Lemma 14.37)
- Generalisation to finite fields (Remark 14.38)
- No orthogonal squares of size six (Theorem 14.39)

* Block design (Definition 15.1)

- Degenerate examples are excluded (Remark 15.2)
- An affine block design (Example 15.3)

* Relations between parameters (Proposition 15.4)
* In any block design, we have $v \leq b$ (Proposition 15.5)

- Symmetric designs (Definition 15.6)

* Quadratic residues (Definition 15.8)

- Coblocks for a quadratic residue design (Remark 15.9)
- The quadratic residue design mod 7 (Example 15.10)
- The quadratic residue design mod 11 (Example 15.11)

* Properties of quadratic residues (Lemma 15.12)
* The sets D and D_x (Definition 15.13)
* Size of the sets D_x (Lemma 15.14)

- The case where $p = 11$ (Example 15.15)

* The quadratic residue design is a block design (Theorem 15.16)