PROBLEMS ON ALGEBRAIC TOPOLOGY

1. Homeomorphisms

Problem 1.1. Let x be a point in $\mathbb{R}^n$, and suppose $\epsilon > 0$. Put $U = \{y \in \mathbb{R}^n \mid \|x - y\| < \epsilon\}$. In lectures we claimed that there is a homeomorphism $f\colon U \to \mathbb{R}^n$ given by

$$f(y) = \frac{y - x}{1 - \|y - x\|^2/\epsilon^2} \qquad\qquad f^{-1}(z) = x + \frac{\sqrt{\epsilon^2 + 4\|z\|^2} - \epsilon}{2\|z\|^2}\epsilon z.$$

Check carefully that these formulae give well-defined and continuous maps with the appropriate domains and ranges, and that they are inverse to each other.

Problem 1.2. Recall that $\mathfrak{u}(n) = \{\beta \in M_n(\mathbb{C}) \mid \beta + \beta^\dagger = 0\}$. Find a basis for $\mathfrak{u}(2)$ over $\mathbb{R}$, and prove that $\mathfrak{u}(2)$ is not a complex vector subspace of $M_2(\mathbb{C})$.

Problem 1.3. Recall our definition of the lens space: we have a complex vector space V of dimension n with inner product, and an integer $d > 1$. We put $C_d = \{\omega \in \mathbb{C} \mid \omega^d = 1\}$, and we let this act on $S(V)$ by multiplication. The lens space is then $M = S(V)/C_d$. What can you say in the special case where $n = 1$, or the special case where $d = 2$?

Problem 1.4. Let V be a finite-dimensional vector space with inner product. In Section 4 of the notes we defined spaces $S(V_+)$, $S'(V_+)$, $S_+(V_+)/S(V)$, S^V, and $B(V)/S(V)$, and gave a table of formulae giving homeomorphisms between all these spaces. Verify a few of these formulae.

Problem 1.5. By quoting a suitable general theorem, prove that $\Delta_1 \times \Delta_2 \times \Delta_3$ is homeomorphic to Δ_6.

Problem 1.6. Consider the square $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x, y \le 1\}$ and the edge $Y = \{(x, 0) \mid 0 \le x \le 1\}$. Prove that X/Y is homeomorphic to B^2.

Problem 1.7. Let X be a space, and Y a closed subspace, and let Z be any other space. Construct a continuous bijection $f\colon (X/Y) \wedge Z_+ \to (X \times Z)/(Y \times Z)$.

(In the cases of interest f^{-1} will be continuous so that f is a homeomorphism, but there are technical subtleties around this point.)

Problem 1.8. If X and Y are finite based sets, with $|X| = n$ and $|Y| = m$, what are $|X \vee Y|$ and $|X \wedge Y|$?

2. Mayer-Vietoris

Problem 2.1. Put $A = \{0, 1, \dots, n-1\} \subseteq \mathbb{R}$ and $U = \mathbb{R}^2 \setminus (A \times \{0\})$. Calculate $H^*(U)$. (Hint: consider the sets $U_\pm = \mathbb{R}^2 \setminus (A \times [0, \pm\infty))$ and use the Mayer-Vietoris sequence.)

3. The Künneth Theorem

Problem 3.1. Consider the spaces $X = \mathbb{C} \setminus \{0, 1\}$ and $Y = \mathbb{C} \setminus \{0, 1, 2\}$. The cohomology of these was described in lectures. Describe $H^n(X \times Y)$ for all n. Show that $a^2 = 0$ for all $a \in H^1(X \times Y)$.

4. Configuration spaces

Problem 4.1. Consider the space

$$X = F_4\mathbb{C} = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \mid z_i \neq z_j \text{ whenever} i \neq j\}.$$

The cohomology of X was described in lectures in terms of generators and relations. Use this to give a basis for $H^*(X)$. (You can check your answer against the following facts: $H^*(F_n\mathbb{C})$ has total rank $n!$, whereas the group $H^{n-1}(F_n\mathbb{C})$ has rank $(n-1)!$, and the groups $H^m(F_n\mathbb{C})$ are zero for $m \geq n$.)

Problem 4.2. Recall that $B_n\mathbb{C}$ is the set of subsets $S \subset \mathbb{C}$ such that $|S| = n$ (topologised as a quotient of $F_n\mathbb{C}$). Prove that $B_2\mathbb{C}$ is homotopy equivalent to S^1.

Problem 4.3. Construct homeomorphisms

$$\begin{aligned} F_2\mathbb{C} &\simeq \mathbb{C} \times \mathbb{C}^\times \\ F_3\mathbb{C} &\simeq \mathbb{C} \times \mathbb{C}^\times \times (\mathbb{C} \setminus \{0, 1\}) \\ B_2\mathbb{C} &\simeq \mathbb{C} \times \mathbb{C}^\times \end{aligned}$$

Describe the cohomology of all these spaces.

Problem 4.4. Let $F_2\mathbb{R}^n$ denote the space of pairs (a, b) with $a, b \in \mathbb{R}^n$ and $a \neq b$. Let $B_2\mathbb{R}^n$ be the quotient of $F_2\mathbb{R}^n$ by the evident action of C_2, so $(a, b) \sim (c, d)$ iff $((a, b) = (c, d)$ or $(a, b) = (d, c)$. Let $\mathbb{R}P^{n-1}$ denote the space of one-dimensional subspaces $L \leq \mathbb{R}^n$. Show that $B_2\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}P^{n-1}$.

5. Matrix groups

Problem 5.1. Give a path joining I to $-I$ in $U(2)$.

Problem 5.2. Put $SU(n) = \{A \in U(n) \mid \det(A) = 1\}$. Define $\alpha\colon SU(3) \to S^5 \times S^5$ by $\alpha(A) = (Ae_0, Ae_1)$ (where $\{e_0, e_1, e_2\}$ is the standard basis of $\mathbb{C}^3$). Prove that α is injective but not surjective.

Problem 5.3. Prove that $SU(2)$ is homeomorphic to S^3, and thus that $U(2)$ is homeomorphic to $S^1 \times S^3$.

Problem 5.4. Prove that the space $GL_2^+(\mathbb{R}) = \{A \in GL_2(\mathbb{R}) \mid \det(A) > 0\}$ is homeomorphic to $\mathbb{R}^3 \times S^1$.

Problem 5.5. Put $J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, and let G be the group of matrices $A \in GL_4(\mathbb{R})$ such that $A^TJA = J$. This is called the *Lorenz group*. Prove that it has at least four path-components.

Problem 5.6. Recall the complex reflection map $\rho\colon S^1 \times \mathbb{C}P^1 \to U(2)$: the matrix $\rho(z, L)$ has eigenvalue z on L, and eigenvalue 1 on $L^\perp$. Consider the following two matrices:

$$A = \frac{1}{2}\begin{bmatrix} i+1 & -i-1 \\ i+1 & i+1 \end{bmatrix} \qquad B = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

One of these has the form $\rho(z, L)$ for some z and L, and the other does not lie in the image of ρ. Work out which is which, and find z and L.

Problem 5.7. Give a formula for the rank of the free abelian group $H^*U(n)$.

Problem 5.8. Give a basis for $\widetilde{H}^*(U(4)/U(2))$. (This should be interpreted as the space obtained from $U(4)$ by collapsing $U(2)$ to a point, not the coset space.)

Problem 5.9. Find an integer n and a class $u \in H^*U(n)$ such that u^2 is a nonzero element of $H^{n^2}U(n)$.

Problem 5.10. Let $\mu\colon U(3)\times U(3) \to U(3)$ be given by $\mu(A,B) = AB$. By quoting facts about μ^* proved in lectures, calculate $\mu^*(a_1a_3a_5) \in H^9(U(3) \times U(3))$.

Problem 5.11. It is known that any manifold M can be embedded as a subspace of a finite dimensional vector space. As an example, exhibit an embedding of PV in $\mathrm{Hom}(V,V)$ (for any Hermitian space V). (Look through the discussion of the topology of $U(V)$ for hints.)

6. Vector bundles

Problem 6.1. If T is the tautological line bundle over $\mathbb{C}P^n$, prove that $S(T \otimes T) = \mathbb{R}P^{2n+1}$.

Problem 6.2. Let $q\colon V \setminus \{0\} \to PV$ be the usual quotient map, and let T denote the tautological bundle over PV. Prove that q^*T is isomorphic to a constant bundle.

Problem 6.3. Let L be the Möbius bundle over S^1, given by $L_z = \{w \in \mathbb{C} \mid w^2 \in z.[0,\infty)\}$. Prove that the bundle $\mathbb{C} \otimes_{\mathbb{R}} L \simeq L \oplus L$ is isomorphic to a constant bundle.

Problem 6.4. Let V and W be complex vector bundles over a base X, and suppose that $\dim_{\mathbb{C}}(W) = 1$. Prove that $P(V \otimes W) \simeq PV$.

Problem 6.5. Let V be a Hermitian space, and let L be the tautological bundle over PV. Interpret $\mathrm{Hom}(L, L^\perp)$ as a bundle over PV, and show that it is isomorphic to the tangent bundle.

Problem 6.6. Let V be a complex vector bundle over $U(n)$ that can be written as a direct sum of line bundles. What can you say about $f_V(t)$?

7. Miscellaneous

Problem 7.1. Recall that the Mobius band can be described as

$$M = \{(z,w) \in \mathbb{C} \times \mathbb{C} \mid |z| = 1 \text{ and } w^2 = tz \text{ for some } t \geq 0\}.$$

Prove that this is homotopy equivalent to S^1.

Problem 7.2. Let G be a path-connected topological group, such that $H^*(G)$ is a finitely generated free abelian group. Prove that every element of $H^1(G)$ is primitive.

Problem 7.3. Let M be the Milnor hypersurface in $\mathbb{C}P^2 \times \mathbb{C}P^3$, and let y and z be the standard generators of H^*M. Give a basis for H^*M, and express $(y+z)^4$ in terms of that basis.

Problem 7.4. Recall that a *Möbius transformation* is a map f from the Riemann sphere $\mathbb{C} \cup \{\infty\} \simeq \mathbb{C}P^1$ to itself that can be written in the form $f(z) = (az+b)/(cz+d)$ for some $a,b,c,d \in \mathbb{C}$ with $ad - bc \neq 0$. Let M be the group of Möbius transformations.

Construct homeomorphisms

$$SL(2,\mathbb{C})/\{\pm 1\} \simeq M \simeq F_3(\mathbb{C} \cup \{\infty\}).$$

By considering $SU(2)$, construct an interesting map $\mathbb{R}P^3 \to F_3(S^2)$. By considering the Gram-Schmidt process, prove that this map is a homotopy equivalence.

Problem 7.5. Let $H_{2,2} = \{([\underline{z}], [\underline{w}] \mid \sum_i z_i w_i = 0\}$ be the standard Milnor hypersurface in $\mathbb{C}P^2 \times \mathbb{C}P^2$, and let $F_3 = \mathrm{Flag}_3(\mathbb{C}^3)$ be the space of flags $0 = W_0 < W_1 < W_2 < W_3 = \mathbb{C}^3$ in $\mathbb{C}^3$. By quoting results from the lectures, write down the cohomology rings of these spaces, and prove by pure algebra that they are isomorphic. Find a homeomorphism $H_{2,2} \to F_3$.

Problem 7.6. For $0 \leq k \leq n+1$ we regard $\mathbb{C}^k$ as a subspace of $\mathbb{C}^{n+1}$ in the usual way. Let B_n be the space of those flags $0 = V_0 < \ldots < V_{n+1} = \mathbb{C}^{n+1}$ in $\mathbb{C}^{n+1}$ for which $V_k \leq \mathbb{C}^{k+1}$ for $k = 0, \ldots, n$. Define line bundles $L_1, \ldots, L_{n+1}$ and $M_1, \ldots, M_n$ over B_n by

$$L_{k,\underline{V}} = V_k \ominus V_{k-1}$$
$$M_{k,\underline{V}} = \mathbb{C}^{k+1} \ominus V_k$$

(here $W \ominus U$ means the orthogonal complement of U in W). Check that $L_k \oplus M_k = \mathbb{C} \oplus M_{k-1}$ and deduce some relations among Euler classes. Show how to regard B_n as a projective bundle over B_{n-1} and deduce a description of $H^* B_n$.

Problem 7.7. Let V be a complex vector space of dimension n and let S be a subset of $\{1, \ldots, n-1\}$, say $S = \{d_1, \ldots, d_{m-1}\}$ with $d_0 := 0 < d_1 < \ldots < d_{m-1} < d_m := n$. Let $\mathrm{Flag}_S(V)$ be the space of sequences $(V_{d_1} < \ldots < V_{d_m} < V)$ such that $\dim(V_{d_k}) = d_k$ for all k. Guess a description of $\mathrm{Hom}(H^* \mathrm{Flag}_S(V), R)$ for any ring R, and outline a proof that your guess is correct.

(Note that when $S = \{k\}$ we have $\mathrm{Flag}_S(V) = \mathrm{Grass}_k(V)$, and when $S = \{1, \ldots, k\}$ we have $\mathrm{Flag}_S(V) = \mathrm{Flag}_k(V)$, and both of these cases have been covered in lectures.)

Problem 7.8. Give a basis for $H^*(\mathrm{Grass}_2(\mathbb{C}^4))$.