EXAMPLES IN THE (CO)HOMOLOGY OF SPACES

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Contents

1. Cohomology

For each topological space X, one can, with considerable effort, define a ring $H^*(X)$, called the *cohomology ring* of X. (More precisely, the version that we will consider is called the singular cohomology ring with integer coefficients. There are other versions, but for the spaces that we consider they would not give interestingly different answers.) The definitions are covered in the companion document `construction.pdf`. In a moment we will recall some basic properties, which are also proved there. With these properties in hand, it is often possible to calculate $H^*(X)$ and deduce topological information without further reference to the definitions. Many courses on algebraic topology spend so much time on setting up the framework that they do not reach many interesting examples, which we feel is a shame. In these notes, we take a more axiomatic approach.

Our initial list of properties is as follows.

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(1) Firstly, $H^*(X)$ is a graded ring. For each integer $j \geq 0$ we have an abelian group $H^j(X)$, and any pair of elements $a \in H^j(X)$ and $b \in H^k(X)$ have a product $ab \in H^{j+k}(X)$. This multiplication is associative and distributes over addition. It is commutative in the graded sense, which means that $ba = (-1)^{jk}ab$. It is convenient to define $H^j(X) = 0$ for $j < 0$.
(2) Secondly, $H^*(X)$ is contravariantly functorial in X. This means that for any continuous map $f\colon X \to Y$ we have a ring homomorphism $f^*\colon H^*(Y) \to H^*(X)$. If we have another map $g\colon Y \to Z$ then $(gf)^* = f^*g^*\colon H^*(Z) \to H^*(X)$. If f is the identity map $X \to X$ then f^* is the identity map on $H^*(X)$.
(3) Thirdly, $H^*(X)$ is homotopy invariant. Two maps $f_0, f_1\colon X \to Y$ are said to be *homotopic* if there is a continuous family of maps $f_t\colon X \to Y$ (for $0 \leq t \leq 1$) interpolating between them. If f_0 and f_1 are homotopic then $f_0^* = f_1^*\colon H^*(Y) \to H^*(X)$; this is what we mean by homotopy invariance.
(4) Finally, if we let 1 denote the space consisting of a single point, then $H^0(1) = \mathbb{Z}$, whereas $H^k(1) = 0$ for all $k \neq 0$. On the other hand, all cohomology groups of the empty space are zero.

Definition 1.1. Suppose we have two spaces X and Y, and thus projections $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$. Given $a \in H^r(X)$ and $b \in H^s(Y)$ we define $a \times b = p^*(a)q^*(b) \in H^{r+s}(X \times Y)$; this is called the *external product* of a and b.

The following statement is a simplified special case of a standard result called the Künneth Theorem.

Proposition 1.2. *Suppose that each of the groups $H^r(X)$ is a finitely generated free abelian group, and similarly for each of the groups $H^s(Y)$. Let $\{a_0, a_1, \dots\}$ be a homogeneous basis for $H^*(X)$ (so each element a_i lies in $H^{r_i}(X)$ for some r_i, and the elements that lie in $H^r(X)$ form a basis for that group). Similarly, let $\{b_0, b_1, \dots\}$ be a homogeneous basis for $H^*(Y)$. Then the external products $a_i \times b_j$ form a homogeneous basis for $H^*(X \times Y)$.* □

Now suppose that we have a space Z that comes as the union of two subsets X and Y that are topologically separate (so $X \cap Y = \emptyset$, and both X and Y are closed in Z). We write $Z = X \amalg Y$ for this situation. Note that we have inclusion maps $i\colon X \to X \amalg Y$ and $j\colon Y \to X \amalg Y$.

The following statement is immediate from the definitions. It can also be regarded as a degenerate case of the Mayer-Vietoris sequence.

Proposition 1.3. *There is an isomorphism*

$$H^k(X \amalg Y) \to H^k(X) \oplus H^k(Y)$$

given by $a \mapsto (i^*(a), j^*(a))$. □

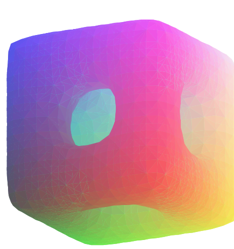
We now want to describe some examples. In Section 2 we will define a number of interesting spaces, and then state some facts about their cohomology rings. In subsequent sections we will prove these statements, developing a number of genera techniques along the way.

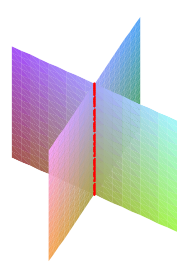
2. Examples of spaces

Many of our examples will be topological manifolds. We recall the definition:

Definition 2.1. A *topological manifold of dimension n* is a second countable, Hausdorff topological space M such that each point $x \in M$ has an open neighbourhood $U \subseteq M$ such that U is homeomorphic to $\mathbb{R}^n$.

Example 2.2.





The space on the left is a manifold of dimension 2; the one on the right is not.

Remark 2.3. Some readers will be familiar with differentiable manifolds. Differentiability will not make any difference for most of what we do so we will ignore it until later.

Remark 2.4. We recall the definitions of the two subsidiary topological conditions mentioned above.

(a) A space X is *Hausdorff* if for all $x, y \in X$ with $x \neq y$ there exist open sets U, V with $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

(b) A space X is *second countable* if there is a countable family $(U_i)_{i \in I}$ of open sets such that for all open sets V and all $y \in V$ there exists i with $y \in U_i \subseteq V$.

For example, any metric space is Hausdorff. Indeed, if $x \neq y$ then we can put $r = d(x, y)/2 > 0$ and take U and V to be the open balls of radius r around x and y respectively. Moreover, the space $\mathbb{R}^n$ is second countable, as we see by considering the family of all open balls of rational radius whose centre has rational coordinates.

Remark 2.5. Suppose that M is a topological space such that each point $x \in M$ has an open neighbourhood $U \subseteq M$ such that U is homeomorphic to $\mathbb{R}^n$. We call such an open set U a *chart domain*. Suppose that for every pair of points $x, y \in M$ we have either

- There is a chart domain U with $x, y \in U$; or
- There are disjoint chart domains U, V with $x \in U$ and $y \in V$.

Then one can check that M is Hausdorff and thus a manifold.

On the other hand, suppose we define $M = \mathbb{R} \times \{0, 1\}/\sim$, where $(x, y) \sim (x', y')$ iff $(x, y) = (x', y')$ or $x = x' \neq 0$.

$\mathbb{R} \times \{0, 1\}$ $\qquad$ $X = (\mathbb{R} \times \{0, 1\})/\sim$

One checks that the images of $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ are homeomorphic to $\mathbb{R}$, so every point of M has a neighbourhood that is a chart domain. Moreover, M is easily seen to be second countable, but the points $(0, 0)$ and $(0, 1)$ cannot be separated in M so M is not Hausdorff.

Convention 2.6. Many examples below will involve vector spaces. Everywhere in these notes, vector spaces are assumed finite dimensional unless otherwise specified, and the scalar field is either $\mathbb{R}$ or $\mathbb{C}$.

Example 2.7.

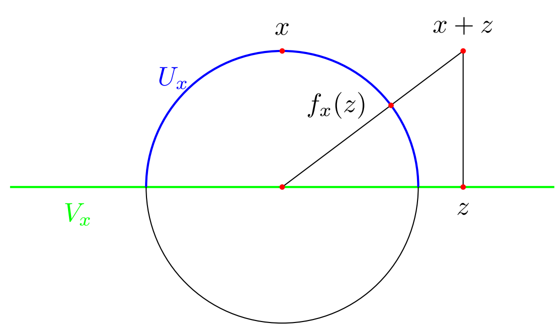
(1) Let U be the open ball of radius $\epsilon > 0$ around a point $x \in \mathbb{R}^n$. Then the function $f(y) = (y - x)/(1 - \|y - x\|^2/\epsilon^2)$ gives a homeomorphism of U with $\mathbb{R}^n$. It follows that any open subspace of $\mathbb{R}^n$ is an n-dimensional topological manifold. An interesting special case is

$$F_n\mathbb{C} := \{z \in \mathbb{C}^n \mid z_i \neq z_j \text{ when } i \neq j\}.$$

This can be viewed as an open subspace of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$; we will study its cohomology later.

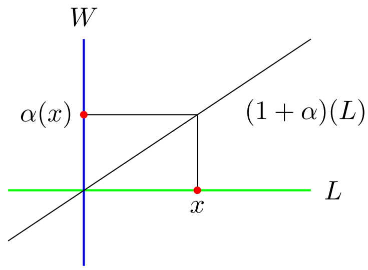
(2) Let V be a vector space of dimension n. There is a natural topology on V (the smallest one for which all linear maps $V \to \mathbb{R}$ are continuous) and with this topology V is homeomorphic to $\mathbb{R}^n$. Thus V is a topological manifold.

(3) Now suppose that V is equipped with an inner product, and define the sphere $S(V)$ as $\{x \in V \mid x^2 = 1\}$. (In this context, we will always write $x^2 = \|x\|^2 = \langle x, x \rangle$). If $x \in S(V)$ we define $U_x = \{y \in S(V) \mid \langle x, y \rangle > 0\}$ and $V_x = \{z \mid \langle x, z \rangle = 0\}$. We also define $f_x \colon V_x \to U_x$ by $f_x(z) = (x+z)/\sqrt{1 + z^2}$.



One can check that this is a homeomorphism, and also V_x is a vector space of dimension $n-1$ so it is homeomorphic to $\mathbb{R}^{n-1}$. It follows that $S(V)$ is a manifold of dimension $n-1$. It is easy to see that it is compact.

(4) Now suppose that V is a complex vector space of dimension m. (In this course we will usually prefer to deal with complex geometry, as the resulting cohomology rings tend to be simpler.) The associated projective space PV is the set of one-dimensional complex subspaces ("lines") in V. Any nonzero vector $x \in V^\times := V \setminus 0$ spans a line in V, which we denote by $[x]$. Every element of PV then has the form $[x]$ for some x, and $[x] = [y]$ iff $y = \lambda x$ for some $\lambda \in \mathbb{C}^\times$. The map $q\colon V^\times \to PV$ (given by $q(x) = [x]$) is thus surjective, and we topologise PV as a quotient of $V^\times$. We claim that it is a topological manifold. To see this, let L be a line and choose a subspace $W < V$ of dimension $m-1$ such that $V = L \oplus W$. Put $U = \{M \in PV \mid M \cap W = 0\}$; this is easily seen to be a neighbourhood of L. Given a linear map $\alpha\colon L \to W$, we have a map $1+\alpha\colon L \to L \oplus W = V$ and thus a subspace $\mathrm{image}(1+\alpha) \leq V$, which is easily seen to be a line. We can thus define $f = f_{L,W}\colon \mathrm{Hom}(L, W) \to PV$ by $f(\alpha) = \mathrm{image}(1+\alpha)$.



One checks that this gives a homeomorphism $\mathrm{Hom}(L, W) \simeq U$ and $\mathrm{Hom}(L, W) \simeq \mathbb{C}^{n-1} \simeq \mathbb{R}^{2n-2}$, as required.

We will also write $\mathbb{C}P^m$ for $P(\mathbb{C}^{m+1})$. In this case, the usual notation is $[z_0 : z_1 : \ldots : z_m]$ for the line spanned by a nonzero vector $(z_0, \ldots, z_m)$.

(5) Suppose that $m \leq n$. The *Milnor hypersurface* in $\mathbb{C}P^m \times \mathbb{C}P^n$ is the space

$$H_{m,n} = \{([z], [w]) \in \mathbb{C}P^m \times \mathbb{C}P^n \mid \sum_{i=0}^{m} z_i w_i = 0\}.$$

(6) Suppose that $d > 2$. The *Fermat hypersurface of degree d* in $\mathbb{C}P^m$ is

$$X_{d,m} = \{[z] \in \mathbb{C}P^m \mid \sum_{i=0}^{m} z_i^d = 0\}.$$

(7) One can also consider the space $\mathrm{Grass}_k(V)$ of k-dimensional complex subspaces ("k-planes") in V (so that $PV = \mathrm{Grass}_1(V)$). These are called Grassmannian spaces. One can check that $\mathrm{Grass}_k(V)$ is a compact manifold of dimension $2k(m-k)$.

(8) A *complete flag* in V is a sequence of complex subspaces $0 = W_0 < W_1 < \ldots < W_m = V$ such that $\dim(W_k) = k$ for all k. The space of complex flags is written $\mathrm{Flag}(V)$; it is again a compact manifold, of dimension $2m(m-1)$.

(9) Now consider the case $V = \mathbb{C}^m$ and let V_k denote the obvious copy of $\mathbb{C}^k$ inside V. We say that a flag W as above is *bounded* if it satisfies $W_k \leq V_{k+1}$ for $k = 0, \ldots, m-1$. We write B_m for the space of bounded flags, which is a compact manifold of dimension $2(m-1)$. It is actually an example of a *toric variety*, which essentially means that the group $(\mathbb{C}^\times)^{m-1}$ acts on B_m and that there is a dense open subspace of B_m which is isomorphic to that group. There is a rich theory of toric varieties, which involves a lot of interesting combinatorics as well as algebra and geometry.

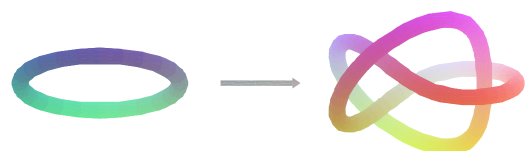
(10) Let V be a complex vector space of dimension m, and suppose that it has a Hermitian inner product (so that $\langle u, v\rangle = \overline{\langle v, u\rangle}$ and $z\langle u, v\rangle = \langle zu, v\rangle = \langle u, \overline{z}v\rangle$ when $z \in \mathbb{C}$ and $u, v \in V$). Any endomorphism α of V has an adjoint $\alpha^\dagger$, characterised by $\langle \alpha(u), v\rangle = \langle u, \alpha^\dagger(v)\rangle$. We write $U(V)$ for the unitary group of V, so

$$U(V) = \{\alpha \in \mathrm{Aut}(V) \mid \alpha^\dagger = \alpha^{-1}\}.$$

We also define
$$\mathfrak{u}(V) = \{\beta \in \mathrm{End}(V) \mid \beta^\dagger = -\beta\}.$$
After choosing an orthonormal basis for V, it is not hard to check that $\mathfrak{u}(V)$ is a real vector space of dimension m^2. Also, if $\beta \in \mathfrak{u}(V)$ we see that the eigenvalues of β are purely imaginary, so that the maps $1 \pm \beta/2$ are invertible. For any $\alpha \in U(V)$ we define $f_\alpha \colon \mathfrak{u}(V) \to \mathrm{Aut}(V)$ by
$$f_\alpha(\beta) = (1 + \beta/2)(1 - \beta/2)^{-1}\alpha.$$
One checks that this gives a homeomorphism of $\mathfrak{u}(V)$ with a neighbourhood of α in $U(V)$. It follows that $U(V)$ is a topological manifold.

(11) Let V be as above, and let C_d denote the cyclic subgroup of order d in $\mathbb{C}^\times$, which is generated by $\omega = e^{2\pi i/d}$. The space $S(V)/C_d$ is then a manifold of dimension $2m-1$ (an example of a *lens space*). To see this, let $\pi \colon S(V) \to S(V)/C_d$ be the projection map, and note that $\pi^{-1}\pi(U) = \bigcup_{k=0}^{d-1} \omega^k U$; this implies that π is an open map. Next put $\epsilon = |\omega - 1|/2$, and for $v \in S(V)$ put $N_\epsilon(v) = \{w \in S(V) \mid \|v - w\| < \epsilon\}$. One checks easily that $\|\omega^k u - u\| \geq 2\epsilon\|u\|$ and thus that $\pi \colon N_\epsilon(v) \to S(V)/C_d$ is injective. It follows that $\pi \colon N_\epsilon(v) \to \pi N_\epsilon(v)$ is a homeomorphism and that the codomain is open in $S(V)$; this shows that $S(V)/C_d$ is a manifold.

(12) Let j be an embedding of the solid torus $S^1 \times D^2$ in $\mathbb{R}^3 \subset S^3$, whose image $K = j(S^1 \times D^2)$ is a knot.



If we remove the interior of K we get a space X_0 with boundary $\partial(X_0) = S^1 \times S^1$, which is the same as the boundary of $X_1 = D^2 \times S^1$. We can thus glue X_0 and X_1 along their boundaries to get a new manifold called X. This is the most basic example of *surgery*: making new manifolds from old by cutting and gluing. There is a well-developed theory to determine whether it is possible to convert one manifold to another by a sequence of surgeries.

Exercise 2.8. In example (3), verify that f_x is a homeomorphism. Also define
$$g_x(z) = \frac{((z/2)^2 - 1)x + z}{(z/2)^2 + 1}$$
and check that this gives a homeomorphism of V_x with the larger open set $W_x = \{y \in S(V) \mid y \neq -x\}$. Interpret this geometrically in terms of stereographic projection.

Exercise 2.9. In example (4), verify that $PV = V^\times/\mathbb{C}^\times$. If V has a complex inner product, show that S^1 acts on $S(V)$ and that $PV = S(V)/S^1$. Deduce that PV is compact.

Exercise 2.10. In example (10), define $\|\beta\| = \sup\{\|\beta(v)\| : v \in S(V)\}$ and $W = \{\beta \in \mathfrak{u}(V) \mid \|\beta\| < \pi\}$. Write $\exp(\beta) = \sum_{k \geq 0} \beta^k/k! \in \mathrm{End}_{\mathbb{C}}(V)$. Prove that exp maps $\mathfrak{u}(V)$ to $U(V)$. Prove also that if $\beta \in W$ then $\exp(\beta)$ has precisely the same eigenspaces as β, and deduce that $\exp \colon W \to U(V)$ is injective. We could have used the function $\beta \mapsto \exp(\beta)\alpha$ instead of $f_\alpha(\beta)$; this generalises more easily to other Lie groups, but is more fiddly.

Example 2.11.

(1) $H^*(S^n)$ is the free abelian group generated by $1 \in H^0(S^n)$ and an element $u_n \in H^n(S^n)$. The ring structure is given by $u_n^2 = 0$. This will be proved in Section 7. For the moment we take it as given, and describe the cohomology of various other spaces by relating them to the spheres S^n.

(2) Suppose we have distinct points $a_1, \ldots, a_n \in \mathbb{C}$ and we define $M = \mathbb{C} \setminus \{a_1, \ldots, a_n\}$. Define $f_i \colon M \to S^1$ by $f_i(z) = (z - a_i)/|z - a_i|$ and put $v_i = f_i^*(u_1)$. Then $H^*(M)$ is the free abelian group generated by $1 \in H^0(M)$ and $v_1, \ldots, v_m \in H^1 M$. The ring structure is given by $v_i v_j = 0$ for all i, j.

(3) Given $i, j \in \{1, \ldots, n\}$ with $i \neq j$ we define $f_{ij} \colon F_n\mathbb{C} \to \mathbb{C}^\times$ by $f_{ij}(z) = z_i - z_j$ and $a_{ij} = f_{ij}^* v_1$. Then $H^*(F_n\mathbb{C})$ is the graded ring freely generated by the elements a_{ij} modulo relations $a_{ij}^2 = 0$ and

$a_{ij} = a_{ji}$ and $a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij} = 0$ for all distinct triples i, j, k. One can also give a basis for this ring as a free abelian group.

(4) Let V be a complex vector space of dimension $m + 1$ (where $m > 0$). We next consider the ring $H^*(PV)$. In the case $V = \mathbb{C}^2$ we have $PV = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} \simeq S^2$, so $H^*(\mathbb{C}P^1) = \mathbb{Z}[u_2]/u_2^2$. In general we can choose a linear embedding $j\colon \mathbb{C}^2 \to V$, giving an embedding $Pj\colon \mathbb{C}P^1 \to PV$. It turns out that there is a unique element $x \in H^2(PV)$ such that $(Pj)^*(x) = u_2$ for all such j. Moreover, it turns out that $H^*(PV) = \mathbb{Z}[x]/x^{m+1}$. Thus, if k is even and $0 \leq k \leq 2m$ then $H^k(PV)$ is a copy of $\mathbb{Z}$ generated by $x^{k/2}$, and $H^k(PV) = 0$ in all other cases.

(5) Now suppose we have another complex vector space W, of dimension $n + 1$, and we consider the space $PV \times PW$. Let $p\colon PV \times PW \to PV$ and $q\colon PV \times PW \to PW$ be the projection maps, and put $y = p^*(x)$ and $z = q^*(x)$ so $y, z \in H^2(PV \times PW)$. It turns out that $H^*(PV \times PW) = \mathbb{Z}[y, z]/(y^{m+1}, z^{n+1})$.

(6) Consider the Milnor hypersurface $H_{m,n} \subset \mathbb{C}P^m \times \mathbb{C}P^n$. By the previous example we have $H^*(\mathbb{C}P^m \times \mathbb{C}P^n) = \mathbb{Z}[y, z]/(y^{m+1}, z^{n+1})$. It turns out that $H^*(H_{m,n})$ is a quotient of this, namely

$$H^*(H_{m,n}) = \mathbb{Z}[y, z]/(y^{m+1}, z^n - yz^{n-1} + \ldots \pm y^n).$$

(7) Consider the Fermat hypersurface $M = X_{d,2n} \subset \mathbb{C}P^{2n}$. The class $x \in H^2(\mathbb{C}P^{2n})$ gives a class $x \in H^2(M)$ by restriction. It turns out that there is a unique class $y \in H^{2n-2}(M)$ such that $x^n = dy$ and $H^*(M)$ has basis $\{1, x, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y\}$. The ring structure is

$$H^*(X_{d,2n}) = \mathbb{Z}[x, y]/(y^2, x^n - dy).$$

(8) Let V be a complex vector space of dimension m, and let $M = \mathrm{Flag}(V)$ be the space of complete flags in V. Choose a complex inner product on V. Given a complete flag $W = (W_0, \ldots, W_m)$ we let L_i be the orthogonal complement of W_{i-1} in W_i. It is easy to see that this is a line and that $V = L_1 \oplus \ldots \oplus L_m$. We can thus define $q_i\colon M \to PV$ by $q_i(W) = L_i$, and we also write $x_i = q^*x \in H^2(M)$. Let c_i be the coefficient of t^{m-i} in the polynomial $f(t) = \prod_{i=1}^m (t + x_i)$, or in other words the i'th elementary symmetric function of the variables $x_1, \ldots, x_m$. It turns out that

$$H^*(M) = \mathbb{Z}[x_1, \ldots, x_m]/(c_1, \ldots, c_m).$$

This is a free abelian group. As a basis, we can take the set of monomials $x_1^{a_1} \ldots x_m^{a_m}$ for which $0 \leq a_i < i$ for each i.

(9) Now fix k with $0 \leq k \leq m$ and consider the Grassmannian space $\mathrm{Grass}_k(V)$. It will be helpful to compare this with the space $\mathrm{Flag}(V)$ considered above, using the map $g\colon \mathrm{Flag}(V) \to \mathrm{Grass}_k(V)$ defined by $g(W) = W_k$. Let a_i be the i'th elementary symmetric function of the variables $x_1, \ldots, x_k$, and let b_j be the j'th elementary symmetric function of the remaining variables $x_{k+1}, \ldots, x_m$. We then have

$$t^m = \sum_{k=0}^m c_i t^{m-i} = \prod_{i=1}^m (t + x_i) = \left(\sum_{i=0}^k a_i t^{k-i}\right)\left(\sum_{j=0}^{m-k} b_j t^{m-k-j}\right).$$

It follows that for $0 < l \leq m$ we have $\sum_{i+j=l} a_i b_j = 0$. In fact, there are no more relations between the a's and b's, so the subring R of $H^*(\mathrm{Flag}(V))$ generated by these elements is isomorphic to

$$\mathbb{Z}[a_1, \ldots, a_k, b_1, \ldots, b_{m-k}]/(\sum_{i+j=l} a_i b_j \mid 0 < l \leq m).$$

It turns out that the map $g^*\colon H^*(\mathrm{Grass}_k(V)) \to H^*(\mathrm{Flag}(V))$ is injective, and it gives an isomorphism of $H^* \mathrm{Grass}_k(V)$ with R.

3. Homotopy

We now discuss some further material related to homotopy invariance. We first consider the basic definition more carefully.

Definition 3.1. Let f_0 and f_1 be continuous maps from a topological space X to another space Y. A *homotopy* from f_0 to f_1 is a continuous map $F\colon [0, 1] \times X \to Y$ such that $F(0, x) = f_0(x)$ and $F(1, x) = f_1(x)$ for all $x \in X$. If there exists such a homotopy, we say that f_0 and f_1 are *homotopic* and write $f_0 \simeq f_1$.

Remark 3.2. A single map $F\colon [0,1]\times X\to Y$ corresponds to a family of maps $f_t\colon X\to Y$ (for $0\leq t\leq 1$) by the prescription $f_t(x)=F(t,x)$, and this is often a conceptually convenient point of view. However, one needs to insist that everything varies continuously, and the simplest way to formulate the required continuity condition is to say that the single map F is continuous (using the usual product topology on $[0,1]\times X$).

Example 3.3. Suppose we have maps $f_0,f_1\colon X\to Y$, where Y is a subset of a real vector space V (with the subspace topology). For $(t,x)\in[0,1]\times X$ we can then define $F(t,x)=(1-t)f_0(x)+tf_1(x)\in V$. If it happens that $F(t,x)\in Y$ for all t and x, then we have a homotopy between f_0 and f_1, which we call the *linear homotopy*. It is distressingly easy to fall into the trap of writing such formulae without verifying that $F(t,x)\in Y$.

In the case where $Y=V$, of course, there is nothing to check: any two maps from X to V are homotopic by a linear homotopy.

Example 3.4. Define maps $f_i\colon S^2\to S^2$ by

$$\begin{aligned} f_0(x,y,z)&=(x,y,z)\\ f_1(x,y,z)&=(-x,y,z)\\ f_2(x,y,z)&=(-x,-y,z). \end{aligned}$$

There is a homotopy between f_0 and f_2 given by

$$F(t,(x,y,z))=(\cos(\pi t)x-\sin(\pi t)y,\sin(\pi t)x+\cos(\pi t)y,z).$$

(In other words, at time t we rotate about the z-axis by an angle πt.) However, it turns out that f_0 is not homotopic to f_1. It is possible but difficult to prove this directly. One approach is as follows. Consider a map $F\colon[0,1]\times S^2\to S^2$, and put $C=F^{-1}\{(0,0,1)\}\subseteq[0,1]\times S^2$. If we merely asume that F is continuous, then C could be very complicated; it could even be fractal, for example, or a knot with infinitely many loops. However, after adjusting F by an arbitrarily small amount, we can arrange that F is continuously differentiable, and that C is a smooth curve, and that a certain kind of coincidental vanishing of derivatives does not happen anywhere on C. Then, by considering how the derivatives vary along C, one can check that it is impossible for F to be a homotopy from f_0 to f_1. A much simpler and more general approach is to use cohomology as follows. Recall that $H^2(S^2)=\mathbb{Z}u_2$. As f_0 is the identity we have $f_0^*(u_2)=u_2$. It will follow from Lemma 7.2 that $f_1^*(u_2)=-u_2$. As cohomology is homotopy invariant, we see that f_1 cannot be homotopic to f_0.

Example 3.5. Consider two points $u_0,u_1\in S^n$, and let L be the line segment joining them, so $L=\{(1-t)u_0+tu_1\mid 0\leq t\leq 1\}$. We claim that L lies in S^n iff $u_0=u_1$, and L passes through 0 iff $u_0=-u_1$. This should be geometrically clear, and can be deduced algebraically from the identities

$$\|(1-t)u_0+tu_1\|^2=1-t(1-t)\|u_0-u_1\|^2=((2t-1)^2\|u_0-u_1\|^2+\|u_0+u_1\|^2)/4,$$

which can be checked by direct expansion.

Now consider two maps $f_0,f_1\colon X\to S^n$, and put $F(t,x)=(1-t)f_0(x)+tf_1(x)\in\mathbb{R}^{n+1}$. As t runs from 0 to 1, the point $F(t,x)$ runs over the line segment from $f_0(x)$ to $f_1(x)$, which is only contained in S^n if $f_0(x)=f_1(x)$. Thus, we see that f_0 and f_1 can only be *linearly* homotopic if they are equal.

Nonetheless, it is still common for f_0 and f_1 to be homotopic by a nonlinear homotopy. Suppose, for example, that for all x we have $f_0(x)+f_1(x)\neq 0$. Then the line segment from $f_0(x)$ to $f_1(x)$ does not pass through zero, so we can regard F as a continuous map from X to $\mathbb{R}^{n+1}\setminus\{0\}$. There is a continuous map $r\colon\mathbb{R}^{n+1}\setminus\{0\}\to S^n$ given by $r(v)=v/\|v\|$, so we can define $G=r\circ F\colon[0,1]\times X\to S^n$. We then have $G(0,x)=r(f_0(x))=f_0(x)/\|f_0(x)\|=f_0(x)$ and similarly $G(1,x)=f_1(x)$, so this gives a homotopy from f_0 to f_1.

Lemma 3.6. *The relation of homotopy is an equivalence relation on the set* $\mathrm{Map}(X,Y)$ *of continuous maps from X to Y.*

Proof. The map $(t,x)\mapsto f_0(x)$ gives a homotopy from f_0 to itself, so the relation is reflexive. If F is a homotopy from f_0 to f_1, then the map $(t,x)\mapsto F(1-t,x)$ gives a homotopy from f_1 to f_0; so the relation is symmetric. Finally, suppose that F is a homotopy from f_0 to f_1, and G is a homotopy from f_1 to f_2. We then define a map $H\colon [0,1]\times X\to Y$ by

$$H(t,x)=\begin{cases}F(2t,x) & \text{if } 0\leq t\leq 1/2\\ G(2t-1,x) & \text{if } 1/2\leq t\leq 1.\end{cases}$$

The two clauses are consistent where they both apply, because $F(1,x)=f_1(x)=G(0,x)$. The domains of the two clauses (namely $[0,1/2]\times X$ and $[1/2,1]\times X$) are both closed in $[0,1]\times X$, and the restriction of H to either domain is clearly continuous. It therefore follows by a standard topological lemma (or an exercise) that H is continuous. This gives the required homotopy from f_0 to f_2. □

Definition 3.7. We write $[X,Y]$ for the set $\mathrm{Map}(X,Y)/\simeq$ of equivalence classes for the above relation. The elements are called *homotopy classes* of maps from X to Y.

The following principle will often be useful for specifying homotopy classes of maps.

Proposition 3.8. *Let P be a path-connected space. Suppose we have a continuous map $F\colon P\times X\to Y$, and we define $f_p\colon X\to Y$ by $f_p(x)=F(p,x)$. Then the homotopy class of f_p is independent of $p\in P$.*

Proof. Suppose we have two different points $p,q\in P$. As P is path-connected, we can choose a continuous map $u\colon [0,1]\to P$ with $u(0)=p$ and $u(1)=q$. We can then define $H\colon [0,1]\times X\to Y$ by $H(t,x)=F(u(t),x)$, and we find that this provides a homotopy between f_p and f_q. □

Lemma 3.9. *Suppose we have maps*

$$X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y \underset{g_1}{\overset{g_0}{\rightrightarrows}} Z$$

where $f_0\simeq f_1$ and $g_0\simeq g_1$. Then $g_0f_0\simeq g_1f_1$.

Proof. Choose a homotopy F from f_0 to f_1, and a homotopy G from g_0 to g_1, and then put $H(t,x)=G(t,F(t,x))$; this gives a homotopy from g_0f_0 to g_1f_1. □

Corollary 3.10. *There is a well-defined composition operation $[Y,Z]\times[X,Y]\to[X,Z]$, given by $([g],[f])\mapsto[gf]$.* □

Remark 3.11. Readers familiar with category theory may like to formulate things as follows. Let Top denote the category of topological spaces and continuous maps. We can then define a new category hTop with the same objects, but with homotopy classes as the morphisms (so $\mathrm{hTop}(X,Y)=[X,Y]$). The homotopy invariance property means that cohomology can be regarded as a contravariant functor from hTop to the category of graded rings.

Definition 3.12. A map $f\colon X\to Y$ is a *homotopy equivalence* if there exists a *homotopy inverse* $g\colon Y\to X$ with $gf\simeq 1_X$ and $fg\simeq 1_Y$. We say that X and Y are *homotopy equivalent* if there exists a homotopy equivalence between them. We say that X is *contractible* if it is homotopy equivalent to a single point.

Proposition 3.13. *If $f\colon X\to Y$ is a homotopy equivalence, then $f^*\colon H^*(Y)\to H^*(X)$ is an isomorphism of graded rings. Thus, if X and Y are homotopy equivalent, then $H^*(X)$ and $H^*(Y)$ are isomorphic. In particular, if X is contractible, then $H^0(X)=\mathbb{Z}$ and $H^i(X)=0$ for $i\neq 0$.*

Proof. Suppose that $f\colon X\to Y$ is a homotopy equivalence, with homotopy inverse $g\colon Y\to X$ say. We then have maps $f^*\colon H^*(Y)\to H^*(X)$ and $g^*\colon H^*(X)\to H^*(Y)$ of graded rings. By functoriality, we have $f^*g^*=(gf)^*\colon H^*(X)\to H^*(X)$. Moreover, we have $gf\simeq 1_X$ and cohomology is homotopy invariant so $(gf)^*=(1_X)^*$, which is just the identity map $1_{H^*(X)}$ by functoriality again, so $f^*g^*=1_{H^*(X)}$. Similarly, we have $g^*f^*=1_{H^*(Y)}$, so f^* and g^* are mutually inverse isomorphisms. The other claims follow immediately from this. □

Before giving some examples, it will be convenient to reformulate the notion of contractibility a little.

Definition 3.14. A *contraction* of a space X is a map $h\colon [0,1]\times X\to X$ such that $h(1,x)=x$ for all x, and $h(0,x)$ is independent of x.

Lemma 3.15. *A space X is contractible if and only if it has a contraction.*

Proof. Let $h\colon [0,1]\times X \to X$ be a contraction. This means that there is a point $x_0 \in X$ such that $h(0,x) = x_0$ for all $x \in X$. Let Y denote the singleton space $\{0\}$. Let f be the unique map from X to Y, given by $f(x) = 0$ for all x. Let g be the map from Y to X given by $g(0) = x_0$. Then fg is equal (and therefore homotopic) to 1_Y, and h gives a homotopy from gf to 1_X. This proves that f is a homotopy equivalence, as required. The converse is essentially the same. □

One other special case of homotopy invariance is worth explaining explicitly.

Lemma 3.16. *Suppose that $f\colon X \to Y$ is homotopic to a constant map. Then for all $k > 0$ and all $a \in H^k(Y)$, we have $f^*(a) = 0$.*

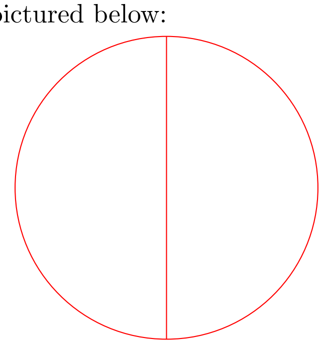
Proof. Let c be a constant map that is homotopic to f, so there is some $y_0 \in Y$ such that $c(x) = y_0$ for all x. Let g be the constant map $X \to \{0\}$, and define $h\colon \{0\} \to Y$ by $h(0) = y_0$, so $c = hg$. This means that $f^*(a) = c^*(a) = g^*(h^*(a))$. However, $h^*(a)$ lies in the group $H^k(\{0\})$, which is zero for $k > 0$. It follows that $h^*(a) = 0$ and so $f^*(a) = 0$. □

Example 3.17. Let V be any vector space; then the map $h(t,x) = tx$ defines a contraction of V. More generally, let X be any subset of V, and let x_0 be a point of X. We say that X is *star-shaped around* x_0 if for all $x \in X$, the line segment from x to x_0 is contained wholly in X. If so, then the formula $h(t,x) = tx + (1-t)x_0$ gives a contraction of X.

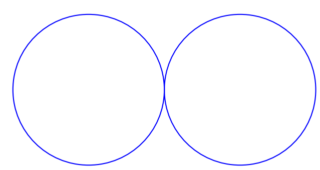
Example 3.18. Let V be a vector space with inner product, and put $V^\times = V \setminus \{0\}$. Let $i\colon S(V) \to V^\times$ be the inclusion, and define $r\colon V^\times \to S(V)$ by $r(v) = v/\|v\|$. Then $ri = 1_{S^n}$, and we can define a homotopy from $1_{V^\times}$ to ir by $h(t,v) = v/\|v\|^t$. This shows that i and r are mutually inverse homotopy equivalences.

Example 3.19. We next claim that $\mathbb{R}\setminus\mathbb{Z}$ is homotopy equivalent to $\mathbb{Z}$. Indeed, we can define $f\colon \mathbb{Z} \to \mathbb{R}\setminus\mathbb{Z}$ by $f(n) = n + 1/2$, and we can define $g\colon \mathbb{R}\setminus\mathbb{Z} \to \mathbb{Z}$ by $g(x) = n$ whenever $n < x < n+1$. Then gf is equal to the identity, and fg is linearly homotopic to the identity.

Example 3.20. For an example where neither fg nor gf is equal to the identity, consider the spaces X and Y pictured below:

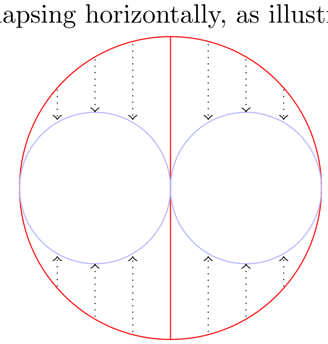


X

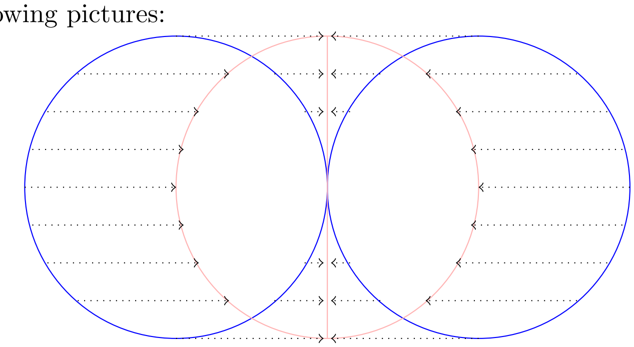


Y

We can define $f\colon X \to Y$ by collapsing vertically, and we can define $g\colon Y \to X$ by doubling and then collapsing horizontally, as illustrated by the following pictures:



f



g

The composites fg and gf are then homotopic to the respective identity maps. The reader should be able to visualise the required homotopies, but we will not attempt to give formulae.

4. Cohomology of configuration spaces

In part (c) of Example 2.11, we constructed certain elements $a_{ij} \in H^1(F_n\mathbb{C})$, and claimed that $H^*(F_n\mathbb{C})$ is generated by these elements subject to certain relations. Here we will prove that these relations do indeed hold in $H^*(F_n\mathbb{C})$. This leaves the task of proving that there are no further generators or relations, which we will not address.

First, we recall the definitions for ease of reference. The space in question is

$$F_n\mathbb{C} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_p \neq z_q \text{ for } p \neq q\}.$$

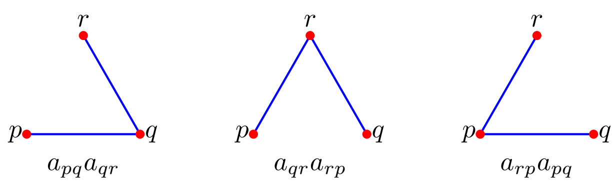
If $p, q \in \{1, \dots, n\}$ with $p \neq q$, we define $f_{pq} \colon F_n\mathbb{C} \to S^1$ by $f_{pq}(z) = (z_p - z_q)/|z_p - z_q|$. We then define $a_{pq} = f_{pq}^*(u_1) \in H^1(F_n\mathbb{C})$, where u_1 is the standard generator of $H^1(S^1)$.

We will prove the following:

Proposition 4.1. *In $H^*(F_n\mathbb{C})$ we have $a_{pq}^2 = 0$ and $a_{pq} = a_{qp}$. Moreover, for any three distinct indices p, q and r we have*

$$a_{pq}a_{qr} + a_{qr}a_{rp} + a_{rp}a_{pq} = 0.$$

The first two claims are easy and will be proved in Lemma 4.8. The last equation is called the *Arnol'd relation*; it will be proved as Proposition 4.12. The combinatorial structure can be remembered using the following picture:



We can generalise the construction of the classes a_{pq} as follows. Firstly, the inclusion $S^1 \to \mathbb{C}^\times$ is a homotopy equivalence, and so induces an isomorphism $H^*(\mathbb{C}^\times) \to H^*(S^1)$ and a bijection $[X, S^1] \to [X, \mathbb{C}^\times]$ for any space X. Under this bijection, f_{pq} corresponds to the map $F_n\mathbb{C} \to \mathbb{C}^\times$ given by $z \mapsto z_p - z_q$. We will work with S^1 or $\mathbb{C}^\times$ as convenient, and not bother to distinguish too carefully between them.

Definition 4.2. For any space X, we consider $\mathrm{Map}(X, \mathbb{C}^\times)$ as a group using pointwise multiplication of functions. We also define $\phi\colon \mathrm{Map}(X, \mathbb{C}^\times) \to H^1(X)$ by $\phi(f) = f^*(u_1)$.

Remark 4.3. Note that if F gives a homotopy from f_0 to f_1, and G gives a homotopy from g_0 to g_1, then the pointwise product of F and G gives a homotopy from $f_0.g_0$ to $f_1.g_1$; so there is an induced group structure on $[X, \mathbb{C}^\times] = [X, S^1]$ given by $[f].[g] = [f.g]$. Moreover, as cohomology is homotopy invariant, there is an induced function $\overline{\phi}\colon [X, \mathbb{C}^\times] \to H^1(X)$ given by $\overline{\phi}([f]) = \phi(f) = f^*(u_1)$. (In fact, for a very large class of spaces X, including all those considered in these notes, the map $\overline{\phi}$ is a bijection; but we will not prove that.)

Example 4.4. In $H^1(F_n\mathbb{C})$ we have $a_{pq} = \phi(f_{pq})$.

Lemma 4.5. *The map ϕ is a homomorphism.*

As the group law on $\mathrm{Map}(X, \mathbb{C}^\times)$ is written multiplicatively, and that on $H^1(X)$ is written additively, the claim is that $\phi(f_1 f_2) = \phi(f_1) + \phi(f_2)$, or equivalently $(f_1.f_2)^*(u_1) = f_1^*(u_1) + f_2^*(u_1)$.

Proof. Define maps as follows:

$$\begin{aligned}
q_1 &\colon \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{C}^\times & q_1(z_1, z_2) &= z_1 \\
q_2 &\colon \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{C}^\times & q_2(z_1, z_2) &= z_2 \\
j_1 &\colon \mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times & j_1(z) &= (z, 1) \\
j_2 &\colon \mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times & j_2(z) &= (1, z) \\
m &\colon \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{C}^\times & m(z_1, z_2) &= z_1 z_2.
\end{aligned}$$

First, Propsition 1.2 tells us that the elements 1, $q_1^*(u_1)$, $q_2^*(u_1)$ and $q_1^*(u_1)q_2^*(u_1)$ give a basis for $H^*(\mathbb{C}^\times \times \mathbb{C}^\times)$, so $q_1^*(u_1)$ and $q_2^*(u_1)$ form a basis for H^1. It follows that for some integers a_1 and a_2 we have $m^*(u_1) =$

$a_1q_1^*(u_1)+a_2q_2^*(u_2)$. By applying j_1^* to this relation we obtain $(mj_1)^*(u_1)=a_1(q_1j_1)^*(u_1)+(q_2j_1)^*(u_1)$. Here mj_1 and q_1j_1 are both just the identity map on $\mathbb{C}^\times$, whereas q_2j_1 is a constant map. Using functoriality and Lemma 3.16, we deduce that $u_1=a_1u_1+0$, so $a_1=1$. If we use j_2 instead of j_1 we find in the same way that $a_2=1$, so $m^*(u_1)=q_1^*(u_1)+q_2^*(u_1)$.

Now consider an arbitrary space X, and maps $f_1,f_2\colon X\to\mathbb{C}^\times$. We can combine these to give a map $g\colon X\to\mathbb{C}^\times\times\mathbb{C}^\times$, namely $g(z)=(f_1(z),f_2(z))$. We now apply g^* to the relation $m^*(u_1)=q_1^*(u_1)+q_2^*(u_1)$ to get $(mg)^*(u_1)=(q_1g)^*(u_1)+(q_2g)^*(u_1)$, or $\phi(mg)=\phi(q_1g)+\phi(q_2g)$. However, we have $q_1g=f_1$ and $q_2g=f_2$ and $mg=f_1.f_2$, so $\phi(f_1.f_2)=\phi(f_1)+\phi(f_2)$ as claimed. □

Remark 4.6. A key step in the above argument was to show that $m^*(u_1)=q_1^*(u_1)+q_2^*(u_1)$. Here $m=q_1.q_2$, so the relation can be rewritten as $\phi(q_1.q_2)=\phi(q_1)+\phi(q_2)$, which is a special case of the result to be proved. The remainder of the argument shows that the general case follows from this special case, which means that the special case is in some sense universal. Readers familiar with category theory may wish to formulate this more precisely using the Yoneda lemma and related ideas.

Lemma 4.7. *For any X and $f\in\mathrm{Map}(X,\mathbb{C}^\times)$ we have $\phi(f)^2=0$.*

Proof. We have $H^2(\mathbb{C}^\times)=0$ so certainly $u_1^2=0$, and f^* is a ring homomorphism, so $\phi(f)^2=f^*(u_1)^2=f^*(u_1^2)=f^*(0)=0$. □

Lemma 4.8. *In $H^*(F_n)$ we have $a_{pq}^2=0$, and $a_{pq}=a_{qp}$.*

Proof. The first claim follows from the previous lemma. Next, it is clear from the definitions that $f_{qp}(z)=-f_{pq}(z)$, so we can define a homotopy between f_{pq} and f_{qp} by $F(t,z)=e^{i\pi t}f_{pq}(z)$. As cohomology is homotopy invariant, this gives $a_{pq}=a_{qp}$. □

Lemma 4.9. *The space $F_1\mathbb{C}$ is homeomorphic to $\mathbb{C}$ and so is contractible, so $H^*(F_1\mathbb{C})=\mathbb{Z}$.*

Proof. This is clear from the definitions: the condition $z_p\neq z_q$ for $p\neq q$ is vacuuously satisfied when $n=1$, so $F_1\mathbb{C}$ is just $\mathbb{C}$. □

Lemma 4.10. *The space $F_2\mathbb{C}$ is homeomorphic to $\mathbb{C}\times\mathbb{C}^\times$ and so homotopy equivalent to S^1. More precisely, the map $f_{12}\colon F_2\mathbb{C}\to\mathbb{C}^\times$ is a homotopy equivalence, and therefore $\{1,a_{12}\}$ is a basis for $H^*(F_2\mathbb{C})$.*

Proof. Define $h\colon F_2\mathbb{C}\to\mathbb{C}\times\mathbb{C}^\times$ by $h(z_1,z_2)=(z_1,z_1-z_2)$. This is a homeomorphism, with inverse $h^{-1}(w_1,w_2)=(w_1,w_1-w_2)$. It follows easily that $F_2\mathbb{C}$ is homotopy equivalent to $\mathbb{C}^\times$ and so to S^1. More explicitly, define $j\colon\mathbb{C}^\times\to F_2(\mathbb{C})$ by $j(w)=(w,0)$. Then $f_{12}j$ is the identity on $\mathbb{C}^\times$, and the map $h(t,(z_1,z_2))=(z_1-z_2+tz_2,tz_2)$ gives a homotopy between jf_{12} and the identity on $F_2\mathbb{C}$. This proves that f_{12} is a homotopy equivalence, so $f_{12}^*\colon H^*(\mathbb{C}^\times)\to H^*(F_2\mathbb{C})$ is an isomorphism, so $\{1,a_{12}\}$ is a basis for $H^*(F_2\mathbb{C})$. □

Lemma 4.11. *There is a homeomorphism $h\colon F_3\mathbb{C}\to\mathbb{C}\times\mathbb{C}^\times\times(\mathbb{C}\setminus\{0,1\})$ given by $h(z_1,z_2,z_3)=(z_1,z_2-z_1,(z_3-z_1)/(z_2-z_1))$.*

Proof. The inverse is $h^{-1}(w_1,w_2,w_3)=(w_1,w_1+w_2,w_1+w_2w_3)$. □

We would like to analyse the effect of this map in cohomology. First, we have maps $d_0,d_1:\mathbb{C}\setminus\{0,1\}\to\mathbb{C}^\times$ given by $d_0(z)=z$ and $d_1(z)=z-1$. We will assume the fact, stated in part (b) of Example 2.11, that the elements 1, $d_0^*(u_1)$ and $d_1^*(u_1)$ give a basis for $H^*(\mathbb{C}\setminus\{0,1\})$ over $\mathbb{Z}$, with $d_0^*(u_1)d_1^*(u_1)=0$. Next, we have projection maps

$$\begin{aligned}
p_1&\colon\mathbb{C}\times\mathbb{C}^\times(\mathbb{C}\setminus\{0,1\})\to\mathbb{C}\\
p_2&\colon\mathbb{C}\times\mathbb{C}^\times(\mathbb{C}\setminus\{0,1\})\to\mathbb{C}^\times\\
p_3&\colon\mathbb{C}\times\mathbb{C}^\times(\mathbb{C}\setminus\{0,1\})\to\mathbb{C}\setminus\{0,1\}.
\end{aligned}$$

We have a basis $\{1\}$ for $H^*(\mathbb{C})$ and a basis $\{1, u_1\}$ for $H^*(\mathbb{C}^\times)$. We can combine these using Proposition 1.2 with our basis for $H^*(\mathbb{C} \setminus \{0,1\})$ to obtain the following bases for the groups $H^k(\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C} \setminus \{0,1\})$:

$$
\begin{aligned}
H^0(\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C} \setminus \{0,1\}) &= \mathbb{Z} \\
H^1(\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C} \setminus \{0,1\}) &= \mathbb{Z}\{p_2^*(u_1), p_3^* d_0^*(u_1), p_3^* d_1^*(u_1)\} \\
H^2(\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C} \setminus \{0,1\}) &= \mathbb{Z}\{p_2^*(u_1) p_3^* d_0^*(u_1), p_2^*(u_1) p_3^* d_1^*(u_1)\}.
\end{aligned}
$$

Now, as h is a homeomorphism, we see that the images under h^* of these elements give a basis for $H^*(F_3\mathbb{C})$. The first basis element for H^1 is $h^* p_2^*(u_1) = (p_2 h)^*(u_1)$, but $p_2 h(z) = z_2 - z_1 = f_{21}(z)$, so $(p_2 h)^*(u_1)$ is just a_{21} (or equivalently a_{12}). The next element is $h^* p_3^* d_0^*(u_1) = (d_0 p_3 h)^*(u_1)$, and $d_0 p_3 h \colon F_3\mathbb{C} \to \mathbb{C}^\times$ is given by $z \mapsto (z_3 - z_1)/(z_2 - z_1) = f_{13}(z)/f_{12}(z)$. This means that

$$h^* p_3^* d_0^*(u_1) = \phi(f_{13}/f_{12}) = a_{13} - a_{12}.$$

Similarly, we have $h^* p_3^* d_1^*(u_1) = \phi(d_1 p_3 h)$, and

$$(d_1 p_3 h)(z) = (z_3 - z_1)/(z_2 - z_1) - 1 = (z_3 - z_2)/(z_2 - z_1) = f_{23}(z)/f_{12}(z),$$

so $h^* p_3^* d_1^*(u_1) = a_{23} - a_{12}$. We now see that our basis for $H^1(F_3\mathbb{C})$ can be rewritten as $\{a_{12}, a_{13} - a_{12}, a_{23} - a_{12}\}$, and it follows that the list $\{a_{12}, a_{13}, a_{23}\}$ is also a basis.

We also remarked previously that $d_0^*(u_1) d_1^*(u_1) = 0$. We can now apply $h^* p_3^*$ to this to obtain the relation $(a_{13} - a_{12})(a_{23} - a_{12}) = 0$. We have seen that $a_{12}^2 = 0$, so this can be expanded as $a_{13}a_{23} - a_{12}a_{23} - a_{13}a_{12} = 0$. Note here that the elements a_{pq} have odd degree, so $a_{pq}a_{rs} = -a_{rs}a_{pq}$, and also $a_{pq} = a_{qp}$. Using these rules we can rearrange the above relation as

$$a_{12}a_{23} + a_{23}a_{31} + a_{31}a_{12} = 0,$$

which is a special case of the Arnol'd relation. It is now easy to deduce the general case:

Proposition 4.12. *For any n, and any three distinct indices $\{p, q, r\} \subseteq \{1, \dots, n\}$, we have*

$$a_{pq}a_{qr} + a_{qr}a_{rp} + a_{rp}a_{pq} = 0 \in H^2(F_n\mathbb{C}).$$

Proof. Define $\pi \colon F_n\mathbb{C} \to F_3\mathbb{C}$ by $\pi(z) = (z_p, z_q, z_r)$. We then have $f_{12}\pi(z) = f_{12}(z_p, z_q, z_r) = z_p - z_q = f_{pq}(z)$, so

$$\pi^* a_{12} = \pi^* f_{12}^*(u_1) = (f_{12}\pi)^*(u_1) = f_{pq}^*(u_1) = a_{pq}.$$

In the same way, we see that $\pi^*(a_{23}) = a_{qr}$ and $\pi^*(a_{31}) = a_{rp}$. We can thus apply the ring homomorphism π^* to the relation

$$a_{12}a_{23} + a_{23}a_{31} + a_{31}a_{12} = 0 \in H^2(F_3\mathbb{C})$$

to obtain

$$a_{pq}a_{qr} + a_{qr}a_{rp} + a_{rp}a_{pq} = 0 \in H^2(F_n\mathbb{C}).$$

□

5. Geometry of balls and spheres

Almost all of the examples mentioned so far are somehow related to the spheres S^n, and the same is true for most other examples of interest. In fact, a large number of naturally occurring spaces are actually homotopy equivalent (or even homeomorphic) to S^n. In this section we will assemble a good collection of examples of this phenomenon, many of which will be useful later.

The most basic definitions are as follows:

$$
\begin{aligned}
B^n &= \{x \in \mathbb{R}^n \mid \sum_i x_i^2 \leq 1\} \\
\mathring{B}^n &= \{x \in \mathbb{R}^n \mid \sum_i x_i^2 < 1\} \\
S^{n-1} &= \{x \in \mathbb{R}^n \mid \sum_i x_i^2 = 1\}.
\end{aligned}
$$

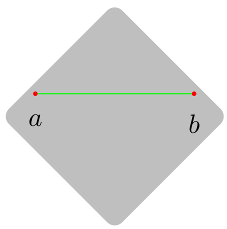
In particular, we have $B^0 = \mathring{B}^0 = \{0\}$ and $S^0 = \{-1, 1\}$ and $S^{-1} = \emptyset$.

More generally, let V be an n-dimensional vector space equipped with an inner product, and define

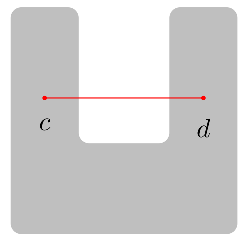
$$
\begin{aligned}
B(V) &= \{x \in V \mid \|x\| \leq 1\} \\
\mathring{B}(V) &= \{x \in V \mid \|x\| < 1\} \\
S(V) &= \{x \in V \mid \|x\| = 1\}.
\end{aligned}
$$

A choice of orthonormal basis identifies these spaces with B^n, $\mathring{B}^n$ and S^{n-1} respectively.

Definition 5.1. Let V be a vector space, and let C be a nonempty subset of V. We say that C is *convex* if whenever $a, b \in C$, the whole line segment $[a, b] = \{(1-t)a + tb \mid 0 \leq t \leq 1\}$ is contained in C.



X is convex



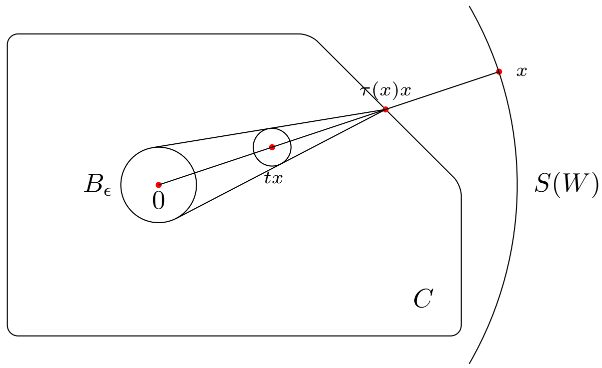
Y is not convex

Remark 5.2. Suppose that C is convex, that $c_0, \ldots, c_n \in C$, and that $t_0, \ldots, t_n \in [0, 1]$ with $\sum_{i=0}^n t_i = 1$. We claim that $\sum_i t_i c_i \in C$. Indeed, this is trivial when $n = 1$, and is the definition of convexity when $n = 2$. For general n we note that the claim is again clear when $t_n = 1$, so we may assume that $\sum_{i=0}^{n-1} t_i = 1 - t_n > 0$. We then put $t'_i = t_i/(1 - t_n)$ and $c' = \sum_{i=0}^{n-1} t'_i c_i$. By induction on n we see that $c' \in C$, so by convexity we have $(1 - t_n)c' + t_n c_n \in C$, but this easily reduces to $\sum_{i=0}^n t_i c_i \in C$ as required.

Remark 5.3. It is easy to see that the set of all inner products on V is a nonempty convex subset of the vector space $\mathrm{Hom}_{\mathbb{R}}(V \otimes_{\mathbb{R}} V, \mathbb{R})$, so in particular it is contractible. To a homotopy theorist, a contractible space of choices is essentially as good as a unique choice, so we will generally assume that vector spaces come equipped with an inner product.

Lemma 5.4. *Let V be as above, and let C be a compact convex subset of V. Let W be the span of the set $\{x - y \mid x, y \in C\}$. Then there is a homeomorphism $r \colon C \to B(W)$, which carries the boundary ∂C to $S(W)$.*

Proof. After translating if necessary we may assume that $0 \in C$ and thus that W is just the span of C. We may then replace V by W and thus assume that C spans V. Choose a basis $\{c_1, \ldots, c_n\}$ contained in C and put $c_0 = 0$. If $t_1, \ldots, t_n > 0$ and $\sum t_i < 1$ then we can put $t_0 = 1 - \sum_{i>0} t_i$ and by convexity we have $\sum_{i>0} t_i c_i = \sum_{i \geq 0} t_i c_i \in C$. It follows that the interior of C is nonempty, and after translating if necessary we can assume that it contains 0. Thus, for some $\epsilon > 0$ the open ball $\mathring{B}_\epsilon$ round zero is contained in C. Suppose $x \in S(W)$, and put $T_x = \{t \geq 0 \mid tx \in C\}$. This is a compact convex subset of $[0, \infty)$ containing $[0, \epsilon)$, so it has the form $[0, \tau(x)]$ for some $\tau(x) \geq \epsilon$. If $t < \tau(x)$ we claim that tx lies in the interior of C. Indeed, we have $t = (1 - s)\tau(x)$ for some $s > 0$ and by convexity $sB_\epsilon + tx = sB_\epsilon + (1 - s)\tau(x)x$ is a neighbourhood of tx contained in C. We conclude that $\{t \mid tx \in \partial C\} = \{\tau(x)\}$. Now define $r \colon \partial C \to S(W)$ by $r(y) = y/\|y\|$; this is easily seen to be a continuous bijection, with inverse $r'(x) = \tau(x)x$. Moreover, a continuous bijection of compact Hausdorff spaces is automatically a homeomorphism. Next, extend r' over $B(W)$ by defining $r'(x) = \|x\| r'(x/\|x\|)$ when $x \neq 0$, and $r'(0) = 0$. It is easy to deduce that this is a homeomorphism $B(W) \to C$, and we define $r \colon C \to B(W)$ to be its inverse.



∎

Example 5.5. Write $I = [0,1] \subset \mathbb{R}$ and

$$\Delta_n = \{(x_0, \dots, x_n) \in I^{n+1} \mid \sum_i x_i = 1\} \subset \mathbb{R}^{n+1}.$$

We could take C to be I or Δ_n or $I^k \times \Delta_m \times \Delta_n \times B^p$.

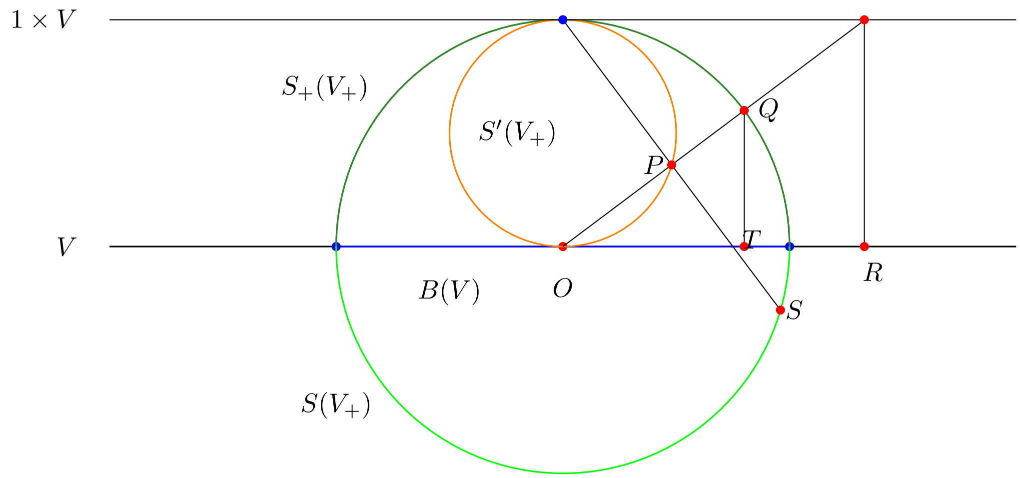
We next consider various kinds of stereographic projection, extending the familiar homeomorphism of the "Riemann sphere" $\mathbb{C} \cup \{\infty\} \simeq \mathbb{R}^2 \cup \{\infty\}$ with S^2. Let V be a real inner product space and define

$$\begin{aligned}
V_+ &= \mathbb{R} \oplus V \\
S^V &= V \cup \{\infty\} = \text{ the one-point compactification of } V \\
S(V_+) &= \{(t,v) \in V_+ \mid t^2 + v^2 = 1\} \\
S_+(V_+) &= \{(t,v) \in S(V_+) \mid t \geq 0\} \\
S'(V_+) &= \{(t,v) \in V_+ \mid (t-1/2)^2 + v^2 = 1/4\} \\
&= \{(t,v) \in V_+ \mid t(1-t) = v^2\}.
\end{aligned}$$

We can define homeomorphisms

$$S'(V_+) \simeq S_+(V_+)/S(V) \simeq S^V \simeq S(V_+) \simeq B(V)/S(V)$$

by letting the points P, Q, R, S and T in the following diagram correspond to each other.



Formulae can be read off from the following table:

P	Q	R	S	T
$S'(V_+)$	$S_+(V_+)/S(V)$	S^V	$S(V_+)$	$B(V)/S(V)$
(p,u)	$(p,u)/\sqrt{p}$	u/p	$(2p-1,2u)$	$u/\sqrt{p}$
$q(q,v)$	(q,v)	v/q	$(2q^2-1,2qv)$	v
$(1,w)/(1+w^2)$	$(1,w)/\sqrt{1+w^2}$	w	$(w^2-1,2w)/(1+w^2)$	$w/\sqrt{1+w^2}$
$(1+s,x)/2$	$(1+s,x)/\sqrt{2(1+s)}$	$x/(1+s)$	(s,x)	$x/\sqrt{2(1+s)}$
$(1-y^2,y\sqrt{1-y^2})$	$(\sqrt{1-y^2},y)$	$y/\sqrt{1-y^2}$	$(1-2y^2,2y\sqrt{1-y^2})$	y

By taking $V=\mathbb{R}^n$ and using the evident homeomorphism $\mathbb{R}^{n+1}\simeq\mathbb{R}\oplus\mathbb{R}^n$ we obtain homeomorphisms

$$S^n\simeq S^n_+/S^{n-1}\simeq\mathbb{R}^n\cup\{\infty\}\simeq B^n/S^{n-1}.$$

Remark 5.6. In these homeomorphisms, the first coordinate behaves differently from the remaining n coordinates. We could obtain a different set of homeomorphisms by regarding $\mathbb{R}^{n+1}$ as $\mathbb{R}^n\oplus\mathbb{R}$, so the last coordinate would behave differently from the first n. This makes little difference except to introduce a sign $(-1)^n$ in various cohomological calculations; some care is needed to keep this sort of thing straight.

6. Geometry of Hermitian spaces

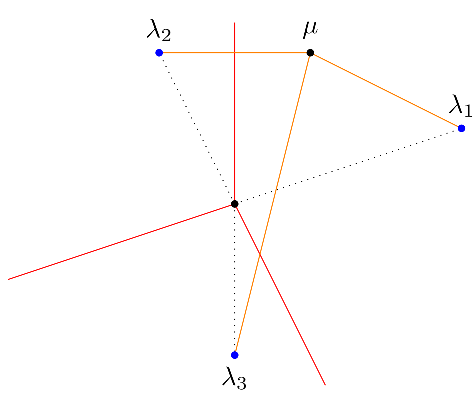
Let V be a complex vector space of dimension n. We always equip $\mathbb{C}^n$ with the inner product $\langle u,v\rangle=\sum_i u_i\overline{v}_i$. By choosing an orthonormal basis in the usual way, we see that any n-dimensional Hermitian space is isomorphic to $\mathbb{C}^n$ with the standard structure.

Definition 6.1. Let V and W be complex vector spaces. We then define $\mathcal{J}(V,W)$ to be the set of complex-linear injective maps $j\colon V\to W$. If V and W have Hermitian structures, we also let $\mathcal{L}(V,W)$ be the set of complex-linear maps $j\colon V\to W$ such that $\langle j(v_0),j(v_1)\rangle=\langle v_0,v_1\rangle$. Note that if $j\in\mathcal{L}(V,W)$ then j preserves norms and is thus injective. We topologise $\mathcal{J}(V,W)$ and $\mathcal{L}(V,W)$ as subspaces of $\mathrm{Hom}_{\mathbb{C}}(V,W)$.

The space $\mathrm{Hom}_{\mathbb{C}}(V,W)$ is a vector space, so it is contractible. The spaces $\mathcal{L}(V,W)$ and $\mathcal{J}(V,W)$ are not contractible, but we will show that they are path-connected (provided that $\dim(V)\leq\dim(W)$) and homotopy equivalent to each other. (This will be very useful in conjunction with Proposition 3.8.)

Lemma 6.2. *If $\dim(V)\leq\dim(W)$ then $\mathcal{J}(V,W)$ is path-connected. (Of course, if $\dim(V)>\dim(W)$ then $\mathcal{J}(V,W)=\emptyset$.)*

Proof. We first claim that $\mathcal{J}(W,W)=\mathrm{Aut}(W)$ is path-connected. Indeed, suppose that $\alpha\in\mathrm{Aut}(W)$, and let $\lambda_1,\dots,\lambda_r$ be the eigenvalues of α. As α is invertible, we know that $\lambda_k\neq 0$ for all k. Choose a number $\mu\in\mathbb{C}^\times$ such that $\arg(\mu)\neq\arg(-\lambda_k)$ for all k (so in the picture below, μ does not lie on any of the the red lines).



This means that the line segment from μ to λ_k (shown above in orange) does not pass through 0. When $0 \leq t \leq 1$, the eigenvalues of the matrix $\alpha_t := t\alpha + (1-t)\mu$ are the numbers $t\lambda_k + (1-t)\mu \neq 0$, so α_t is invertible, and these matrices form a path from α to μ times the identity map. It is easy to join μ to 1 by a path in $\mathbb{C}^\times$ and thus conclude that α can be joined to the identity in $\mathrm{Aut}(W)$.

Now choose a linear embedding $j \in \mathcal{J}(V, W)$; this is possible because $\dim(V) \leq \dim(W)$. Define $f\colon \mathrm{Aut}(W) \to \mathcal{J}(V, W)$ by $f(\alpha) = \alpha \circ j$; this is continuous, and elementary linear algebra shows that it is surjective. As $\mathrm{Aut}(W)$ is path-connected, we see that the same is true of $\mathcal{J}(V, W)$. □

Construction 6.3. Let V and W be as above, with $\dim(V) \leq \dim(W)$. If $\alpha\colon V \to W$ is injective and L is a line in V then $\alpha(L)$ is a line in W. We can thus define $J\colon \mathcal{J}(V, W) \times PV \to PW$ by $J(\alpha, L) = \alpha(L)$, and it is easy to see that this is continuous. Thus, if we choose an α and put $j_{VW}(L) = J(\alpha, L) = \alpha(L)$ then the homotopy class of j_{VW} is independent of the choice of α. It is also easy to see that $j_{VV} = 1$ and $j_{UW} = j_{VW} \circ j_{UV}$ up to homotopy, whenever this makes sense.

Definition 6.4. Let V be a Hermitian space, and $\alpha\colon V \to V$ a linear map. We say that α is *self-adjoint* if $\alpha^\dagger = \alpha$, or equivalently $\langle \alpha(x), y\rangle = \langle x, \alpha(y)\rangle$ for all $x, y \in V$. We say that α is *nonnegative* if for all $x \in V$, the number $\langle \alpha(x), x\rangle$ is real and nonnegative.

Remark 6.5. If $\alpha = \beta^\dagger\beta$ for some $\beta\colon V \to W$ then $\alpha^\dagger = \beta^\dagger\beta^{\dagger\dagger} = \beta^\dagger\beta = \alpha$, so α is self-adjoint. Moreover, we have $\langle \alpha(x), x\rangle = \langle \beta(x), \beta(x)\rangle = \|\beta(x)\|^2 \geq 0$, so α is nonnegative.

Lemma 6.6. *α is self-adjoint iff $\langle \alpha(x), x\rangle$ is real for all $x \in V$. (In particular, if α is nonnegative then it is self-adjoint.)*

Proof. First suppose that α is self-adjoint, so $\langle \alpha(x), y\rangle = \langle x, \alpha(y)\rangle$ for all x and y. Put $y = x$ and use the rule $\langle u, v\rangle = \overline{\langle v, u\rangle}$ to see that $\langle \alpha(x), x\rangle = \overline{\langle \alpha(x), x\rangle}$, so $\langle \alpha(x), x\rangle$ is real.

Conversely, suppose that $\langle \alpha(x), x\rangle$ is always real. Put $f(x, y) = \langle \alpha(x), y\rangle - \langle x, \alpha(y)\rangle$, so the claim is that $f(x, y) = 0$ for all x and y. We first show that $f(x, y)$ is real. Indeed, the numbers $\langle \alpha(x+y), x+y\rangle$ and $\langle \alpha(x), x\rangle$ and $\langle \alpha(y), y\rangle$ are real, because α is positive. Also, using the rule $\langle v, u\rangle = \overline{\langle u, v\rangle}$ we see that $\langle u, v\rangle + \langle v, u\rangle$ is always real, so in particular $\langle \alpha(y), x\rangle + \langle x, \alpha(y)\rangle$ is real. One can also check that

$$f(x, y) = \langle \alpha(x+y), x+y\rangle - \langle \alpha(x), x\rangle - \langle \alpha(y), y\rangle - (\langle \alpha(y), x\rangle + \langle x, \alpha(y)\rangle),$$

which is real as claimed. This holds for all x and y, so we can replace x by ix to see that $f(ix, y)$ is also real. However, we see from the definition that $f(ix, y) = if(x, y)$, so both $f(x, y)$ and $if(x, y)$ are real, so $f(x, y) = 0$. □

Lemma 6.7. *If α is self-adjoint, then every eigenvalue of α is real. If α is nonnegative, then every eigenvalue of α is nonnegative. In either case, the eigenspaces for different eigenvalues are orthogonal.*

Proof. Let $x \in V$ be a nonzero eigenvector with eigenvalue λ, so that $\alpha(x) = \lambda x$. We can replace x by $x/\|x\|$ and thus assume that $\|x\| = 1$. We then have $\langle \alpha(x), x\rangle = \langle \lambda x, x\rangle = \lambda\langle x, x\rangle = \lambda$. If α is self-adjoint (resp. nonnegative), we deduce that λ is real (resp. nonnegative). Now suppose we have another eigenvector y with a different eigenvalue, say μ. It will again be harmless to assume that $\|y\| = 1$, and again we have $\mu \in \mathbb{R}$. We now have

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle \alpha(x), y\rangle = \langle x, \alpha(y)\rangle = \langle x, \mu y\rangle = \mu\langle x, y\rangle,$$

so $(\lambda - \mu)\langle x, y\rangle = 0$. As $\lambda \neq \mu$, it follows that $\langle x, y\rangle = 0$, as required. □

Lemma 6.8. *If $\alpha\colon V \to V$ is self-adjoint, then one can choose an orthonormal basis $e_1, \dots, e_n$ for V and real numbers $\lambda_1, \dots, \lambda_n$ such that $\alpha(e_k) = \lambda_k e_k$ for all k.*

Proof. This is trivial if $V = 0$, so suppose that $\dim(V) = n > 0$. By standard linear algebra, the map α has at least one eigenvector, so we can find $e_n \in V \setminus \{0\}$ and $\lambda_n \in \mathbb{C}$ with $\alpha(e_n) = \lambda_n e_n$. After replacing e_n by $e_n/\|e_n\|$ if necessary, we may assume that $\|e_n\| = 1$. Now put $V' = e_n^\perp = \{v \in V \mid \langle v, e_n\rangle = 0\}$. If $v \in V'$ then

$$\langle \alpha(v), e_n\rangle = \langle v, \alpha^\dagger e_n\rangle = \langle v, \alpha(e_n)\rangle = \langle v, \lambda_n e_n\rangle = \overline{\lambda_n}\langle v, e_n\rangle = 0,$$

so $\alpha(v) \in V'$. This means that α restricts to give an endomorphism of V', which is again self-adjoint. Thus, by induction on n, we can find an orthonormal basis $e_1, \dots, e_{n-1}$ for V' and real numbers $\lambda_1, \dots, \lambda_{n-1}$ such that $\alpha(e_i) = \lambda_i e_i$ for $i = 1, \dots, n-1$. By appending e_n and λ_n to these lists, we obtain the required basis for V. $\square$

Proposition 6.9. *Let $P = P(V)$ be the space of nonnegative maps $\beta\colon V \to V$. There is a homeomorphism $\phi\colon P \to P$ given by $\phi(\beta) = \beta^2$.*

Proof. First, note that if $\beta \in P$ then $\beta^2 = \beta^\dagger\beta$ which is nonnegative, so the formula $\phi(\beta) = \beta^2$ does indeed define a map $P \to P$, which is clearly continuous.

Now suppose we have an element $\alpha \in P$. Let the distinct eigenvalues of α be $\lambda_1, \dots, \lambda_r$, and let $V_1, \dots, V_r$ be the corresponding eigenspaces. We see from Lemmas 6.7 and 6.8 that V the numbers λ_i are real and nonnegative, and that V is the orthogonal direct sum of the spaces V_i. We can thus let μ_i be the nonnegative real square root of λ_i, and put

$$\beta = \bigoplus_i (\mu_i.1_{V_i})\colon V = \bigoplus_i V_i \to \bigoplus_i V_i = V.$$

It is then easy to see that $\beta \in P$ and

$$\beta^2 = \bigoplus_i \mu_i^2.1_{V_i} = \bigoplus_i \lambda_i.1_{V_i} = \alpha,$$

so $\phi(\beta) = \alpha$.

Suppose we have some other element $\gamma \in P$ with $\gamma^2 = \alpha$. It is then clear that $\gamma\alpha = \gamma^3 = \alpha\gamma$. Thus, for $x \in V_i$ we have $\alpha\gamma(x) = \gamma\alpha(x) = \gamma(\lambda_i x) = \lambda_i\gamma(x)$, so $\gamma(x) \in V_i$. This means that γ decomposes as the direct sum of certain maps $\gamma_i\colon V_i \to V_i$. Now $\gamma_i^2 = \alpha|_{V_i} = \lambda_i.1_{V_i} = \mu_i^2.1_{V_i}$. If $\mu_i = 0$ this gives $\|\gamma_i(x)\| = \langle\gamma_i(x), \gamma_i(x)\rangle = \langle\gamma_i^2(x), x\rangle = 0$, so $\gamma_i = 0$, which is the same as $\beta|_{V_i}$. If $\mu_i > 0$ then we instead note that $(\gamma_i + \mu_i)(\gamma_i - \mu_i) = 0$, and $\gamma_i + \mu_i$ has strictly positive eigenvalues and so is invertible, so $\gamma_i - \mu_i = 0$, so again $\gamma_i = \beta|_{V_i}$. It therefore follows that $\gamma = \beta$.

We now see that $\phi\colon P \to P$ is a continuous bijection. We still need to check that the inverse is continuous. First note that given a polynomial $f(x) = \sum_{i=0}^d c_i x^i$ (with $c_i \in \mathbb{R}$) and a linear map $\alpha \in \mathrm{End}(V)$, we can define $f(\alpha) = \sum_i c_i\alpha^i \in \mathrm{End}(V)$. In this way we can interpret f as a map $\mathrm{End}(V) \to \mathrm{End}(V)$, which is clearly continuous. If $\alpha \in P$ has eigenvalues and eigenspaces as before, we find that $f(\alpha)$ acts on V_i as $f(\lambda_i)$ times the identity. If we can find a polynomial $f(x)$ that is close to $\sqrt{x}$ in a suitable sense, then $f(\alpha)$ will be close to $\phi^{-1}(\alpha)$, and this can be used to prove continuity.

We can fill in the details as follows. Put $p_n(x) = \sum_{k=1}^n a_k x^k$, where

$$a_k = (-1)^{k-1}\begin{pmatrix} 1/2 \\ k \end{pmatrix} = (-1)^{k-1}\frac{\frac{1}{2}\,\frac{-1}{2}\,\frac{-3}{2}\cdots\frac{3-2k}{2}}{k!}.$$

From this we see that $a_k > 0$ for all k, and that $p_k(x)$ is a truncated Taylor series for the function $q(x) = 1 - \sqrt{1-x}$. This function is analytic on the open unit disc, and it follows by standard complex analysis that $p_n(x)$ converges to $q(x)$ uniformly on $[0, r]$ for any $r < 1$. We need to know that we actually have uniform convergence on $[0, 1]$. If $0 \leq x < 1$ we know that $p_n(x)$ converges to $q(x)$, and all the coefficients are positive, so $p_n(x) \leq q(x)$, and from the definition we see that $q(x) \leq 1$, so $p_n(x) \leq 1$. As p_n is continuous everywhere, we can now let x tend to one to see that $p_n(1) \leq 1$. As this holds for all n we see that the series $\sum_k a_k$ is convergent, so the series $\sum_k a_k x^k$ converges absolutely and uniformly on $[0, 1]$ to some continuous function. This function must agree with $q(x)$ on $[0, r]$ for all $r < 1$, so by continuity it is equal to $q(x)$ everywhere. It follows that the polynomials $1 - p_n(1-x)$ converge uniformly on $[0, 1]$ to $1 - q(1-x) = \sqrt{x}$. Now fix a large constant R, and consider the polynomials $m_{R,k}(x) = \sqrt{R}(1 - p_k(1 - x/R))$; we deduce that they converge uniformly on $[0, R]$ to $\sqrt{R}(1 - q(1 - x/R)) = \sqrt{R}\sqrt{x/R} = \sqrt{x}$.

Now put

$$X_R = \{\alpha \in P \mid \text{ all eigenvalues are at most } R\}.$$

The polynomials $m_{R,k}$ give continuous maps $X_R \to \mathrm{End}(V)$ that converge uniformly to ϕ^{-1}; it follows that ϕ^{-1} is continuous as required. $\square$

Remark 6.10. Thes sequence of polynomials $1 - p_n(1-x)$ used above does not actually converge very rapidly to $\sqrt{x}$. Convergence is much faster if we use the polynomials

$$f_n(x) = \sum_{j=0}^{n}\sum_{k=0}^{n}(-1)^{j+k}\frac{(2n+2j+1)!(2n+2k+1)!}{2^{4n}(2j+2k+1)(2k+2)(n+j)!(n-j)!(2j)!(n+k)!(n-k)!(2k)!}x^j,$$

but it is not so easy to prove that.

Definition 6.11. Given $\alpha \in P$ we write $\sqrt{\alpha}$ for $\phi^{-1}(\alpha)$, so $\sqrt{\alpha}$ is the unique element $\beta \in P$ such that $\beta^2 = \alpha$.

Remark 6.12. If we do not insist that $\beta \in P$, then there are very many different maps β with $\beta^2 = \alpha$ (not just $\sqrt{\alpha}$ and $-\sqrt{\alpha}$). For example, take $\alpha = 1_V$. Then for every splitting $V = V_0 \oplus V_1$ we have an endomorphism $\beta = 1_{V_0} \oplus (-1_{V_1})$ with $\beta^2 = 1$.

Proposition 6.13. *Let V and W be Hermitian spaces, and let $P'(V)$ be the space of strictly positive self-adjoint endomorphisms of V. Then there is a natural homeomorphism*

$$\mu\colon \mathcal{L}(V,W) \times P'(V) \to \mathcal{J}(V,W)$$

given by $\mu(\gamma,\alpha) = \gamma\alpha$, with inverse

$$\mu^{-1}(\beta) = (\beta \circ (\sqrt{\beta^\dagger\beta})^{-1}, \sqrt{\beta^\dagger\beta}).$$

Proof. First, it is clear that the formula $\mu(\gamma,\alpha) = \gamma\alpha$ does indeed give a continuous map $\mathcal{L}(V,W)\times P'(V) \to \mathcal{J}(V,W)$. In the opposite direction, suppose we have $\beta \in \mathcal{V}(V,W)$. For $v \in V \setminus \{0\}$ we then have $\beta(v) \neq 0$ so $\langle \beta^\dagger\beta(v), v\rangle = \|\beta(v)\|^2 > 0$. This proves that $\beta^\dagger\beta$ is strictly positive, and it follows that the same is true of the map $\alpha = \sqrt{\beta^\dagger\beta}$. In particular, we see that α is invertible, so we can put $\gamma = \beta\alpha^{-1}\colon V \to V$. We now have $\gamma^\dagger = (\alpha^\dagger)^{-1}\beta^\dagger = \alpha^{-1}\beta^\dagger$, so $\gamma^\dagger\gamma = \alpha^{-1}\beta^\dagger\beta\alpha^{-1} = \alpha^{-1}\alpha^2\alpha^{-1} = 1$. This shows that $\gamma \in \mathcal{L}(V,W)$, so it is valid to define a map $\nu\colon \mathcal{J}(V,W) \to \mathcal{L}(V,W) \times P'(V)$ by

$$\nu(\beta) = (\beta \circ (\sqrt{\beta^\dagger\beta})^{-1}, \sqrt{\beta^\dagger\beta}).$$

It is clear that $\mu\nu(\beta) = \beta$. Conversely, suppose we start with $\gamma \in \mathcal{L}(V,W)$ and $\alpha \in P'(V)$, and we define $\beta = \gamma\alpha$. As $\gamma \in \mathcal{L}(V,W)$ we have $\gamma^\dagger\gamma = 1$, so $\beta^\dagger\beta = \alpha\gamma^\dagger\gamma\alpha = \alpha^2$, so $\sqrt{\beta^\dagger\beta} = \alpha$. From this we see that $\nu\mu = 1$, so μ and ν are mutually inverse homeomorphisms. □

Lemma 6.14. *The spaces $P(V)$ and $P'(V)$ are contractible.*

Proof. For $t \in [0,1]$ and $\alpha \in P'(V)$, put $h(t,\alpha) = t\alpha + (1-t).1_V$. We then have $\langle h(t,\alpha)v, v\rangle = t\langle\alpha(v), v\rangle + (1-t)\|v\|^2$. Suppose that $v \neq 0$. If $t > 0$ then the term $t\langle\alpha(v), v\rangle$ is strictly positive, and if $t < 1$ then $(1-t)\|v\|^2$ is strictly positive, so in all cases $\langle h(t,\alpha)v, v\rangle > 0$. This shows that $h(t,\alpha) \in P'(V)$ for all $(t,\alpha) \in [0,1] \times P'(V)$, and clearly $h(0,\alpha) = 1_V$ for all α, so we have the required contraction. The same formula also works for $P(V)$. □

Corollary 6.15. *The space $\mathcal{L}(V,W)$ is homotopy equivalent to $\mathcal{J}(V,W)$, and thus is connected when* $\dim(V) \leq \dim(W)$.

Proof. As $P'(V)$ is homotopy equivalent to a point, we see that $\mathcal{L}(V,W)$ is homotopy equivalent to $\mathcal{L}(V,W)\times P'(V)$, and thus to $\mathcal{J}(V,W)$. □

Exercise 6.16. A homotopy equivalence between X and Y gives a bijection $\pi_0(X) \simeq \pi_0(Y)$ (where $\pi_0(X)$ is the set of path-components of X). In particular, check that if X is path-connected and Y is homotopy equivalent to X then Y is path-connected.

Construction 6.17. Let V and W be Hermitian spaces, with $\dim(V) \leq \dim(W)$. If $\alpha \in \mathcal{L}(V,W)$ we find that $\alpha^\dagger\alpha = 1_V$ and thus that the map $\epsilon = \alpha\alpha^\dagger\colon W \to W$ satisfies $\epsilon^2 = \epsilon$. In fact, it's not hard to check that ϵ is the orthogonal projection of W onto the subspace $\alpha(V)$. Given an automorphism β of V we have an automorphism $\alpha\beta\alpha^{-1}$ of $\alpha(V)$ and thus an automorphism $\beta' = \beta \oplus 1$ of $\alpha(V) \oplus \alpha(V)^\perp = W$. It's not hard to check the formula

$$\beta' = 1 - \alpha\alpha^\dagger + \alpha\beta\alpha^\dagger,$$

and thus that the definition $J_\alpha(\beta) = J(\alpha, \beta) = \beta'$ gives a continuous map $\mathcal{L}(V, W) \times U(V) \to U(W)$. Just as in Construction 6.3, we therefore get a well-defined homotopy class of maps $j_{VW} \colon U(V) \to U(W)$.

Note that the map J_α is actually a group homomorphism. If α is just the usual inclusion of $\mathbb{C}^n$ in $\mathbb{C}^{n+m}$ then J_α is just the map

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

of matrix groups.

Construction 6.18. Let V be a nonzero Hermitian space. Given $v \in S(V)$, we can define a map $\epsilon_v \colon U(V) \to S(V)$ by $\epsilon_v(\alpha) = \alpha(v)$. The homotopy class of ϵ_v is independent of v (because $S(V)$ is connected and the map $(v, \alpha) \mapsto \alpha(v)$ is continuous). We may thus denote this homotopy class by ϵ_V. If $\dim(W) < \dim(V)$ and $\beta \in \mathcal{L}(W, V)$ then we can choose v to be orthogonal to $\beta(W)$, and then $\epsilon_v J_\beta(\gamma) = v$ for all $\gamma \in U(W)$. It follows that $\epsilon_V \circ j_{WV}$ is the homotopy class of a constant map $U(W) \to S(V)$. It is also easy to check that when $\dim(W) = \dim(V)$ we have $\epsilon_V j_{WV} = \epsilon_W$ up to homotopy.

Construction 6.19. We next define the "complex reflection map" $r \colon S^1 \times PV \to U(V)$. We regard S^1 as $\{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Given $\lambda \in S^1$ and $L \in PV$ we let $r(\lambda, L) \colon V \to V$ be the map with eigenvalue λ on L and 1 on $L^\perp$; this clearly gives an element of $U(V)$. Moreover, if v is a generator of L we have

$$r(\lambda, L)(x) = x + (1 - \lambda)\langle x, v\rangle v / v^2.$$

Using this formula, we see that the map $r \colon S^1 \times PV \to U(V)$ is continuous. Clearly $r(1, L) = 1_V$ for all L, so r induces a map $r \colon \Sigma PV_+ = (S^1 \times PV)/(1 \times PV) \to U(V)$. Using some elementary theory of eigenvalues we see that this induced map is injective.

Lemma 6.20. *Let V be a Hermitian space, and v a point of $S(V)$. Write $W = (\mathbb{C}v)^\perp < V$. Then the composite*

$$S^1 \times PV \xrightarrow{r} U(V) \xrightarrow{\epsilon_v} S(V)$$

induces a homeomorphism

$$\frac{S^1 \times PV}{(S^1 \times PW) \cup (1 \times PV)} \to S(V).$$

Proof. Write $X = (S^1 \times PV)/(S^1 \times PW \cup 1 \times PV)$. It is easy to check that $\epsilon_v r(S^1 \times PW \cup 1 \times PV) = \{v\}$, so we do indeed get an induced map $f \colon X \to S(V)$. This is a continuous map of compact Hausdorff spaces, so it suffices to check that it is bijective, or equivalently that $\epsilon_v r$ gives a bijection

$$X \setminus \{0\} = (S^1 \setminus \{1\}) \times PV \setminus PW \to S(V) \setminus \{v\}.$$

Suppose we have $x \in S(V) \setminus \{v\}$. Then $v - x \neq 0$ so $L = \mathbb{C}(v - x) \in PV$. As $\|v\| = \|x\| = 1$ and $v \neq x$ we must have $\langle v, x\rangle \neq 1$ and so $\langle v, v - x\rangle \neq 0$ so $v - x \notin W$ so $L \notin PW$. We can split V as $L^\perp \oplus L$, so there is a unique way to write $v = a + b$ with $a \in L^\perp$ and $b \in L$. As $v - x \in L$ we see that $x = a + c$ for some $c \in L$. As $\langle v, v - x\rangle \neq 0$ we see that $v \notin L^\perp$ so $b \neq 0$, so b is a basis for the one-dimensional space L. It follows that there is a unique $\lambda \in \mathbb{C}$ such that $c = \lambda b$. We also have $1 = v^2 = a^2 + b^2$ and $1 = x^2 = a^2 + |\lambda|^2 b^2$; it follows that $|\lambda| = 1$. Also, as $x \neq v$ we have $\lambda \neq 1$. Thus $(\lambda, L) \in X \setminus \{0\}$ and $\epsilon_v r(\lambda, L) = r(\lambda, L)(v) = x$. This shows that f is surjective. It is easy to check that our choice of L and v was forced, so in fact f is bijective. $\square$

Corollary 6.21. *There is a unique homeomorphism $r' \colon S(V) \wedge U(W)_+ \to U(V)/U(W)$ making the following diagram commute:*

$$\begin{tikzcd}
S^1 \times PV \times U(W) \arrow[r, "r \times 1"] \arrow[d, two heads, "\epsilon r \times 1"'] & U(V) \times U(W) \arrow[r, "\mu"] & U(V) \arrow[d, two heads] \\
S(V) \wedge U(W)_+ \arrow[rr, "r'"'] & & U(V)/U(W)
\end{tikzcd}$$

Moreover, we have $\epsilon r'(x \wedge \beta) = x$ for $x \in S(V)$, $\beta \in U(W)$.

Remark 6.22. Here the symbol $U(V)/U(W)$ means the quotient space of $U(V)$ in which $\alpha \sim \beta$ iff $\alpha = \beta$ or $\alpha, \beta \in U(W)$, so that the subspace $U(W)$ is collapsed to a single point. In other contexts the symbol $U(V)/U(W)$ means the coset space, in which $\alpha \sim \beta$ iff $\alpha^{-1}\beta \in U(W)$. In our case, one can check that the map $\epsilon\colon U(V) \to S(V)$ induces a homeomorphism of the coset space with $S(V)$.

Proof. It is easy to check that

$$S(V) \wedge U(W)_+ = (S(V) \setminus \{v\}) \times U(W) \amalg \{0\}$$

as sets. Given $x \in S(V) \setminus \{v\}$ and $\beta \in U(W)$ we note (from the lemma) that there is a unique $(\lambda, L) \in (S^1 \setminus \{1\}) \times (PV \setminus PW)$ such that $r(\lambda, L)(v) = x$. We define $r'(x \wedge \beta) = r(\lambda, L) \circ \beta$. If we also define $r'(0) = 0$ we get a function $S(V) \wedge U(W)_+ \to U(V)/U(W)$, and it is easy to check that this makes the diagram commute. The composite going across and then down is continuous, and the left hand vertical map is a surjection of compact Hausdorff spaces and thus a quotient map. It follows that r' is continuous. Now suppose that $\alpha \in U(V) \setminus U(W)$, so $x := \alpha(v) \neq v$. As before we have a unique (λ, L) such that $r(\lambda, L)(v) = x$, and we put $\beta = r(\lambda, L)^{-1}\alpha$. Then $\beta(v) = r(\lambda, L)^{-1}(x) = v$, so $\beta \in U(W)$, and clearly $r'(x \wedge \beta) = \alpha$. Suppose we also have $\alpha = r'(x' \wedge \beta')$ with $x' \neq v$, say $x' = \epsilon r(\lambda', L')$ with $\lambda' \neq 1$ and $L' \notin PW$. Then $\alpha = r(\lambda', L') \circ \beta'$ so

$$x := \alpha(v) = r(\lambda', L')(\beta'(v)) = r(\lambda', L')(v) = x',$$

so $L = L'$ and $\lambda = \lambda'$, so $\beta = r(\lambda, L)^{-1}\alpha = \beta'$. Thus r' gives a bijection

$$(S(V) \wedge U(W)_+) \setminus \{0\} \to U(V) \setminus U(W),$$

and thus a bijection

$$(S(V) \wedge U(W)_+) \setminus \{0\} \to U(V)/U(W),$$

as required. □

7. Cohomology of balls and spheres

Firstly, we recall that $H^*(1) = \mathbb{Z}$ (where 1 denotes the one-point space $\{0\}$). If X is homeomorphic to B^n for some $n > 0$ then the unique map $X \to 1$ is a homotopy equivalence, so $H^*(X) = \mathbb{Z}$ also, and thus $\widetilde{H}^*(X) = 0$.

We next consider the Euclidean spheres $S^n \subset \mathbb{R}^{n+1}$. We index the usual basis for $\mathbb{R}^{n+1}$ as $\{e_0, \dots, e_n\}$, and we consider e_0 as the basepoint of S^n. In the case $n = 0$ we have $S^0 = \{-1, 1\}$, based at 1. We have

$$H^*(S^0) = H^*(\{-1\}) \times H^*(\{1\}) = \mathbb{Z} \times \mathbb{Z}$$

Write $u_0 = (1, 0) \in \widetilde{H}^0(S^0)$, so $\widetilde{H}^*(S^0)$ is freely generated by $\{u_0\}$.

It is now easy to compute $\widetilde{H}^* S^n$ by induction on n. We divide S^n into two hemispheres (say X_+ and X_-) and choose a basepoint x_0 lying in the equatorial sphere $X_+ \cap X_- = S^{n-1}$. We then apply the Mayer-Vietoris sequence for the pair (X_+, X_-), or the long exact sequence of the triple $(S^n, X_-, \{x_0\})$, to get an isomorphism $\widetilde{H}^* S^n = \widetilde{H}^{*-1} S^{n-1}$. This shows that $\widetilde{H}^* S^n \simeq \mathbb{Z}$, concentrated in degree n. There are various possible choices of details: we can work with S^n or $\mathbb{R}^n \cup \{\infty\}$ or B^n/S^{n-1} or $\Delta_n/\partial\Delta_n$, we can use either the first or the last coordinate to define the two hemispheres, and so on. Any choice of details provides a generator of $\widetilde{H}^n S^n$, and this group is isomorphic to $\mathbb{Z}$ so it only has two generators (the one you first thought of, and its negative). Thus, the details we choose will affect the answer only up to sign.

We next give a more complete argument, with the details chosen so as to make the multiplicative structure work nicely.

Given $n > 0$, write $A = \mathbb{R}^n$ and $B = (\mathbb{R}^n)^\times = \mathbb{R}^n \setminus \{0\}$ and

$$C = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 > 0 \text{ or } (x_2, \dots, x_n) \neq 0\}.$$

The homotopy $h_t(x_1, x') = ((1-t)x_1 + t, x')$ shows that C is homotopy equivalent to $1 \times \mathbb{R}^{n-1}$ and thus is contractible, as of course is A; this implies that $H^*(A, C) = 0$. The long exact sequence of the triple (A, B, C) now gives an isomorphism

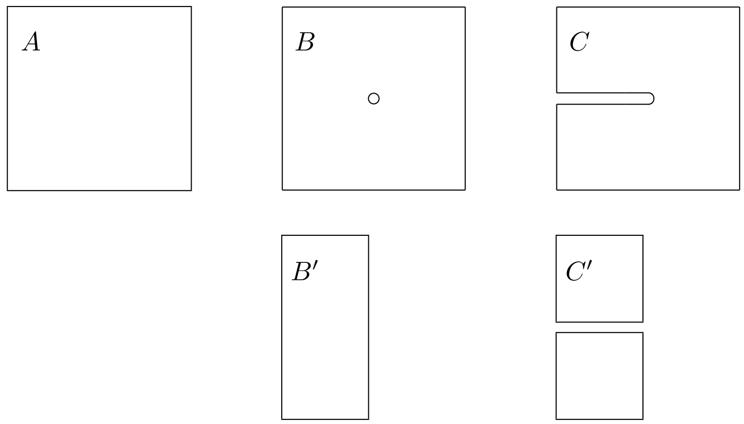
$$\delta_{A,B,C}\colon H^*(B, C) \to H^{*+1}(A, B).$$

Now put $B' = (-\infty, 0) \times \mathbb{R}^{n-1} \subset B$ and $C' = (-\infty, 0) \times (\mathbb{R}^{n-1})^\times$. Note that (B', C') is obtained from (B, C) by removing the subset $([0, \infty) \times \mathbb{R}^{n-1}) \setminus \{0\}$, which is closed in B and contained in the interior of

C. It follows by excision that the restriction map $H^*(B,C) \to H^*(B',C')$ is an isomorphism. On the other hand, the projection $\mathbb{R}\times\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ evidently gives a homotopy equivalence $(B',C') \to (\mathbb{R}^{n-1},(\mathbb{R}^{n-1})^\times)$ and thus an isomorphism in cohomology. Putting all these together, we get an isomorphism

$$\sigma\colon H^*(\mathbb{R}^{n-1},(\mathbb{R}^{n-1})^\times) \to H^{*+1}(\mathbb{R}^n,(\mathbb{R}^n)^\times).$$

The base case is $H^*(\mathbb{R}^0,(\mathbb{R}^0)^\times) = H^*(1,\emptyset) = \mathbb{Z}$; we write u_0 for the canonical generator. We then recursively define $u_n = \sigma(u_{n-1}) \in H^n(\mathbb{R}^n,(\mathbb{R}^n)^\times)$ for $n > 0$; it is clear from the above that $H^*(\mathbb{R}^n,(\mathbb{R}^n)^\times)$ is freely generated by $\{u_n\}$. The following diagram illustrates the various spaces considered, in the case $n = 2$.



Next, it is easy to see that the inclusion $S^{n-1} \to (\mathbb{R}^n)^\times$ is a homotopy equivalence, and thus that $H^*((\mathbb{R}^n)^\times, S^{n-1}) = 0$. Similarly, we have $H^*(\mathbb{R}^n, B^n) = 0$. It follows using the long exact sequences of various triples, that the evident inclusions give isomorphisms

$$H^n(\mathbb{R}^n,(\mathbb{R}^n)^\times) = H^n(\mathbb{R}^n,S^{n-1}) = H^n(B^n,S^{n-1}) = \widetilde{H}^n S^n.$$

Moreover, we can use the homeomorphisms of Section 5 to obtain isomorphisms

$$\widetilde{H}^n S^n = H^n(S^n_+,S^{n-1}) = \widetilde{H}^n(\mathbb{R}^n \cup \{\infty\}) = H^n(B^n,S^{n-1}).$$

We will write u_n for the image of u_n under any of the above isomorphisms.

We next investigate the behaviour of our elements u_n under external products. Given pairs (X,Y) and (X',Y') we write

$$(X,Y)\wedge(X',Y') = (X\times X', X\times Y' \cup Y\times X'),$$

and note that we have

$$(\mathbb{R}^n,(\mathbb{R}^n)^\times)\wedge(\mathbb{R}^m,(\mathbb{R}^m)^\times) = (\mathbb{R}^{n+m},(\mathbb{R}^{n+m})^\times).$$

The external product construction gives a map

$$H^*(X,Y)\otimes H^*(X',Y') \to H^*((X,Y)\wedge(X',Y')).$$

Lemma 7.1. *This product satisfies* $u_n u_m = u_{n+m} \in H^{n+m}(\mathbb{R}^{n+m},(\mathbb{R}^{n+m})^\times)$.

Proof. The Künneth Theorem tells us that the product map is an isomorphism, which means that $u_n u_m = \pm u_{n+m}$; the only problem is to get the sign right. Using the associativity of external products, we reduce easily to the case $n = 1$. In the construction above, one checks that

$$(A,B) = (\mathbb{R},\mathbb{R}^\times)\wedge(\mathbb{R}^m,(\mathbb{R}^m)^\times)$$
$$(A,C) = (\mathbb{R},\mathbb{R}^+)\wedge(\mathbb{R}^m,(\mathbb{R}^m)^\times)$$
$$(B,C) = (\mathbb{R}^\times,\mathbb{R}^+)\wedge(\mathbb{R}^m,(\mathbb{R}^m)^\times).$$

We have $u_{m+1} = \delta(u_m) = \delta(u_0 u_m) = \delta(u_0)u_m = u_1 u_m$, as required. □

We next observe that $GL_n(\mathbb{R})$ acts on the pair $(\mathbb{R}^n,(\mathbb{R}^n)^\times)$ and thus on $H^*(\mathbb{R}^n,(\mathbb{R}^n)^\times) = \mathbb{Z}\{u_n\}$.

Lemma 7.2. *If* $\alpha \in GL_n(\mathbb{R})$ *then* $\alpha^*(u_n) = \mathrm{sign}(\det(\alpha))u_n$.

Proof. First, $\alpha^*(u_n)$ must be a generator of $H^*(\mathbb{R}^n, (\mathbb{R}^n)^\times)$, so $\alpha^*(u_n) = c(\alpha)u_n$ for some map $c\colon GL_n(\mathbb{R}) \to \{\pm 1\}$. It is easy to check that c is a group homomorphism, and it follows that $c(\alpha) = 1$ if α can be written as a product of matrices of the form β^2, or as a product of commutators. The theory of Gaussian elimination shows that $GL_n(\mathbb{R})$ is generated by elementary matrices and permutation matrices, and using this one can easily deduce that $c(\alpha) = 1$ when $\det(\alpha) > 0$. If we write $GL_n^+(\mathbb{R}) = \{\alpha \in GL_n(\mathbb{R}) \mid \det(\alpha) > 0\}$ then this is a normal subgroup and the map $\alpha \mapsto \operatorname{sign}(\det(\alpha))$ induces an isomorphism $GL_n(\mathbb{R})/GL_n^+(\mathbb{R}) \to \{\pm 1\}$. It follows that $\alpha^*(u_n) = \operatorname{sign}(\det(\alpha))^\epsilon u_n$ for some $\epsilon \in \{0, 1\}$. To show that $\epsilon = 1$, it suffices to exhibit one α for which $\alpha^*(u_n) = -u_n$. We take α to be the map

$$(x_1, \ldots, x_n) \mapsto (-x_1, x_2, \ldots, x_n).$$

Using the equation $u_n = u_1 u_{n-1}$ it is not hard to reduce to the case $n = 1$. In this case, we have $u_1 = \delta_{\mathbb{R},\mathbb{R}^\times,\mathbb{R}^+}(1)$. Note that $H^*(\mathbb{R}^\times) = H^*(\mathbb{R}^-) \times H^*(\mathbb{R}^+) = \mathbb{Z} \times \mathbb{Z}$, and the action of α is just $\alpha^*(a, b) = (b, a)$. Using the inclusion $(\mathbb{R}, \mathbb{R}^\times, \emptyset) \subseteq (\mathbb{R}, \mathbb{R}^\times, \mathbb{R}^+)$ and the naturality of coboundary maps we see that $u_1 = \delta_{(\mathbb{R},\mathbb{R}^\times,\emptyset)}(1, 0)$. We now abbreviate $\delta_{(\mathbb{R},\mathbb{R}^\times,\emptyset)}$ to δ. An evident naturality argument shows that $\alpha^*(u_1) = \delta(0, 1)$. On the other hand, the element $(1, 1) \in H^0(\mathbb{R}^\times)$ lies in the image of $H^0(\mathbb{R})$, so the long exact sequence shows that

$$u_1 + \alpha^* u_1 = \delta(1, 0) + \delta(0, 1) = \delta(1, 1) = 0.$$

Thus $\alpha^*(u_1) = -u_1$, as required. □

7.1. **Complex spheres.** Let V be a complex vector space of dimension n. Choose a basis $\{v_1, \ldots, v_n\}$ for V over $\mathbb{C}$; then the list

$$\{v_1, iv_1, \ldots, v_n, iv_n\}$$

is a basis over $\mathbb{R}$, giving rise to a homeomorphism $f\colon (V, V^\times) \to (\mathbb{R}^{2n}, (\mathbb{R}^{2n})^\times)$. We define $u_V = f^* u_{2n} \in H^{2n}(V, V^\times)$. We also write u_V for the corresponding elements of $\widetilde{H}^{2n} S^V$ and (if V has a Hermitian structure) $H^{2n}(B(V), S(V))$. The space of possible bases $\{v_i\}$ is easily identified with $\mathcal{J}(\mathbb{C}^n, V)$, which is path-connected; so our definition of u_V is independent of the choices made. It is also easy to check that $u_V u_W = u_{V+W}$ for any pair of complex vector spaces V, W.

Also, if V is Hermitian, we can choose the basis $\{v_i\}$ to be orthonormal, and we get a homeomorphism $f\colon S(V) \to S^{2n-1}$. We then define $u'_V = f^* u_{2n-1} \in \widetilde{H}^{2n-1} S(V)$.

Remark 7.3. Let $X := S^{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. The group $GL(2, \mathbb{C})$ acts on X by Möbius transformations: the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ corresponds to the map $z \mapsto (az + b)/(cz + d)$. It is well-known that these are all the complex analytic automorphisms of X. As $GL(2, \mathbb{C})$ is connected, we see that all these maps are homotopic to the identity and thus induce the identity map in cohomology. Now let Y be an arbitrary compact Riemann surface of genus 0. Then there is an analytic isomorphism $f\colon Y \to X$, and we write $u_Y = f^* u_{\mathbb{C}} \in H^2 Y$. The preceeding remarks show that this is independent of the choice of f, and clearly $H^* Y = \mathbb{Z}\{1, u_Y\} = \mathbb{Z}[u_Y]/u_Y^2$.

8. Cohomology of unitary groups

Theorem 8.1. *There is a unique way to define elements $a_{V,2k-1} \in H^{2k-1} U(V)$ for $0 < k \leq \dim(V)$ such that*

(a) $a_{V,2n-1} = \epsilon_V^* u'_V$ *if* $n = \dim(V)$
(b) *If* $\dim(W) \leq \dim(V)$ *then*

$$j_{WV}^* a_{V,2k-1} = \begin{cases} a_{W,2k-1} & \text{if } k \leq \dim(W) \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we have $H^* U(V) = E[a_{V,1}, \ldots, a_{V,2n-1}]$.

Proof. If $V = 0$ then this says that $H^* U(V)$ is an exterior algebra on no generators, in other words $H^* U(V) = \mathbb{Z}$, which is true because $U(V) = \{1\}$. Now suppose that $n := \dim(V) > 0$, and that the theorem is true for spaces of smaller dimension. Define $a_{V,2n-1} = \epsilon_V^* u'_V$. From the remarks in Construction 6.18, we see that $j_{WV}^* a_{V,2n-1} = 0$ if $\dim(W) < n$, and $j_{WV}^* a_{V,2n-1} = a_{W,2n-1}$ if $\dim(W) = n$. Note also that $(u'_V)^2 = 0$ and thus $a_{V,2n-1}^2 = 0$.

Choose $v \in S(V)$ and put $W = v^{\perp}$, so $H^*U(W) = E[a_{W,1}, \dots, a_{W,2n-3}]$. We know from Corollary 6.21 that $U(V)/U(W) \simeq S(V) \wedge U(W)_+$, and so the Künneth Theorem and the excision axiom tell us that $H^k(U(V), U(W)) = H^{k-2n+1}U(W)$. In particular, we have $H^k(U(V), U(W)) = 0$ for $k < 2n-1$; it follows from the evident long exact sequence that $j_{WV}^* \colon H^kU(V) \to H^kU(W)$ is an isomorphism for $k < 2n-2$. It follows that for $i < n$ there is a unique element $a_{V,2i-1} \in H^*U(V)$ such that $j_{WV}^* a_{V,2i-1} = a_{W,2i-1}$. It is easy to check that these elements are independent of our choice of v, and that this is the unique way to define elements satisfying (a) and (b).

Next, observe that the image of j_{WV}^* is a subring containing the elements $a_{W,2i-1}$ for all i. It follows that j_{WV}^* is surjective, and thus that our long exact sequence reduces to a short exact sequence

$$0 \to H^{k-2n+1}U(W) \to H^kU(V) \to H^kU(W) \to 0.$$

The two end terms are free abelian groups, and it follows easily that the same is true of the middle term. As the elements $a_{V,2i-1}$ have odd dimension we know that $a_{V,2i-1}a_{V,2j-1} = -a_{V,2j-1}a_{V,2i-1}$ and in particular $2a_{V,2i-1}^2 = 0$. As everything is torsion-free, we conclude that $a_{V,2i-1}^2 = 0$. It follows that the subring generated by $\{a_{V,2i-1} \mid i < n\}$ maps isomorphically to $H^*U(W)$.

We next need to consider the map $H^{*-2n+1}U(W) \to H^*U(V)$ in more detail. We claim that this sends $j_{WV}^*(x)$ to $a_{V,2n-1}x$ for all $x \in H^*U(V)$. Assuming this, we see easily that $H^*U(V)$ is freely generated by $\{1, a_{V,2n-1}\}$ as a module over $E[a_{V,2i-1} \mid i < n]$, and thus that $H^*U(V) = E[a_{V,2i-1} \mid i \leq n]$ as required.

To prove the claim, note that the map $\alpha \mapsto (\epsilon(\alpha), \alpha)$ induces a map $\theta \colon U(V)/U(W) \to S(V)_+ \wedge U(V)$. The element $a_{V,2n-1}x$ is the pullback of the external product $u'_V x \in \widetilde{H}^*(S(V)_+ \wedge U(V))$ along the map $\theta\pi$, where $\pi \colon U(V) \to U(V)/U(W)$ is the obvious quotient map. We thus need to identify the image of $\theta^*(u'_V x)$ under our isomorphism $H^*(U(V), U(W)) = H^*(S(V) \wedge U(W)_+)$ coming from our homeomorphism $r' \colon S(V) \wedge U(W)_+ \to U(V)/U(W)$. It will suffice to check that $(r')^*\theta^*(u'_V x) = u'_V j_{WV}^*(x)$. To do this, we consider two maps $s, s' \colon S(V) \to S(V) \wedge U(V)_+$. The first is just $s(x) = x \wedge 1$. For the second, when $x \neq v$ we define $s'(x) = x \wedge r(\lambda, L)$, where (λ, L) is the unique pair with $\epsilon r(\lambda, L) = x$. This extends to a continuous map $S(V) \to S(V) \wedge U(V)_+$ by the argument of Corollary 6.21. We also consider the map $t \colon S(V) \wedge U(V)_+ \to S(V)$ defined by $t(x \wedge \alpha) = x$, and note that $ts = ts'$. We claim that $s^* = (s')^* \colon \widetilde{H}^k(S(V) \times U(V)_+) \to \widetilde{H}^k S(V)$. Indeed, if $k \neq 2n-1$ then the target group is zero and there is nothing to prove. If $k = 2n-1$ then the Künneth theorem tells us that $t^* \colon \widetilde{H}^{2n-1}S(V) \to \widetilde{H}^{2n-1}(S(V) \wedge U(V)_+)$ is an isomorphism; as $ts = ts'$ we have $s^*t^* = (s')^*t^*$ and the claim follows. Now consider the maps

$$s \wedge 1, s' \wedge 1 \colon S(V) \wedge U(W)_+ \to S(V) \wedge U(V)_+ \wedge U(W)_+.$$

The Künneth theorem assures us that $(s \wedge 1)^* = (s' \wedge 1)^*$. The multiplication map $\mu \colon (\alpha, \beta) \mapsto \alpha\beta$ gives a map

$$1 \wedge \mu \colon S(V) \wedge U(V)_+ \wedge U(W)_+ \to S(V) \wedge U(V)_+.$$

One checks from the definitions that $\theta r' = (1 \wedge \mu)(s' \wedge 1)$ and that $1 \wedge j_{WV} = (1 \wedge \mu)(s \wedge 1)$. It follows that

$$\begin{aligned}
(r')^*\theta^*(u'_V x) &= (s' \wedge 1)^*(1 \wedge \mu)^*(u'_V x) \\
&= (s \wedge 1)^*(1 \wedge \mu)^*(u'_V x) \\
&= (1 \wedge j_{WV})^*(u'_V x) \\
&= u'_V j_{WV}^*(x)
\end{aligned}$$

as required. □

Notation 8.2. From now on, we just write a_{2k-1} rather than $a_{V,2k-1}$.

9. Hopf algebras

Observe that $U(V)$ is not merely a manifold but also a topological group; we next investigate the consequences of this fact for the structure of $H^*U(V)$. The group structure is given by a multiplication map $\mu \colon U(V) \times U(V) \to U(V)$ (sending (α, β) to $\alpha\beta$) and a unit map $\eta \colon 1 \to U(V)$. These give rise to ring maps

$$\begin{aligned}
\psi &= \mu^* \colon H^*U(V) \to H^*U(V) \otimes H^*U(V) \\
\epsilon &= \eta^* \colon H^*U(V) \to \mathbb{Z}.
\end{aligned}$$

The associativity law says that

$$\mu(\mu \times 1) = \mu(1 \times \mu)\colon U(V)^3 \to U(V),$$

and this implies that

$$(\psi \otimes 1)\psi = (1 \otimes \psi)\psi\colon H^*U(V) \to H^*U(V)^{\otimes 3}.$$

Similarly, the unit laws imply that

$$(\epsilon \otimes 1)\psi = 1 = (1 \otimes \epsilon)\psi\colon H^*U(V) \to H^*U(V).$$

Definition 9.1. A graded commutative ring A^* equipped with ring maps

$$A^* \otimes A^* \xleftarrow{\psi} A^* \xrightarrow{\epsilon} \mathbb{Z}$$

satisfying these identities is called a *Hopf algebra*.

If A^* is a Hopf algebra and $x \in A^*$, we say that x is *primitive* if $\psi(x) = x \otimes 1 + 1 \otimes x$. It is easy to see that the primitive elements form an additive subgroup of A^*.

Exercise 9.2. Check that if x is primitive then $\epsilon(x) = 0$.

The Hopf algebra structure of $H^*U(V)$ is actually quite simple, as revealed by the following result.

Proposition 9.3. *The generators $a_{2k-1} \in H^{2k-1}U(V)$ are primitive.*

Proof. Let $\pi_0, \pi_1\colon U(V)^2 \to U(V)$ be the two projections. Put $b_{2i-1} = \pi_0^*(a_{2i-1}) = a_{2i-1} \otimes 1$ and $c_{2i-1} = \pi_1^*(a_{2i-1}) = 1 \otimes a_{2i-1}$. The Künneth Theorem tells us that

$$H^*U(V)^2 = H^*U(V) \otimes H^*U(V) = E[b_1, b_3, \dots, b_{2n-1}, c_1, \dots, c_{2n-1}].$$

We need to show that $\mu^*(a_{2k-1}) = b_{2k-1} + c_{2k-1}$. We start with the case $k = n$, so $a_{2n-1} = \epsilon_V^* u'_V$. Choose $v \in S(V)$ and put $W = v^\perp$ as before. We then have $\mu^* a_{2n-1} = (\epsilon_v \mu)^* u'_V$. Note that

$$\epsilon_v \mu(\alpha, \beta) = \alpha\beta(v) = \begin{cases} \epsilon_v(\alpha) & \text{if } \beta \in U(W) \\ \epsilon_v(\beta) & \text{if } \alpha = 1. \end{cases}$$

It follows that $\mu^* a_{2n-1}$ becomes b_{2n-1} if we restrict to $U(V) \times U(W)$, and it becomes c_{2n-1} if we restrict to $1 \times U(V)$. The kernel of restriction to $U(V) \times U(W)$ is the ideal (c_{2n-1}), and the kernel of the other restriction is $(b_1, \dots, b_{2n-1})$. It follows that

$$\mu^*(a_{2n-1}) - b_{2n-1} - c_{2n-1} \in (c_{2n-1}) \cap (b_1, \dots, b_{2n-1}),$$

and one checks that the indicated ideal has no nontrivial elements in degree $2n-1$. Thus $\mu^*(a_{2n-1}) = b_{2n-1} + c_{2n-1}$ as claimed.

Now suppose that $k < n$. Choose a subspace $W < V$ with $\dim(W) = k$; the above argument shows that $a_{2k-1} \in H^{2k-1}U(V)$ restricts to give a primitive element of $H^{2k-1}U(W)$. On the other hand, we see easily from our calculations that the restriction map $H^*U(V)^2 \to H^*U(W)^2$ is an isomorphism in degrees up to $2k-2$, and it follows easily that $a_{V,2k-1}$ is itself primitive. □

10. Cohomology of projective spaces

We next turn our attention to the projective space PV associated to a complex vector space V. First, we observe that there is a homeomorphism $f\colon P(\mathbb{C}^2) \to \mathbb{C} \cup \{\infty\}$ given by $f([z : w]) = z/w$. We define $x_{\mathbb{C}^2} = f^* u_{\mathbb{C}} \in H^2 P(\mathbb{C}^2)$.

Theorem 10.1. *There is a unique way to define elements $x_V \in H^2 PV$ for all $V \neq 0$ such that*

(a) *$x_{\mathbb{C}^2}$ is as above*
(b) *$j_{WV}^* x_V = x_W$ whenever $\dim(W) \leq \dim(V)$.*

Moreover, we have $H^*PV = \mathbb{Z}[x_V]/x_V^{\dim(V)}$.

The proof will be given after some lemmas.

Let V and W be finite dimensional complex vector spaces, and define

$$E = \{(L, \alpha) \mid L \in PV \ , \ \alpha\colon L \to W\}$$

There is a surjective map $PV \times \mathrm{Hom}(V, W) \to E$ defined by $(L, \alpha) \mapsto (L, \alpha|_L)$, and we topologise E in such a way that this is a quotient map. We also let $i\colon PV \to P(V \oplus W) \setminus PW$ be the obvious inclusion, and we define $j\colon PV \to E$ by $j(L) = (L, 0)$.

Lemma 10.2. *There is a natural homeomorphism $f\colon E \to P(V \oplus W) \setminus PW$ such that $fj = i$.*

Proof. Given $(L, \alpha) \in E$ we have

$$1 + \alpha\colon L \to L \oplus W \leq V \oplus W.$$

We define $f(L, \alpha) = (1 + \alpha)(L) \in P(V \oplus W)$. In the other direction, suppose that $M \in P(V \oplus W) \setminus PW$. Then $M \not\leq W = \ker(\pi\colon V \oplus W \to W)$, so $L := \pi(M)$ is one-dimensional and thus lies in PV. Moreover, we see that $\pi\colon M \to L$ is an isomorphism, so for each $v \in L$ there is a unique $w \in W$ such that $(v, w) \in M$. By defining $\alpha(v) = w$ we obtain a map $\alpha\colon L \to W$, and we define $g(M) = (L, \alpha)$. This defines a map $g\colon P(V \oplus W) \setminus PW \to E$, which one checks is inverse to f. $\square$

Corollary 10.3. *The inclusion $PV \to P(V \oplus W) \setminus PW$ is a homotopy equivalence.* $\square$

Lemma 10.4. *If V has dimension n then there is a canonical basis $\{1 = y_{V,0}, \dots, y_{V,n-1}\}$ for H^*PV, with $y_{V,k} \in H^{2k}PV$. Moreover, if $k < \dim(W) \leq \dim(V)$ then $j^*_{WV}(y_{V,k}) = y_{W,k}$.*

Proof. We work by induction on $n = \dim(V)$, the case $n = 1$ being trivial. Suppose $n > 1$, and choose a splitting $V = L \oplus W$, where L is a line. We then have a homotopy equivalence $PW \simeq PV \setminus \{L\}$, so $H^*(PV, PW) = H^*(PV, \{L\}^c)$. On the other hand, the proof of the previous lemma gives a homeomorphism $PV \setminus PW \to U := \mathrm{Hom}(L, W)$, under which $\{L\}^c \setminus PW$ becomes $U^\times$. As PW is a closed subset of PV in the interior of $\{L\}^c$, we have an excision isomorphism

$$H^*(PV, PW) \simeq H^*(PV, \{L\}^c) \simeq H^*(U, U^\times).$$

We know already that $H^*(U, U^\times) = \mathbb{Z}\{u_U\}$, concentrated in degree $\dim_{\mathbb{R}}(U) = 2n - 2$. The long exact sequence of the pair (PV, PW) now gives isomorphisms $H^rPV = H^rPW$ for $r \neq 2n - 2$ and $H^{2n-2}PV = H^{2n-2}(U, U^\times) = \mathbb{Z}\{u_U\}$. If $k < n - 1$ we let $y_{V,k}$ be the element corresponding to $y_{W,k}$ under the first isomorphism; and we let $y_{V,n-1}$ be the element corresponding to u_U under the second isomorphism. If we chose a different splitting $V = L' \oplus W'$ then we could find some $\alpha \in \mathrm{Aut}(V)$ with $\alpha(L') = L$ and $\alpha(W') = W$ and deduce that $\alpha^*(y_{V,k}) = y'_{V,k}$. On the other hand, α can be joined to 1 by a path in $\mathrm{Aut}(V)$, so $\alpha^* = 1$, so our definition of $y_{V,k}$ is independent of the choices made. $\square$

Notation 10.5. We write $x_V = y_{V,1} \in H^2PV$.

Lemma 10.6. *Suppose that $\dim(V) = n$ and that x_V^{n-1} is a generator of the group $H^{2n-2}PV \simeq \mathbb{Z}$. Then $H^*PV = \mathbb{Z}[x_V]/x_V^n$.*

Proof. As $H^{2k}PV = \mathbb{Z}\{y_k\}$ for $k < n$, there are unique integers d_k with $x^k = d_k y_k$ for $0 \leq k < n$. Note that d_k divides x^k, so it divides x^{n-1}. As x^{n-1} is a generator of $H^{2n-2}PV \simeq \mathbb{Z}$, it must be indivisible, so $d_k = \pm 1$. This implies that x^k is a generator of $H^{2k}PV$. We also have $H^{2n}PV = 0$ so $x^n = 0$, and the claim follows easily. $\square$

We next sketch an argument showing that the hypothesis of the lemma is always satisfied. We will give more details of this kind of argument later, when we come to discuss degree theory and Poincaré duality. We will also give a completely separate argument for the ring structure of H^*PV.

Let $V_k < \mathbb{C}[t]$ be the space of polynomials of degree at most k, so $V_k \simeq \mathbb{C}^{k+1}$ and $PV_k \simeq \mathbb{C}P^k$. By multiplying polynomials we get a map $V_j^\times \times V_k^\times \to V_{j+k}^\times$, which induces a map

$$\mu\colon PV_j \times PV_k \to PV_{j+k}.$$

Similarly, we get a map

$$\mu\colon (\mathbb{C} \cup \{\infty\})^n = (PV_1)^n \to PV_n.$$

The Künneth theorem tells us that

$$H^*(PV_1)^n = \mathbb{Z}[x_1,\dots,x_n]/(x_1^2,\dots,x_n^2),$$

and thus $H^2(PV_1)^n = \mathbb{Z}\{x_1,\dots,x_n\}$.

Lemma 10.7. *We have $\mu^*(x) = x_1+\dots+x_n$, and thus $\mu^*(x^n) = (\sum_i x_i)^n = n!\prod_i x_i$.*

Proof. Let $i_k\colon PV_1 \to (PV_1)^n$ be the inclusion of the k'th axis, so $i_k^* x_j = \delta_{jk}x$. The maps

$$i_k^*\colon H^2(PV_1)^n \to H^2 PV_1$$

are thus jointly injective, so it suffices to check that $i_k^*\mu^*(x) = i_k^*(\sum_j x_j) = x$. This is clear because μi_k is just the inclusion $PV_1 \to PV_n$. $\square$

Lemma 10.8. *We have $\mu^*(y_n) = n!\prod_i x_i \in H^{2n}(PV_1)^n$.*

Proof. Choose distinct numbers $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ and put $g_i = t-\lambda_i \in V_1$, and let $[g]$ denote the corresponding point of $(PV_1)^n$. More generally, given any permutation $\sigma \in \Sigma_n$, let $\sigma[g] \in (PV_1)^n$ correspond to the list $(g_{\sigma^{-1}(1)},\dots,g_{\sigma^{-1}(n)})$. Put $f = \prod_i g_i$, so that $\mu(\sigma[g]) = [f]$ for all σ. As $\mathbb{C}[t]$ has unique factorisation, we actually see that $\mu^{-1}\{[f]\} = \{\sigma[g] \mid \sigma \in \Sigma_n\}$. This is really the key point, that a generic element of PV has $n!$ preimages in $(PV_1)^n$. By using the splitting $V_n = V_{n-1}\oplus \mathbb{C}f$ in our construction of the elements y_k, we see that y_n generates the image of the restriction map $H^{2n}(PV_n,\{[f]\}^c) \to H^{2n}PV_n$. The map μ induces a map

$$\mu^*\colon \mathbb{Z}y_n = H^{2n}(PV_n,\{[f]\}^c) \to H^{2n}((PV_1)^n,\{\sigma[g] \mid \sigma\in\Sigma_n\}^c).$$

One checks that for fixed σ we have $H^{2n}((PV_1)^n,\{\sigma[g]\}^c) = \mathbb{Z}$, and that the evident restriction maps give an isomorphism

$$H^{2n}((PV_1)^n,\{\sigma[g] \mid \sigma\in\Sigma_n\}^c) \to \prod_{\sigma\in\Sigma_n} H^{2n}((PV_1)^n,\{\sigma[g]\}^c) \simeq \prod_\sigma \mathbb{Z}.$$

This has an evident basis $\{e_\sigma \mid \sigma \in \Sigma_n\}$. One checks that $\mu^*(y_n) = \sum_\sigma e_\sigma$ and that the restriction map $H^*(((PV_1)^n,\{\sigma[g] \mid \sigma\in\Sigma_n\}^c) \to H^*(PV_1)^n$ sends each e_σ to $\prod_i x_i$. The claim follows easily. $\square$

We know that $x^n = d_n y_n$ for some $d_n \in \mathbb{Z}$; by combining the last two lemmas we see that $d_n = 1$, and it follows as explained previously that $H^*PV_n = \mathbb{Z}[x]/x^{n+1}$. As any vector space of dimension $n+1$ is isomorphic to V_n, this proves the theorem.

Remark 10.9. Consider the complex reflection map $r\colon S^1\times PV \to U(V)$. Clearly $r(1\times PV) = \{1\}$ and $(S^1\times PV)/(1\times PV) = S^1\wedge PV_+$, so we have an induced map $S^1\wedge PV_+ \to U(V)$. We claim that $r^*a_{2k+1} = \pm u_1 x^k \in \widetilde{H}^{2k+1}(S^1\wedge PV_+)$. Indeed, it is easy to reduce to the case $\dim(V) = k+1$, using the evident commutative square

$$\begin{tikzcd} S^1\wedge PW_+ \arrow[r, "r"] \arrow[d, "1\wedge j_{WV}"'] & U(W) \arrow[d, "j_{WV}"] \\ S^1\wedge PV_+ \arrow[r, "r"'] & U(V). \end{tikzcd}$$

In that case, we can choose $v \in S(V)$ and put $W = v^\perp$. We observe that $\epsilon_v r$ induces a homeomorphism $S^1\wedge(PV/PW) \to S(V)$, and that the evident map $\widetilde{H}^{2k}(PV/PW) \to \widetilde{H}^{2k}PV$ is an isomorphism. It follows easily that $r^*a_{2k+1} = (\epsilon_v r)^* u_V$ is a generator of $\widetilde{H}^{2k+1}(S^1\wedge PV_+) = \mathbb{Z}\{u_1x^k\}$, so $r^*a_{2k+1} = \pm u_1 x^k$ as claimed. One could doubtless determine the sign with a bit of work.

11. Vector bundles

Definition 11.1. A *pre-vector bundle* over a space X consists of a collection of vector spaces V_x (one for each point $x\in X$) with some extra structure that we now describe. The *total space* of V is the set

$$EV = \coprod_x V_x = \{(x,v) \mid x\in X, v\in V_x\}.$$

There is an obvious map $\pi\colon EV \to X$ defined by $\pi(x,v) = x$; the extra structure is that EV should have a specified topology such that π is continuous. The spaces V_x are called the *fibres* of V.

If $Y \subseteq X$ then we define $V|_Y$ to be the collection $\{V_y \mid y \in Y\}$, with the set $E(V|_Y) = \pi^{-1}(Y)$ topologised as a subspace of EV.

If W is a vector space then we have a pre-bundle with total space $X \times W$ (topologised as a product), so every fibre is W. This is called the *trivial* pre-bundle with fibre W. If V is isomorphic (in the evident sense) to a trivial pre-bundle, then we say that it is *trivialisable*.

A *vector bundle* over X is a pre-bundle V such that every point $x \in X$ has a neighbourhood U such that $V|_U$ is trivialisable. A *linear map* between vector bundles V and W is a collection of linear maps $f_x \colon V_x \to W_x$ such that the resulting map $Ef \colon EV \to EW$ is continuous.

Remark 11.2. If V is a vector bundle over X, it is easy to see that the function $x \mapsto \dim(V_x)$ is locally constant on X, and thus constant on each connected component; but it need not be globally constant.

Remark 11.3. We can obviously replace $\mathbb{R}$ by $\mathbb{C}$ everywhere to define the notion of a complex vector bundle. Alternatively, a complex vector bundle is a real vector bundle equipped with a linear map $i \colon V \to V$ such that $i^2 = -1$.

Construction 11.4. Suppose we have a space X and a collection of vector spaces V_x for each $x \in X$, but no topology on EV as yet. Suppose we have an open covering $\{U_\alpha \alpha \in I\}$ of X, vector spaces W_α, and linear isomorphisms $\phi_\alpha(x) \colon V_x \to W_\alpha$ for $x \in U_\alpha$. We'd like to use these to define a topology on EV making V into a vector bundle. For this to work, we need the following compatibility condition:

> For each $\alpha, \beta \in I$ the equation $\theta_{\alpha\beta}(x) = \phi_\beta(x)\psi_\alpha(x)^{-1}$ defines a continuous map $\theta_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{Hom}(W_\alpha, W_\beta)$.

Assuming this, we define a bijective map

$$\psi_\alpha \colon \pi^{-1}W_\alpha \to W_\alpha \times U_\alpha$$

by $\psi_\alpha(x, v) = (x, \phi_\alpha(x)(v))$. Note that the map

$$\psi_{\alpha\beta} := \psi_\beta\psi_\alpha^{-1} \colon (W_\alpha \cap W_\beta) \times U_\alpha \to (W_\alpha \cap W_\beta) \times U_\beta$$

is a homeomorphism. We declare that a subset $A \subseteq EV$ is open if and only if $\psi_\alpha(A \cap \pi^{-1}W_\alpha)$ is open in $W_\alpha \times U_\alpha$ for all α. It is easy to check that this gives a topology making V into a vector bundle, such that $V|_{W_\alpha}$ is trivialisable for all α.

Example 11.5. Define a map $f \colon \mathbb{R} \times (-1/2, 1/2) \to \mathbb{R}^3$ by

$$f(\theta, t) = (1 + t\cos(\theta))(\cos(2\theta), \sin(2\theta), 0) + (0, 0, t\sin(\theta)).$$

This satisfies $f(\theta + \pi, t) = f(\theta, -t)$, and its image B is a Möbius band. We'll exhibit a vector bundle V over S^1 and a homeomorphism $EV \to B$. Regard S^1 as $\{z \in \mathbb{C} \mid |z| = 1\}$ and put $V_z = \{w \in \mathbb{C} \mid w^2 \in \mathbb{R}_+ z\}$. If w is any square root of z then it is easy to see that $V_z = \mathbb{R}w$, and thus that we have a vector bundle V over S^1. Put $E'V = \{(z, w) \in EV \mid |w| < 1/2\}$; the map $(z, w) \mapsto (z, w/2\sqrt{1 + |w|^2})$ gives a homeomorphism $EV \to E'V$. We next define $g \colon E'V \to \mathbb{R}^3$ by

$$g(x + iy, a + ib) = (1 + a)(x, y, 0) + (0, 0, b).$$

It is easy to check that this gives the required homeomorphism.

Example 11.6. Fix a vector space W, and put $X = \{\alpha \in \mathrm{End}(W) \mid \alpha^2 = \alpha\}$. For $\alpha \in W$ we put $V_\alpha = \mathrm{image}(\alpha) \leq W$, and we topologise $EV = \{(\alpha, w) \in X \times W \mid w \in \mathrm{image}(\alpha)\}$ as a subspace of $X \times W$. We claim that this gives a vector bundle over X. To see this, put $V'_\alpha = \ker(\alpha)$. One checks using $\alpha^2 = \alpha$ that $V_\alpha = \ker(1 - \alpha)$ and $V'_\alpha = \mathrm{image}(1 - \alpha)$ and $V = V_\alpha \oplus V'_\alpha$. With respect to this decomposition we have $\alpha = 1 \oplus 0$. This construction actually gives a bijection $X \simeq \{(U, U') \mid W = U \oplus U'\}$.

We can write any $\beta \in \mathrm{End}(V) = \mathrm{Hom}(V_\alpha \oplus V'_\alpha, V_\alpha \oplus V'_\alpha)$ in matrix form with respect to our decomposition, say

$$\beta = \begin{pmatrix} \beta_{00} & \beta_{10} \\ \beta_{01} & \beta_{11} \end{pmatrix}$$

so for example $\beta_{01} \colon V_\alpha \to V'_\alpha$ is the composite

$$V_\alpha \to V \xrightarrow{\beta} V \to V'_\alpha.$$

Put

$$N(\alpha) = \{\beta \in X \mid \beta_{00} \text{ and } \beta_{11} \text{ are isomorphisms. }\}$$

For any vector space U, it is well-known that $\mathrm{Aut}(U)$ is open in $\mathrm{End}(U)$, and it follows easily that $N(\alpha)$ is an open neighbourhood of α in X. If $\beta \in N(\alpha)$ then $\beta(V_\alpha) \leq V_\beta$ and the projection $\beta(V_\alpha) \to V_\alpha$ is an isomorphism so certainly the projection $V_\beta \to V_\alpha$ is surjective. Similarly, the projection $V'_\beta \to V'_\alpha$ is surjective. As $W = V_\alpha \oplus V'_\alpha = V_\beta \oplus V'_\beta$, a dimension count tells us that the projection $V_\beta \to V_\alpha$ is an isomorphism for all $\beta \in N(\alpha)$. This trivialises V over $N(\alpha)$, so V is a vector bundle as claimed. Note that $\dim(V_\alpha) = \mathrm{trace}(\alpha)$; this is locally but not globally constant on X.

Example 11.7. Take

$$X = B_n\mathbb{C} = \{\text{finite subsets } S \subset \mathbb{C} \text{ such that } |S| = n\}.$$

Recall that

$$F_n\mathbb{C} = \{\underline{z} \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}.$$

There is clearly surjective map $(z_1, \ldots, z_n) \mapsto \{z_1, \ldots, z_n\}$ from $F_n\mathbb{C}$ to $B_n\mathbb{C}$, and we give $B_n\mathbb{C}$ the quotient topology. We next define a vector bundle V over $B_n\mathbb{C}$ by

$$V_S = \mathrm{Map}(S, \mathbb{R}) = \{ \text{ all functions } x\colon S \to \mathbb{R} \ .$$

Given $(S, x) \in EV$ we can consider the set

$$S' = \{(s, x(s)) \mid s \in S\} \subset \mathbb{C} \times \mathbb{R}.$$

This construction gives an embedding $EV \to B_n(\mathbb{C} \times \mathbb{R})$, which we use to topologise EV as a subspace of $B_n(\mathbb{C} \times \mathbb{R})$, which is itself topologised as a quotient of $F_n(\mathbb{C} \times \mathbb{R})$.

An interesting feature of this example is that the bundle $\mathbb{C} \otimes V$ is trivial, although V itself is not. To see this, note that

$$E(\mathbb{C} \otimes V) = \{(S, z) \mid S \in B_n\mathbb{C} \ , \ z\colon S \to \mathbb{C}\}.$$

Let P_n be the space of complex polynomials $f(t)$ of degree less than n, and define $\phi\colon B_n\mathbb{C} \times P_n \to E(\mathbb{C} \otimes V)$ by

$$\phi(S, f) = (S, f|_S).$$

Fix S and consider the resulting map $\phi_S\colon P_n \to V_S$. The kernel consists of polynomials of degree less than n that vanish on each of the n points in S, so they must be zero. As $\dim(P_n) = \dim(V_S) = n$ we conclude that ϕ_S is a linear isomorphism and thus that ϕ gives a trivialisation of $\mathbb{C} \otimes V$ as claimed.

Example 11.8. Let W be a finite dimensional vector space with inner product, and put $X = S(W)$. Given $x \in X$ we put $T_xX = x^\perp < W$; this is the hyperplane in W tangent to X at x. These spaces form a vector bundle TX over $S(V)$, called the tangent bundle.

Example 11.9. Let W be a finite dimensional complex vector space, and put $X = PW$, so that a point of X is a line $L < W$. One can define vector bundles V over X by rules such as $V_L = L \oplus L^*$, or $V_L = L \otimes L$ or just $V_L = L$. The last example is called the *tautological bundle* over PW. Note that in that case we have

$$EV = \{(L, w) \mid w \in L \in PW\}.$$

We can make part of the last example more systematic as follows.

Construction 11.10. Let $\mathcal{V}$ be the category of vector spaces, and let $F\colon \mathcal{V} \to \mathcal{V}$ be a functor. Suppose also that F is continuous, in the sense that the maps

$$F\colon \mathrm{Hom}_{\mathbb{R}}(V, W) \to \mathrm{Hom}_{\mathbb{R}}(FV, FW)$$

are continuous for all V and W. Let V be a vector bundle over a space X; we would like to assemble the spaces $F(V_x)$ into a vector bundle over X, to be (abusively) denoted by FV. Let U be an open set such that $V|_U$ is trivial, and let ϕ be a choice of trivialisation, which amounts to a continuous family of isomorphisms $\phi(x)\colon V_x \to \mathbb{R}^n$ for $x \in U$. We then have a family of isomorphisms $F(\phi(x))\colon F(V_x) \to F(\mathbb{R}^n)$. If (W, ψ) is another local trivialisation, then we have a continuous map $\theta\colon U \cap W \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n)$ given by $x \mapsto \psi(x)\phi(x)^{-1}$. As $F\colon \mathrm{End}(\mathbb{R}^n) \to \mathrm{End}(F(\mathbb{R}^n))$ is continuous, we deduce that the map $x \mapsto F(\psi(x))F(\phi(x))^{-1}$ is also continuous. Thus, we can use Construction 11.4 to impose the required topology on $EFV = \coprod_{x \in X} FV_x$.

Example 11.11. Given a bundle V over X, we can use this to define bundles $V \oplus V$, $V \otimes V$, $\Lambda^k V$ and so on.

Remark 11.12. There are a number of obvious variants on the above construction. Firstly, we could work everywhere with complex bundles instead of real ones. Alternatively, we could observe that we really only need a continuous functor F defined on the category $\mathcal{V}_{\text{iso}}$, which has the same objects as $\mathcal{V}$ but the morphisms are restricted to be isomorphisms. The basic example is to take $FV = V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$ and $F\alpha = (\alpha^*)^{-1} \colon V^* \to W^*$ when $\alpha \colon V \to W$. This allows us to define the dual bundle V^* of a bundle V. Finally, we could consider functors $\mathcal{V}^k \to \mathcal{V}$ for any $k \geq 0$, instead of just functors $\mathcal{V} \to \mathcal{V}$. In the case $k = 2$, this allows us to start with bundles V and W over the same space X, and construct bundles $V \oplus W$ and $V \otimes W$ with fibres $(V \oplus W)_x = V_x \oplus W_x$ and $(V \otimes W)_x = V_x \otimes W_x$.

We now want to relate bundles defined over different spaces.

Construction 11.13. Let V be a vector bundle over X, and let $f \colon Y \to X$ be a continuous map; we will define a new bundle f^*V over Y, called the *pullback* of V. The fibres are just $(f^*V)_y = V_{f(y)}$. We thus have natural bijections

$$\begin{aligned} E(f^*V) &= \{(y, v) \mid y \in Y \text{ and } v \in V_{f(y)}\} \\ &= \{(y, a) \in Y \times EV \mid f(y) = \pi(a)\}. \end{aligned}$$

This is naturally thought of as a subspace of $Y \times EV$, and we topologise it as such. A trivialisation of $V|_U$ (where U is open in X) gives a trivialisation of $(f^*V)|_{f^{-1}U}$ in an obvious way. Using this, we see that f^*V is indeed a vector bundle.

Remark 11.14. Let W and V be vector bundles over Y and X, and let $f \colon Y \to X$ be continuous. A *map of vector bundles covering* f means a continuous map $\phi \colon EW \to EV$ such that the following diagram commutes

$$\begin{tikzcd} EW \arrow[r, "\phi"] \arrow[d, "\pi"'] & EV \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X, \end{tikzcd}$$

and the resulting maps $\phi \colon W_y \to V_{f(x)}$ are linear. Such maps biject with homomorphisms $\phi' \colon W \to f^*V$ of vector bundles over Y.

We next discuss inner products on vector bundles.

Definition 11.15. Let V be a vector bundle over a space X, and suppose we have an inner product $\langle , \rangle_x$ on V_x for each x. Put

$$EV \times_X EV = \{(x, u, v) \mid x \in X \ , \ u, v \in V_x\} = \{(a, b) \in EV \times EV \mid \pi(a) = \pi(b)\},$$

and topologise this as a subspace of $EV \times EV$. We say that the given data constitute an *inner product on* V if the map $(x, u, v) \to \langle u, v \rangle_x$ (from $EV \times_X EV$ to $\mathbb{R}$) is continuous.

Remark 11.16. In Example 11.5, the standard inner product on $\mathbb{C} \simeq \mathbb{R}^2$ restricts to give an inner product on $V_z < \mathbb{C}$; this gives an inner product on V. Similarly, in examples 11.6, 11.8 and 11.9, a choice of inner product on the fixed space W gives rise to an inner product on the bundle V. In example 11.7 we can define $\langle x, y \rangle_S = \sum_{z \in S} x(s) y(s)$ to get an inner product on V.

Remark 11.17. There is an evident analogous way to define the notion of a Hermitian inner product on a complex bundle.

Remark 11.18. The space of inner products on a given bundle V is easily seen to be convex and thus either contractible or empty. It is non-empty in all reasonable cases, for example if the base space X is paracompact; this holds if X is locally compact Hausdorff, or metrisable, or a manifold, or a real or complex algebraic variety, or a simplicial complex with the weak topology. (Of course, some of the conditions in this list are implied by other ones.)

We next define the *Thom space* of a vector bundle V over a space X, which will be written as X^V. If X is compact we can just define X^V to be the one-point compactification of EV. For more general X, we need to be more careful about the topology. We define a space Y (the *fibrewise one-point compactification* of EV) as follows. As a set, we have

$$Y = EV \amalg \{\infty_x \mid x \in X\} = \coprod_{x\in X} (V_x \cup \{\infty_x\}).$$

This has an evident projection $\pi\colon Y \to X$. If $U \subseteq X$ then a trivialisation of $V|_U$ gives rise to a bijection $\pi^{-1}U \to U \times (\mathbb{R}^n \cup \{\infty\})$. It is easy to see that there is a unique way to topologise Y such that these bijections are always homeomorphisms. Moreover, the map $i\colon x \mapsto \infty_x$ embeds X as an NDR in Y. We define

$$X^V := Y/i(X).$$

If we choose an inner product on V, we can define the ball and sphere bundles:

$$D(V) = \{(x, v) \in EV \mid \|v\| \leq 1\}$$
$$S(V) = \{(x, v) \in EV \mid \|v\| = 1\}$$

By applying the constructions of Section 5 to each fibre, we get a homeomorphism $X^V \simeq B(V)/S(V)$.

Example 11.19. Let V and W be finite dimensional vector spaces over $\mathbb{C}$. Let L be the tautological line bundle over PV, so that $\mathrm{Hom}(L, W)$ is a vector bundle over PV. Then we claim that there is a canonical homeomorphism

$$P(V \oplus W)/PW \simeq PV^{\mathrm{Hom}(L,W)}.$$

Indeed, Lemma 10.2 gives a homeomorphism $P(V \oplus W) \setminus PW \simeq E(\mathrm{Hom}(L, W))$, and the claim follows by taking one-point compactifications.

Example 11.20. Let V be the bundle over S^1 discussed in Example 11.5, so that EV is homeomorphic to the Möbius band. Let $Y = P_{\mathbb{R}}(\mathbb{C} \oplus \mathbb{R})$ be the space of one-dimensional real subspaces of $\mathbb{C} \oplus \mathbb{R}$. We claim that $(S^1)^V$ is homeomorphic to Y. To see this, put $L = 0 \oplus \mathbb{R} < \mathbb{C} \oplus \mathbb{R}$, so that $L \in Y$, and put $Y' = Y \setminus \{L\}$. It will be enough to exhibit a homeomorphism $f\colon EV \to Y'$ and pass to one-point compactifications. To do this, recall that

$$EV = \{(z, w) \in \mathbb{C}^2 \mid \; |z| = 1 \text{ and } w^2/z \in [0, \infty)\}.$$

Given $(z, w) \in EV$ we can choose $\xi \in S^1$ with $\xi^2 = z$, and note that $w/\xi \in \mathbb{R}$. We can thus put $f(z, w) = \mathbb{R}.(\xi, w/\xi) \in Y'$, and one checks that this is a well-defined homeomorphism, as required.

Example 11.21. If V is a trivial bundle, say $V = W \times X$, we have $X^V = X_+ \wedge S^V$.

12. Smooth structures

Definition 12.1. Let M be a manifold of dimension n. A *chart* on M is a pair (U, ϕ), where U is open in M, and $\phi\colon U \to \mathbb{R}^n$ is an open, continuous and injective map (and thus a homeomorphism $U \to \phi(U)$). Two charts (U, ϕ) and (V, ψ) are *compatible* if the transition function

$$\phi\psi^{-1}\colon \psi(U \cap V) \to \phi(U \cap V)$$

is smooth (which means infinitely differentiable, or C^∞). An *atlas* on M is a set of compatible charts (U_α, ϕ_α) such that the sets U_α cover X. An atlas is *complete* if whenever (V, ψ) is a chart that is compatible with each (U_α, ϕ_α), there exists β such that $(V, \psi) = (U_\beta, \phi_\beta)$. A complete atlas is also called a *smooth structure* on M. A *smooth manifold* is a manifold with a specified smooth structure. When we talk about charts on a smooth manifold, we implicitly mean charts contained in the specified atlas.

Remark 12.2. Let $\mathcal{A}$ be an atlas on M, and let $\overline{\mathcal{A}}$ be the set of all charts (V, ψ) that are compatible with (U, ϕ) for all $(U, \phi) \in \mathcal{A}$. One can check that $\overline{\mathcal{A}}$ is a complete atlas, and that it is the unique complete atlas that contains $\mathcal{A}$. In practise we will specify smooth structures by giving an incomplete atlas and implicitly completing it in this way.

Remark 12.3. We will also allow ourselves to consider charts $\phi\colon U \to W$, where W is an arbitrary real vector space of dimension n. We can of course choose a linear isomorphism $W \to \mathbb{R}^n$ to get charts as originally defined.

Example 12.4. We now reexamine the manifolds listed in Example 2.7. For example (a), note that if U is an open subset of $\mathbb{R}^n$ and $j\colon U \to \mathbb{R}^n$ is the inclusion then $\{(U, j)\}$ is an atlas making U a smooth manifold. For example (b), if V is an n-dimensional real vector space and

$$\mathcal{A} = \{(V, \phi) \mid \phi \text{ is a linear isomorphism } V \to \mathbb{R}^n\}$$

then $\mathcal{A}$ is an atlas on V, making it a smooth manifold. In each of examples (c), (d) and (j) we exhibited various charts to prove that the relevant spaces were manifolds. It is easy to check that in each case the given charts are all compatible, so they form an atlas.

Definition 12.5. Let $f\colon M \to N$ be a continuous map between smooth manifolds. We say that f is *smooth* if for all charts (U, ϕ) on M and all charts (V, ψ) on N, the map

$$\mathbb{R}^m \supseteq \phi(U \cap f^{-1}(V)) \xrightarrow{\phi^{-1}} f^{-1}(V) \xrightarrow{f} V \xrightarrow{\psi} \psi(V) \subseteq \mathbb{R}^n$$

is smooth.

Remark 12.6. If the smooth structures arise as in Remark 12.2, it suffices to check the above condition when (U, ϕ) and (V, ψ) are contained in the original incomplete atlases.

Definition 12.7. Let M be a smooth manifold of dimension m, and let x be a point of M. A *germ* at x is an equivalence class of pairs (U, f), where U is an open neighbourhood of x and $f\colon U \to \mathbb{R}$ is smooth. Two such pairs (U, f) and (V, g) are equivalent if there is a neighbourhood W of x such that $W \subseteq U \cap V$ and $f|_W = g|_W$. We write $\mathcal{O}_x$ or $\mathcal{O}_{M,x}$ for the set of germs; this is a commutative $\mathbb{R}$-algebra in an obvious way. There is a ring map $\epsilon\colon \mathcal{O}_x \to \mathbb{R}$ defined by $\epsilon[U, f] = f(x)$, and we write $\mathfrak{m}_x = \mathfrak{m}_{M,x}$ for the kernel.

We next analyse the structure of the ring $A_n = \mathcal{O}_{\mathbb{R}^n,0}$. First, we write x_i for the germ of the i'th projection map $(a_1, \ldots, a_n) \mapsto a_i$, so $x_i \in \mathfrak{m} := \mathfrak{m}_{\mathbb{R}^n,0}$. Also, we can define a map $v_i\colon A_n \to \mathbb{R}$ by $v_i[U, f] = (\partial f/\partial x_i)|_0$. Note that this satisfies $v_i(fg) = \partial_i(f)\epsilon(g) + \epsilon(f)\partial(g)$, and thus that $v_i(\mathfrak{m}^2) = 0$.

Proposition 12.8. *The ideal* $\mathfrak{m}$ *is generated by* $\{x_1, \ldots, x_n\}$. *Moreover,* $\mathfrak{m}^2 = \{f \in \mathfrak{m} \mid \partial_i(f) = 0 \text{ for all } i\ \}$, *and* $\{x_1, \ldots, x_n\}$ *gives a basis for* $\mathfrak{m}/\mathfrak{m}^2$ *over* $\mathbb{R}$.

Proof. Consider a germ $[U, f] \in \mathfrak{m}$, so $f(0) = 0$. We may assume without loss that U is an open ball centred at 0. Define

$$g_k(\underline{a}) = \int_{t=0}^{1} \frac{\partial f}{\partial x_k}(t\underline{a})dt.$$

One can check that this is a smooth function on U. Moreover, we have $\epsilon(g_k) = \partial_k(f)$ and

$$\sum_k a_k g_k(\underline{a}) = \int_{t=0}^{1} \frac{d}{dt} f(t\underline{a})dt = f(\underline{a}) - f(0) = f(\underline{a}).$$

This means that $f = \sum_k x_k g_k \in A_n$, and it follows immediately that the set $X = \{x_1, \ldots, x_n\}$ generates $\mathfrak{m}$. It follows in turn that X generates $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $A_n/\mathfrak{m} = \mathbb{R}$. Moreover $v_i(\mathfrak{m}^2) = 0$ so we get an induced map $\partial_i\colon \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{R}$ and this satisfies $v_i(x_j) = \delta_{ij}$; it follows that X is linearly independent modulo $\mathfrak{m}^2$, as required. ∎

Definition 12.9. Let M be a smooth manifold, and x a point of M. The *cotangent space* to M at x is $T_x^*M = \mathfrak{m}_x/\mathfrak{m}_x^2$. The *tangent space* to M at x is $T_xM = \operatorname{Hom}_{\mathbb{R}}(T_x^*M, \mathbb{R})$. If U is a neighbourhood of x and $f\colon U \to \mathbb{R}$ is smooth, we write $d_x(f)$ for the image of the germ $[U, f - f(0)]$ in T_x^*M.

Remark 12.10. A *derivation* from $\mathcal{O}_x$ to $\mathbb{R}$ is an $\mathbb{R}$-linear map $v\colon \mathcal{O}_x \to \mathbb{R}$ such that $v(fg) = v(f)\epsilon(g) + \epsilon(f)v(g)$. This implies immediately that $v(\mathfrak{m}^2) = 0$. By taking $f = g = 1$ we see that $v(1) = 0$ and thus $v(f) = 0$ when f is the germ of a constant map. Note also that $\mathcal{O}_x = \mathbb{R} \oplus \mathfrak{m}_x$. Using these remarks it is easy to identify $T_xM = \operatorname{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{R})$ with the space of derivations.

Construction 12.11. Let M be a manifold, and let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a smooth map (for some $\epsilon > 0$) with $\gamma(0) = x$ say. If (U, f) is a germ at x then $f \circ \gamma(t)$ is defined when $|t|$ is sufficiently small, so we can define $v_\gamma(f) = (f \circ \gamma)'(0)$. This gives a derivation $v_\gamma\colon \mathcal{O}_{M,x} \to \mathbb{R}$, and thus an element $v_\gamma \in T_xM$. This makes contact with another popular definition of T_xM as a set of equivalence classes of curves; the two definitions of course turn out to be equivalent.

Construction 12.12. Let M be a smooth manifold, and let (U, ϕ) be a chart on M, so

$$\phi = (\phi_1, \dots, \phi_m) \colon U \to \mathbb{R}^m.$$

One can see easily from Proposition 12.8 that for $x \in U$, the list $(d_x(\phi_1), \dots, d_x(\phi_m))$ is a basis for $T_x^* M$. There is thus a unique linear map $\phi'(x) \colon T_x U \to \mathbb{R}^n$ with

$$\phi'(x)(\sum_i a_i d_x(\phi_i)) = (a_1, \dots, a_n).$$

Now let (V, ψ) be another chart. We then have a smooth map

$$\theta = \phi\psi^{-1} \colon \psi(U \cap V) \to \phi(U \cap V) \subseteq \mathbb{R}^m,$$

and one checks that $\phi'(x)\psi'(x)^{-1} \in \mathrm{End}(\mathbb{R}^m) = M_m(\mathbb{R})$ is just the matrix of partial derivatives of θ, evaluated at $\psi(x)$. Using this, we find that the assignment $x \mapsto \phi'(x)\psi'(x)^{-1}$ gives a smooth map $U \cap V \to \mathrm{Aut}(\mathbb{R}^m)$. As in Construction 11.4, this allows us to collect the cotangent spaces $T_x^* M$ into a vector bundle $T^* M$ over M. Similarly, we can collect the tangent spaces together to form a vector bundle TM, called the tangent bundle.

Example 12.13.
(a) Let V be a finite dimensional vector space, regarded as a smooth manifold in the obvious way. If $\phi \in V^*$ then we can regard ϕ as a smooth function $V \to \mathbb{R}$, and thus define $d_x(\phi) \in T_x^* V$ for all $x \in V$. For any fixed $x \in V$, the map $\phi \mapsto d_x(\phi)$ is easily seen to be an isomorphism $V^* \to T_x^* V$. It follows that $T^* V$ is just the trivial bundle $V \times V^*$ over V. By dualising, we see that TV is the trivial bundle $V \times V$ over V. A pair (x, v) corresponds to the derivation $\mathcal{O}_{V,x} \to \mathbb{R}$ given by

$$f \mapsto \frac{d}{dt} f(x + tv) \bigg|_{t=0}.$$

(b) Let V be a complex vector space, and suppose $L \in PV$; let $j \colon L \to V$ be the inclusion. We claim that there is a canonical isomorphism $T_L PV = \mathrm{Hom}(L, V)/\mathbb{C}j$. **prove this cleanly**.

13. The Thom isomorphism theorem

In this section we determine the cohomology of the Thom space of an oriented vector bundle. Before doing this, we need to explain what we mean by an orientation.

Definition 13.1. Let V be a vector space of dimension n. An *orientation* of V is a generator of the group $H^n(V, V^\times) \simeq \mathbb{Z}$. We write $\mathrm{Or}(V)$ for the set of orientations of V (so $|\mathrm{Or}(V)| = 2$). Next, let V be a vector bundle over a space X such that $\dim(V_x) = n$ for all x. A *local orientation* of V is a pair (x, u) with $x \in X$ and $u \in \mathrm{Or}(V_x)$; we write $\mathrm{LOr}(V)$ for the set of such pairs. There is a projection map $\pi \colon \mathrm{LOr}(V) \to X$, and a local trivialisation $V|_Y \simeq Y \times W$ gives rise to a bijection $\pi^{-1} Y \to Y \times \mathrm{Or}(W)$. There is a unique way to topologise $\mathrm{LOr}(V)$ such that π is continuous and these bijections are homeomorphisms. If $Y \subseteq X$ we write $\Gamma(Y, \mathrm{LOr}(V))$ for the set of maps $u \colon Y \to \mathrm{LOr}(V)$ such that $\pi u = 1 \colon Y \to X$. In other words, an element of $\Gamma(Y, \mathrm{LOr}(V))$ is a choice of orientation $u_y \in \mathrm{Or}(V_y)$ for each $y \in Y$, such that u_y varies continuously with y.

Remark 13.2. A basis $\{e_1, \dots, e_n\}$ for a vector space V gives an isomorphism $f \colon V \to \mathbb{R}^n$ and thus an orientation $u_e := f^*(u_n)$ of V. One checks that if $e'_i = \sum_j A_{ij} e_j$ then $u_{e'} = \mathrm{sign}(\det(A)) u_e$, and thus that u_e depends only on the path component of $e_1 \wedge \ldots \wedge e_n$ in the space $(\Lambda^n V) \setminus 0 \simeq \mathbb{R}^\times$. This construction gives a bijection

$$\pi_0((\Lambda^n V) \setminus 0) \simeq \mathrm{Or}(V).$$

There is also a more global way to define orientations for vector bundles, which we now examine.

Definition 13.3. We write $EV^\times = \{(x, v) \in EV \mid v \neq 0\} = EV \setminus X$. For each $x \in X$ we let

$$i_x \colon (V_x, V_x^\times) \to (EV, EV^\times)$$

be the evident inclusion of pairs, which gives a map $i_x^* \colon H^n(EV, EV^\times) \to H^n(V_x, V_x^\times) \simeq \mathbb{Z}$. A *global orientation* or *Thom class* of V is a class $u \in H^n(EV, EV^\times)$ such that $i_x^*(u)$ is a generator for all x. We write $\mathrm{Thom}(V)$ for the set of such u. There is an evident map $\theta \colon \mathrm{Thom}(V|_Y) \to \Gamma(Y, \mathrm{LOr}(V))$ defined by $\theta(u)_y = i_y^*(u)$.

We next observe that the inclusions $S(V) \to EV^{\times}$ and $B(V) \to EV$, and the projection $EV \to X$ are all homotopy equivalences. This gives canonical isomorphisms $H^*EV = H^*BV = H^*X$ and

$$H^*(EV, EV^{\times}) \simeq H^*(B(V), S(V)) \simeq \widetilde{H}^*(B(V)/S(V)) = \widetilde{H}^*(X^V).$$

We also note that $H^*(EV, EV^{\times})$ is a module over H^*EV, so $\widetilde{H}^*(X^V)$ is a module over H^*X.

Theorem 13.4. *Let V be a vector bundle over X such that $\dim(V_x) = n$ for all x. Then*

(a) $\widetilde{H}^k X^V = 0$ *for $k < n$.*
(b) *The map $\theta\colon \mathrm{Thom}(V) \to \Gamma(X, \mathrm{LOr}(V))$ is a bijection.*
(c) *If $u \in \mathrm{Thom}(V)$ then $\widetilde{H}^* X^V$ is a free module over H^*X on the single generator u (so that $\widetilde{H}^k X^V \simeq H^{k-n}X$).*

Proof. First suppose that V is a trivial bundle, say $V = X \times W$, and choose $w \in \mathrm{Or}(W)$. Then the Künneth theorem gives $H^*(EV, EV^{\times}) = H^*X \otimes H^*(W, W^{\times}) = H^{*-n}(X).w$. In particular, we have $H^k(EV, EV^{\times}) = 0$ for $k < n$ and $H^n(EV, EV^{\times}) = H^0(X).w$. We can identify $H^0(X)$ with the set of locally constant functions $t\colon X \to \mathbb{Z}$, and one checks easily that $tw \in \mathrm{Thom}(V)$ iff $t(x) \in \{\pm 1\}$ for all x, iff t is invertible in $H^0(X)$, iff the map $x \mapsto t(x)w$ lies in $\Gamma(X, \mathrm{LOr}(V))$. It is easy to deduce that the theorem holds in this case.

Let $\mathcal{X}$ be the collection of subspaces $Y \subseteq X$ such that the theorem is true for the bundle $V|_Y$ over Y. For any $Y \subseteq X$ we put $A^kY = H^{k-n}(EV|_Y, EV|_Y^{\times})$; thus $\mathrm{Thom}(V|_Y) \subset A^0Y$, and if $Y \in \mathcal{X}$ and $k < 0$ we have $A^kY = 0$.

By the first paragraph, if $V|_Y$ is trivial we see that $Y \in \mathcal{X}$. Now suppose we have two open subspaces $Y, Z \subseteq X$ such that Y, Z and $Y \cap Z$ all lie in $\mathcal{X}$. We then have $EV|_{Y\cup Z} = EV|_Y \cup EV|_Z$ and so on, and we obtain a Mayer-Vietoris sequence

$$\ldots \to A^{k-1}(Y \cap Z) \to A^k(Y \cup Z) \to A^kY \times A^kZ \to A^k(Y \cap Z) \to \ldots.$$

when $k < 0$ we have $A^kY = A^kZ = A^{k-1}(Y \cap Z) = 0$ so $A^k(Y \cup Z) = 0$. In the case $k = 0$, we see that the map $u \mapsto (u|_Y, u|_Z)$ gives a bijection from $A^0(Y \cup Z)$ to the set of pairs $(v, w) \in A^0Y \times A^0Z$ such that $v|_{Y\cap Z} = w|_{Y\cap Z}$. Clearly the subset $\mathrm{Thom}(V|_{Y\cup Z}) \subset A^0(Y \cup Z)$ bijects with

$$\{(v, w) \in \mathrm{Thom}(V|_Y) \times \mathrm{Thom}(V|_Z) \mid v|_{Y\cap Z} = w|_{Y\cap Z}\}.$$

One checks directly that the analogous construction gives a bijection

$$\Gamma(Y \cup Z, \mathrm{LOr}(V)) \to \{(v, w) \in \Gamma(Y, \mathrm{LOr}(V)) \times \Gamma(Z, \mathrm{LOr}(V)) \mid v|_{Y\cap Z} = w|_{Y\cap Z}\}.$$

As θ_Y, θ_Z and $\theta_{Y\cap Z}$ are bijections, we deduce that $\theta_{Y\cup Z}$ is also a bijection.

Next, suppose we have $u \in \mathrm{Thom}(V|_{Y\cup Z})$, so that $u|_Y \in \mathrm{Thom}(V|_Y)$ and $u|_Z \in \mathrm{Thom}(V|_Z)$. Consider the following diagram of long exact sequences:

$$\begin{tikzcd}
H^{k-1}Y \times H^{k-1}Z \arrow[r] \arrow[d] & H^{k-1}(Y \cap Z) \arrow[r] \arrow[d] & H^k(Y \cup Z) \arrow[r] \arrow[d] & H^kY \times H^kZ \arrow[r] \arrow[d] & H^k(Y \cap Z) \arrow[d] \\
A^{k-1}Y \times A^{k-1}Z \arrow[r] & A^{k-1}(Y \cap Z) \arrow[r] & A^k(Y \cup Z) \arrow[r] & A^kY \times A^kZ \arrow[r] & A^k(Y \cap Z)
\end{tikzcd}$$

The rows are Mayer-Vietoris sequences, and the vertical maps have the form $a \mapsto au$. As restriction maps in cohomology respect multiplication, we see that the first, third and fourth squares commute. With a little more work one shows that the second square commutes as well; we will not give the details here. All the vertical maps except the middle one are isomorphisms, because Y, Z and $Y \cap Z$ lie in $\mathcal{X}$. It follows by the five-lemma that the middle vertical map is an isomorphism as well, so $Y \cup Z \in \mathcal{X}$.

Now suppose we have open sets $Y_1, \ldots, Y_n$ such that $V|_{Y_k}$ is trivial for each k. Put $Z = Y_1 \cup \ldots \cup Y_{n-1}$; by induction we may assume that $Z \in \mathcal{X}$. As $V|_{Y_n}$ is trivial, the same is true of $V|_{Z\cap Y_n}$, so Y_n and $Y_n \cap Z$ lie in $\mathcal{X}$. It follows that $Y_1 \cup \ldots Y_n \in \mathcal{X}$.

Now suppose that X is compact. We can then choose $Y_1, \ldots, Y_n$ as above such that $Y_1 \cup \ldots \cup Y_n = X$, proving the theorem in this case. If X is non-compact we still deduce that $Y \in \mathcal{X}$ whenever $Y \subseteq X$ is compact. We can then show that $X \in \mathcal{X}$ by passing to limits; we have not really studied the cohomology of limits, so we omit the details here. $\square$

Remark 13.5. If W is a complex vector space, then section 7.1 gives a canonical orientation of W. If V is a complex vector bundle of dimension n over X then we can do this fibrewise to get a canonical Thom class $u(V) \in \widetilde{H}^{2n}X^V$.

Definition 13.6. Let V be an oriented vector bundle of dimension n over a space X, so we have a specified Thom class $u \in \widetilde{H}^n(X^V)$. Let $i\colon X \to X^V$ be defined by $i(x) = (x, 0) \in EV \subset X^V$, and put $e(V) = i^*(u)$. This is called the *Euler class* of V.

Example 13.7. Let V be a complex vector space, and let L be the tautological bundle over PV. Example 11.19 gives a homeomorphism $P(V \oplus \mathbb{C}) = P(V \oplus \mathbb{C})/P\mathbb{C} = (PV)^{L^*}$. We claim that the element $u = x_{V\oplus\mathbb{C}} \in \widetilde{H}^2(PV)^{L^*}$ is the Thom class associated to the canonical orientation of L^*. This reduces to the claim that for each point $M \in PV$, the restriction of u to $\widetilde{H}^2S^{M^*} = \widetilde{H}^2P(M \oplus \mathbb{C}) \simeq \mathbb{Z}$ is the canonical generator, and this is easy given our knowledge of the cohomology of projective spaces. The Euler class $e(L^*)$ is just the pullback of the Thom class along the inclusion $PV \to P(V \oplus \mathbb{C})$, so $e(L^*) = x_L \in H^2PV$.

Remark 13.8. There is a point here that can cause some confusion. In the above example, we used a homeomorphism $P(M \oplus \mathbb{C}) = S^{M^*}$. On the other hand, the map $v \mapsto [v, 1]$ gives a homeomorphism $M \to P(M \oplus \mathbb{C}) \setminus P\mathbb{C}$, which extends to give a homeomorphism $S^M = P(M \oplus \mathbb{C})$. By composing these, we get a homeomorphism $\chi\colon S^M \to S^{M^*}$. If $v \in M \setminus \{0\}$ then there is a unique linear map $v^{-1}\colon M \to \mathbb{C}$ with $v^{-1}(v) = 1$, and the construction $v \mapsto v^{-1}$ gives a homeomorphism $M \setminus \{0\} \to M^* \setminus \{0\}$. If we define $0^{-1} = \infty$ and $\infty^{-1} = 0$ then this extends to give a homeomorphism $S^M \to S^{M^*}$, and it is not hard to see that this is the same as χ. Moreover, all the spaces here are Riemann surfaces and all the maps are complex analytic. It follows using Remark 7.3 that $\chi^*u_{M^*} = u_M$.

Remark 13.9. Suppose we have an oriented bundle V over X, and a map $f\colon Y \to X$. We then have an obvious map of pairs $(E(f^*V), E(f^*V)^\times) \to (EV, EV^\times)$, and by pulling back our Thom class for V we get a Thom class for f^*V. It is easy to see that if we use this to orient f^*V, we have $e(f^*V) = f^*e(V)$.

Remark 13.10. Suppose we have two oriented bundles V and W over the same base X, with Thom classes $v \in H^n(EV, EV^\times)$ and $w \in H^m(EW, EW^\times)$ say. This gives an external product

$$v \times w \in H^{n+m}(EV \times EW, (EV \times EW) \setminus (X \times X)).$$

There is an evident inclusion of pairs

$$(E(V \oplus W), E(V \oplus W)^\times) \to (EV \times EW, (EV \times EW) \setminus (X \times X)),$$

and by restricting $v \times w$ we get a class in $H^{n+m}(E(V \oplus W), E(V \oplus W)^\times)$, which we just denote by vw. It is not hard to check that this is a Thom class. Using this, we find that $e(V \oplus W) = e(V)e(W)$, and we also have $e(0) = 1 \in H^0X$.

Remark 13.11. Suppose that an oriented bundle V has a nowhere-vanishing section s. We claim that $e(V) = 0$. To see this, define $i_t(x) = (x, ts(x)) \in EV \subset X^V$ (for $t \in \mathbb{R}$) and $i_\infty(x) = \infty \in X^V$. This gives a continuous map $[0, \infty] \times X \to X^V$. By definition $e(V) = i_0^*(u)$ but i_0 is homotopic to i_∞ (because $[0, 1] \simeq [0, \infty]]$) and $i_\infty^*(u) = 0$ so $e(V) = 0$. In particular, this holds when V is a trivial bundle (unless $V = 0$).

Proposition 13.12. *Let V be an oriented bundle of dimension n over a space X. Then there is a natural map $\delta\colon H^*S(V) \to H^{*+1-n}X$ giving a long exact sequence as follows (called the* Gysin sequence*):*

$$\ldots \to H^{k-n}X \xrightarrow{e(V)} H^kX \xrightarrow{\pi^*} H^kS(V) \xrightarrow{\delta} H^{k+1-n}X \to \ldots$$

(Here $e(V)$ refers to the map $x \mapsto xe(V)$.)

Proof. The pair $(B(V), S(V))$ has a long exact sequence as follows.

$$\ldots \to H^k(B(V), S(V)) \to H^kB(V) \to H^kS(V) \xrightarrow{\delta} H^{k+1}(B(V), S(V)) \to \ldots.$$

The projection $B(V) \to X$ and the inclusion of the zero section $X \to B(V)$ are mutually inverse homotopy equivalences, so we can replace $H^*B(V)$ by H^*X. We also have

$$H^*(B(V), S(V)) = \widetilde{H}^*(X^V) = H^{*-n}(X).u,$$

where u is the Thom class. The map $H^*(B(V), S(V)) \to H^*B(V)$ is a map of $H^*(X)$-modules and sends u to $e(V)$, so it sends xu to $xe(V)$. With these identifications, we easily recover the Gysin sequence. $\square$

Example 13.13. Let V be a Hermitian space of dimension n, and let L be the tautological bundle over PV. As in Example 11.19, we have a homeomorphism $(PV)^{L^*} = P(V \oplus \mathbb{C})$ and with this identification the element $x_{V\oplus\mathbb{C}} \in \widetilde{H}^2 P(V \oplus \mathbb{C})$ is a Thom class. Recall that

$$S(L^*) = \{(M, \alpha) \mid M \in PV \ \ , \ \ \alpha\colon M \to \mathbb{C} \ \text{ and } \ \|\alpha\| = 1\}.$$

An element $v \in S(V)$ gives a map $v^*\colon V \to \mathbb{C}$ by $v^*(u) = \langle u, v\rangle$. One checks that the definition $f(v) = (\mathbb{C}v, v^*|_{\mathbb{C}v})$ gives a homeomorphism $S(V) \to S(L^*)$. The Gysin sequence thus has the form

$$H^{*-3}PV \to H^{*-2}PV \xrightarrow{x} H^*PV \to H^*S(V) \to H^{*-1}PV.$$

This implies that multiplication by x is surjective except possibly where the target dimension is 0 or $2n-1$. In conjunction with Lemma 10.4 this easily gives a new proof of Theorem 10.1.

Exercise 13.14. Give a proof of Theorem 10.1 based solely on the Gysin sequence.

14. Classifying vector bundles

Definition 14.1. $\mathrm{Vect}(X)$ is the set of isomorphism classes of complex vector bundles over X, and $\mathrm{Vect}_k(X)$ is the subset of bundles of constant dimension k. We also write $\mathrm{Pic}(X) = \mathrm{Vect}_1(X)$. If V is a bundle then we write $[V]$ for the isomorphism class of V.

Remark 14.2. If X is connected then $\mathrm{Vect}(X) = \coprod_k \mathrm{Vect}_k(X)$.

Remark 14.3. We can make $\mathrm{Vect}(X)$ into a commutative semiring by defining $[V] + [W] = [V \oplus W]$ and $[V][W] = [V \otimes W]$. However, we have no additive inverses. On the other hand, if L is one-dimensional then there is a natural isomorphism $L^* \otimes L \simeq \mathrm{Hom}(L, L) \simeq \mathbb{C}$, so $[L^*][L] = 1 \in \mathrm{Pic}(X)$, so $\mathrm{Pic}(X)$ is a group under multiplication.

Our main task in this section is to define a certain space G_k, and give a natural isomorphism $\mathrm{Vect}_k(X) = [X, G_k]$ for compact spaces X.

Definition 14.4. Let V be a complex vector space, and suppose $0 \le k \le n$. Write

$$\mathrm{Grass}_k(V) = \{W \le V \mid \dim(W) = k\}.$$

Recall that $\mathcal{J}(\mathbb{C}^k, V)$ denotes the space of linear injective maps $j\colon \mathbb{C}^k \to V$. The construction $j \mapsto j(\mathbb{C}^k)$ gives rise to a bijection $\mathcal{J}(\mathbb{C}^k, V)/GL(k, \mathbb{C}) \to \mathrm{Grass}_k(V)$, and we topologise $\mathrm{Grass}_k(V)$ so as to make this a homeomorphism.

Proposition 14.5. *The space* $\mathrm{Grass}_k(V)$ *is a compact manifold of dimension* $2k(n-k)$*, where* $n = \dim(V)$*. (It is called the* k*'th* Grassmann manifold *of* V*.)*

Proof. For any surjective linear map $\alpha\colon V \to \mathbb{C}^k$, define

$$T_\alpha = \{j \in \mathcal{J}(\mathbb{C}^k, V) \mid \alpha j = 1_{\mathbb{C}^k}\}$$
$$\widetilde{U}_\alpha = \{j \in \mathcal{J}(\mathbb{C}^k, V) \mid \alpha j \text{ is iso }\}$$
$$U_\alpha = \{W \in \mathrm{Grass}_k(V) \mid \alpha(W) = \mathbb{C}^k\}.$$

We see that $\widetilde{U}_\alpha \simeq T_\alpha \times GL(k, \mathbb{C})$ is an open subset of $\mathcal{J}(\mathbb{C}^k, V)$ that is preserved by the action of $GL(k, \mathbb{C})$, and that our quotient map $q\colon \mathcal{J}(\mathbb{C}^k, V) \to \mathrm{Grass}_k(V)$ carries $\widetilde{U}_\alpha$ onto U_α. It follows that U_α is open in $\mathrm{Grass}_k(V)$, and is homeomorphic to T_α. If we fix $j \in T_\alpha$ then we get a homeomorphism $\mathrm{Hom}(\mathbb{C}^k, \ker(\alpha)) \to T_\alpha$ by $\beta \mapsto \beta + j$; this shows that $\mathrm{Grass}_k(V)$ is locally homeomorphic to $\mathbb{C}^{k(n-k)} = \mathbb{R}^{2k(n-k)}$ as required. It is now enough to show that $\mathrm{Grass}_k(V)$ is a compact Hausdorff space.

Choose a Hermitian structure on V, so we can define the space $\mathcal{L}(\mathbb{C}^k, V) \subset \mathcal{J}(\mathbb{C}^k, V)$ of isometric linear embeddings $j\colon \mathbb{C}^k \to V$. The map $j \mapsto j(\mathbb{C}^k)$ identifies $\mathrm{Grass}_k(V)$ with $\mathcal{J}(\mathbb{C}^k, V)/GL_k(\mathbb{C})$. Using Proposition 6.13 we can also identify this quotient with $\mathcal{L}(\mathbb{C}^d, V)/U(d)$. The space $\mathcal{L}(\mathbb{C}^k, V)$ can be identified with a closed subspace of $S(V)^k$, so it is compact, and it follows that $\mathrm{Grass}_k(V)$ is compact. Next, put

$$\mathrm{Proj}_k(V) = \{\pi \in \mathrm{End}(V) \mid \pi = \pi^2 = \pi^\dagger \text{ and } \mathrm{trace}(\pi) = k\}.$$

This is clearly a closed subspace of $\mathrm{End}(V)$, and thus is Hausdorff. Given $W \in \mathrm{Grass}_k(V)$, define $h(W)\colon V \to V$ to be the map $1 \oplus 0\colon W \oplus W^\perp \to W \oplus W^\perp$, or in other words the orthogonal projection onto W. This gives a bijection $h\colon \mathrm{Grass}_k(V) \to \mathrm{Proj}_k(V)$, with inverse $h^{-1}(\alpha) = \mathrm{image}(\alpha)$. If $w_1, \ldots, w_k$ is an orthonormal basis of W then $h(W)(v) = \sum_j \langle v, w_j \rangle w_j$, which depends continuously on the vectors w_j; it follows that the composite $hgf\colon \mathcal{L}(\mathbb{C}^k, V)/U(k) \to \mathrm{Proj}_k(V)$ is continuous. A continuous bijection from a compact to a Hausdorff space is a homeomorphism, so we deduce that h is a homeomorphism and that $\mathrm{Grass}_k(V)$ is Hausdorff. □

Exercise 14.6. Suppose that $V = V_0 \oplus V_1$, where $\dim(V_0) = k$. Define $U = \{W \in \mathrm{Grass}_k(V) \mid U \cap V_1 = 0\}$, and define $\phi\colon \mathrm{Hom}(V_0, V_1) \to U$ by $\phi(\alpha) = (1+\alpha)(V_0)$. Check that U is open and that ϕ is a homeomorphism.

Exercise 14.7. Now suppose that $V_1 = V_0^\perp$, and that $\alpha\colon V_0 \to V_1$. Show that the map $1 + \alpha^\dagger\alpha\colon V_0 \to V_0$ is invertible, and write β for its inverse. An endomorphism of $V = V_0 \oplus V_1$ can be written as a 2×2 matrix with entries in $\mathrm{Hom}(V_0, V_0)$, $\mathrm{Hom}(V_1, V_0)$, $\mathrm{Hom}(V_0, V_1)$, and $\mathrm{Hom}(V_1, V_1)$. Show that the orthogonal projection onto $\phi(\alpha)$ corresponds to the matrix $\left(\begin{smallmatrix} \beta & \beta\alpha^\dagger \\ \alpha\beta & \alpha\beta\alpha^\dagger \end{smallmatrix}\right)$.

Lemma 14.8. *There is a tautological vector bundle T of dimension k over $\mathrm{Grass}_k(V)$, with*

$$ET = \{(W, w) \mid W \in \mathrm{Grass}_k(V) \text{ and } w \in W\}.$$

Proof. The only point that needs checking is local triviality. It will be enough to check that $T|_U$ is trivial whenever U is as in Exercise 14.6, or equivalently that ϕ^*T is a trivial bundle over $\mathrm{Hom}(V_0, V_1)$. Here

$$E(\phi^*T) = \{(\alpha, v) \mid \alpha \in \mathrm{Hom}(V_0, V_1) \text{ and } v \in (1+\alpha)(V_0)\}.$$

This is homeomorphic to $\mathrm{Hom}(V_0, V_1) \times V_0$ by the map $(\alpha, u) \mapsto (\alpha, (1+\alpha)(u))$, which gives the required trivialisation. □

To get the space G_k, we need to generalise the above construction so as to allow V to have infinite dimension. For this, we need some discussion of colimit topologies.

Definition 14.9. Let X be a set, and suppose we have subsets $X_0 \subseteq X_1 \subseteq \ldots \subseteq X$ with $X = \bigcup_n X_n$. Suppose that each set X_n has a specified Hausdorff topology such that X_n is closed in X_m when $n \leq m$, and that the specified topology on X_n is the same as its topology as a subspace of X_m. We then declare that a subset $Y \subseteq X$ is closed iff $Y \cap X_n$ is closed in X_n for all n. This gives a topology on X, which we call the *colimit topology.*

Exercise 14.10. Check that for any space Z, the continuous maps $f\colon X \to Z$ biject with systems of continuous maps $f_k\colon X_k \to Z$ such that $f_k|_{X_{k-1}} = f_{k-1}$ for all k. (This means that X is the colimit of the spaces X_k in the sense of category theory.)

Lemma 14.11. *Let $\{X_k\}$ and X be as above, and suppose that $K \subseteq X$ is compact. Then $K \subseteq X_n$ for some n.*

Proof. Put $K_n = K \cap X_n$ (so that $K_0 \subseteq K_1 \subseteq \ldots \subseteq K$) and $S = \{n \in \mathbb{N} \mid K \cap K_n \neq K_{n-1}\}$ (where we take $K_{-1} = \emptyset$). It will be enough to show that S is finite. For each $n \in S$ choose $x_n \in K_n \setminus K_{n-1}$ and put $L = \{x_n \mid n \in S\}$; it will now be enough to show that L is finite. For any subset $M \subseteq L$, we see that $M \cap X_n$ is finite and thus (by the Hausdorff property) closed in X_n. This holds for all n, so we see that M is closed in X. This holds for all $M \subseteq L$, so we see that L is a discrete closed subspace of the compact space K, and thus is finite as claimed. □

Definition 14.12. We write $\mathbb{C}^\infty$ for the set of sequences $(z_0, z_1, \ldots)$ such that $z_k = 0$ for $k \gg 0$; this is a complex vector space with an obvious basis $\{e_0, e_1, \ldots\}$. We can think of $\mathbb{C}^k$ as the span of the vectors $\{e_i \mid i < k\}$ and thus as a subset of $\mathbb{C}^\infty$, with $\mathbb{C}^\infty = \bigcup_k \mathbb{C}^k$. We topologise $\mathbb{C}^\infty$ as the colimit.

Remark 14.13. There are various obvious metrics on $\mathbb{C}^\infty$, but none of them gives rise to the colimit topology. Indeed, there is no countable basis of neighbourhoods of 0 in $\mathbb{C}^\infty$, so the colimit topology cannot arise from *any* metric.

Definition 14.14. We write G_k for the set of k-dimensional complex subspaces of $\mathbb{C}^\infty$. This is naturally thought of as $\bigcup_n \mathrm{Grass}_k(\mathbb{C}^n)$, and we give it the colimit topology. One can check that the tautological bundles over the spaces $\mathrm{Grass}_k(\mathbb{C}^n)$ fit together to give a vector bundle over G_k. **details?**

Theorem 14.15. *There is a natural isomorphism $[X, G_k] \simeq \mathrm{Vect}_k(X)$ for compact spaces X.*

The proof will follow after some preliminary results.

Definition 14.16. Let X be a compact Hausdorff space, and let $\mathcal{U} = \{U_\alpha\}$ be a family of open sets that cover X. Given a map $\phi\colon X \to I$, we write $\mathrm{supp}(\phi)$ for the closure of the set $\phi^{-1}(0,1]$. A *partition of unity subordinate to $\mathcal{U}$* is a finite list $\phi_1, \dots, \phi_n$ of maps $\phi_i\colon X \to I$ such that $\sum_i \phi_i = 1$ and for each i there exists α such that $\mathrm{supp}(\phi) \subseteq U_\alpha$.

Lemma 14.17. *Let X and $\mathcal{U}$ be as above. Then there always exists a partition of unity subordinate to $\mathcal{U}$.*

Proof. For any $x \in X$ we can choose α such that $x \in U_\alpha$. By standard general topology we can then find an open neighbourhood V of x such that $\overline{V} \subseteq U_\alpha$. By Urysohn's lemma we can find $\psi\colon X \to I$ with $\psi = 0$ on $X \setminus V$ and $\psi(x) = 1$. Clearly $\mathrm{supp}(\psi) \subseteq \overline{V} \subseteq U_\alpha$. We can do this for every x, and the resulting open sets $\psi^{-1}(0,1]$ cover X; we can thus choose a finite subcover. This gives functions $\psi_1, \dots, \psi_n$ and indices $\alpha_1, \dots, \alpha_n$ such that $\mathrm{supp}(\psi_i) \subseteq U_{\alpha_i}$, and the sets $\psi_i^{-1}(0,1]$ cover X. This means that the function $\psi = \sum_i \psi_i$ is strictly positive everywhere, and thus $1/\psi$ is defined and continuous. We put $\phi_i = \psi_i/\psi$; it is easy to see that these form a partition of unity as required. □

Lemma 14.18. *Let X be a compact Hausdorff space, and Y a closed subspace. Let V be a vector bundle over X, and let s be a section of $V|_Y$. Then there exist sections t of V such that $t|_Y = s$.*

Proof. First suppose that V is trivial, say $EV = \mathbb{R}^n \times X$. Then s is just given by an n-tuple of functions $s_i\colon Y \to \mathbb{R}$, and the problem is to extend these to get functions $t_i\colon X \to \mathbb{R}$. This is possible by Tietze's extension theorem.

Now suppose that V need not be trivial. Nonetheless, X can be covered by open sets U for which $V|_{\overline{U}}$ is trivial, and we can choose a partition of unity $\phi_1, \dots, \phi_n$ subordinate to such a cover. By construction there are closed sets $X_i = \overline{U_i}$ such that $V|_{X_i}$ is trivial and $\mathrm{supp}(\phi_i)$ is contained in the interior of X_i. Put $Y_i = X_i \cap Y$ and $s_i = s|_{Y_i}$. By the trivial case, we can choose sections r_i of V over X_i such that $r_i|_{Y_i} = s_i$. Now define a section t_i of V over X by

$$t_i(x) = \begin{cases} \phi_i(x) r_i(x) & \text{if } x \in U_i \\ 0 & \text{if } x \in X \setminus \mathrm{supp}(\phi_i). \end{cases}$$

The two clauses are consistent where they both apply, and the domains of definition are both open; it follows easily that t_i is continuous. Now put $t = \sum_i t_i$ to get the required extension of s. □

Corollary 14.19. *Let X be a compact Hausdorff space, and Y a closed subspace. Let V and W be vector bundles over X, and let $\alpha\colon V|_Y \to W|_Y$ be a homomorphism of bundles. Then there is a homomorphism $\beta\colon V \to W$ such that $\beta|_Y = \alpha$.*

Proof. Apply the lemma to the bundle $\mathrm{Hom}(V, W)$. □

Lemma 14.20. *Let V and W be vector bundles over a space X, and let $\alpha\colon V \to W$ be a morphism between them. Then the set $\{x \mid \alpha_x\colon V_x \to W_x$ is iso $\}$ is open.*

Proof. Suppose that α_x is iso for some fixed x; we need to prove that α_y is iso when y is near x. For this we may replace X by a small neighbourhood of x over which V and W are trivial, and thus assume that $EV = EW = X \times \mathbb{R}^n$. This means that there is some continuous function $\beta\colon X \to M_n(\mathbb{R})$ such that $\alpha(x, v) = \beta(x).v$. Our task is to prove that $\{x \mid \beta(x)$ is invertible $\}$ is open in X, but this is easy because $GL_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$. □

Lemma 14.21. *Let X be a compact Hausdorff space, and let V be a vector bundle over X. Then there exists a bundle W over X such that $V \oplus W$ is trivial.*

Proof. We'll say that a closed subspace $Y \subseteq X$ is *good* if there is a finite list $s_1, \ldots, s_n$ of sections of V such that for all $y \in Y$, the vectors $s_1(y), \ldots, s_n(y)$ span V_y. From this it is clear that finite unions of good subspaces are good. Now suppose $x \in X$, and choose a closed neighbourhood Y of x such that $V|_Y$ is trivial. This means that we can choose sections $t_1, \ldots, t_m$ of $V|_Y$ such that for all $y \in Y$, the vectors $t_1(y), \ldots, t_m(y)$ form a basis of V_y. By Lemma 14.18, we can choose sections s_k of V such that $s_k|_Y = t_k$; this shows that Y is good. A compactness argument now shows that X is good, giving a surjective map $s\colon \mathbb{R}^n \to V$ of bundles over X for some n. Let W_x be the kernel of $s(x)\colon \mathbb{R}^n \to V_x$; one checks that these spaces fit together to form a vector bundle and that the restriction of s to $W^\perp$ gives an isomorphism $W^\perp \to V$. It follows that $V \oplus W \simeq \mathbb{R}^n$ as required. $\square$

Lemma 14.22. *Let X and Y be spaces such that X is compact, and let W be a vector bundle over Y. If $f_0, f_1\colon X \to Y$ are homotopic maps, then $f_0^*W \simeq f_1^*W$ as bundles over X.*

Proof. Choose a homotopy $F\colon I \times X \to Y$ with $F(t,x) = f_t(x)$ when $t \in \{0,1\}$, and define $f_t(x) = F(t,x)$ for all t. Define an equivalence relation on I by $t \sim s$ iff $f_t^*W \simeq f_s^*W$. As I is connected, it will suffice to show that the equivalence classes are open. Fix $t \in I$, and define a bundle U over $I \times X$ by $U_{(s,x)} = W_{F(t,x)}$. There is an obvious isomorphism $\alpha\colon U|_{t\times X} \to (F^*W)|_{t\times X}$, and by Corollary 14.19 we can choose $\beta\colon U \to F^*V$ such that $\beta|_{t\times X} = \alpha$. Lemma 14.20 shows that the set $N := \{(s,x) \mid \beta_{(s,x)} \text{ is iso }\}$ is open in $I \times X$, and it clearly contains $t \times X$. As X is compact, a little general topology shows that $M \times X \subseteq N$ for some neighbourhood M of t in I. When $s \in M$, the map $\beta|_{s\times X}$ gives an isomorphism $f_t^*W \to f_s^*W$, so M is contained in the equivalence class of t. Thus, the equivalence classes are open, as required. $\square$

Proof of Theorem 14.15. Let T be the tautological bundle over G_k, and define $\phi_X\colon [X, G_k] \to \mathrm{Vect}_k(X)$ by $\phi([f]) = [f^*T]$; this is well-defined by Lemma 14.22. Let V be a bundle of dimension k over X. Using Lemma 14.21, we can choose a continuous map $\alpha\colon EV \to \mathbb{C}^n$ which is linear and injective on each fibre. Define $f\colon X \to \mathrm{Grass}_k(\mathbb{C}^n) \subset G_k$ by $f(x) = \alpha_x(V_x) \leq \mathbb{C}^n$, so

$$E(f^*T) = \{(x,w) \mid x \in X \text{ and } w \in \alpha_x(V_x)\}.$$

Thus, the map $(x,v) \mapsto (x, \alpha_x(v))$ identifies EV with $E(f^*T)$ and thus V with f^*T, proving that ϕ_X is surjective. Now suppose that $f_0^*T \simeq V \simeq f_1^*T$ say; we need to show that $f_0 \simeq f_1$. First, Lemma 14.11 shows that there exists n such that $f_0(X) \cup f_1(X) \subseteq \mathrm{Grass}_k(\mathbb{C}^n)$. Now regard $\mathbb{C}^{2n}$ as $\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^n$, and define $\rho_\theta = \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right) \otimes 1\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$; this induces a map $\rho'_\theta\colon \mathrm{Grass}_k(\mathbb{C}^{2n}) \to \mathrm{Grass}_k(\mathbb{C}^{2n})$. After replacing f_1 by the homotopic map $\rho'_{\pi/2} \circ f_1$, we may assume that $f_0(X) \subseteq \mathrm{Grass}_k(\mathbb{C}^n \oplus 0)$ and $f_1(X) \subseteq \mathrm{Grass}_k(0 \oplus \mathbb{C}^n)$. As $f_0^*T \simeq V \simeq f_1^*V$, there exist continuous families of isomorphisms $\alpha_0(x)\colon V_x \to f_0(x) < \mathbb{C}^n \oplus 0$ and $\alpha_1(x)\colon V_x \to f_1(x) < 0 \oplus 0\mathbb{C}^n$. Define $\alpha_t(x) := (1-t)\alpha_0(x) + t\alpha_1(x)\colon V_x \to \mathbb{C}^{2n}$; it is easy to see that this is injective, so we can also define $f_t(x) = \alpha_t(x)(V_x) \in \mathrm{Grass}_k(\mathbb{C}^{2n}) \subset G_k$. This gives the required homotopy from f_0 to f_1, proving that ϕ_X is injective. $\square$

15. Classifying line bundles

Recall that $\mathrm{Pic}(X)$ is the group of isomorphism classes of complex line bundles over X. If X is compact then Theorem 14.15 identifies $\mathrm{Pic}(X)$ with $[X, G_1]$. If V is a finite dimensional complex vector space then it is clear that $\mathrm{Grass}_1(V) = PV$, so $G_1 = \bigcup_n \mathbb{C}P^n$; we also write $\mathbb{C}P^\infty$ for this space. In this section we relate the isomorphism $[X, \mathbb{C}P^\infty] \simeq \mathrm{Pic}(X)$ to the group structure on $\mathrm{Pic}(X)$.

We can identify the vector space $\mathbb{C}^\infty$ with the polynomial ring $\mathbb{C}[t]$, by sending the basis vector e_k to t^k. If $L$and M are one-dimensional subspaces of $\mathbb{C}[t]$, we write $LM = \{fg \mid f \in L \text{ and } g \in M\}$. This is again a one-dimensional subspace, so we can define $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ by $\mu(L,M) = LM$. This makes $\mathbb{C}P^\infty$ into a commutative topological monoid; the unit element is the obvious copy of $\mathbb{C}$ in $\mathbb{C}[t]$. To make it a topological group, we would need an inversion map $\chi\colon L \mapsto L^{-1}$ such that $LL^{-1} = \mathbb{C}$, or in other words the following diagram (in which Δ is the diagonal map) would have to commute:

$$\begin{tikzcd} \mathbb{C}P^\infty \arrow[r, "\Delta"] \arrow[d] & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "1\times\chi"] & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[d, "\mu"] \\ 1 \arrow[rr] & & [2]\mathbb{C}P^\infty. \end{tikzcd}$$

One can check that no such map exists. However, we do have a homotopical substitute.

Lemma 15.1. *The monoid $\mathbb{C}P^\infty$ is actually a group up to homotopy, in the sense that there is an "inversion map" $\chi\colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ such that $\mu\circ(1\times\chi)\circ\Delta$ is homotopic to the constant map with value 1.*

Proof. Given $f=\sum_k a_kt^k\in\mathbb{C}[t]^\times$ and $s\in[0,1]$ we define

$$\begin{aligned}\tilde{\chi}([f]) &= \sum_k \overline{a_k}t^k \in \mathbb{C}[t]^\times\\ \|f\|^2 &= \sum_k |a_k|^2\in\mathbb{R}\\ \tilde{\zeta}_s([f]) &= s\|f\|^2+(1-s)f\overline{f}\in\mathbb{C}[t].\end{aligned}$$

If $\lambda\in\mathbb{C}^\times$, we find that $\tilde{\chi}(\lambda f)=\overline{\lambda}\tilde{\chi}(f)$ and $\tilde{\zeta}_s(\lambda f)=|\lambda|^2\tilde{\zeta}_s(f)$, so $[\tilde{\chi}(\lambda f)]=[\tilde{\chi}(f)]$ and $[\tilde{\zeta}_s(\lambda f)]=[\tilde{\zeta}_s(f)]$ in $\mathbb{C}P^\infty$. We therefore get induced maps $\chi,\zeta_s\colon\mathbb{C}P^\infty\to\mathbb{C}P^\infty$. Given this, we just observe that $\mu\circ(1\times\chi)=\zeta_0$ is homotopic to the constant map ζ_1, as required. □

Corollary 15.2. *For any space X, the set $[X,\mathbb{C}P^\infty]$ has a natural group structure.*

Proof. Given elements $a,b\in[X,\mathbb{C}P^\infty]$ represented by maps $u,v\colon X\to\mathbb{C}P^\infty$ say, we have a map

$$w=(X\xrightarrow{\Delta}X^2\xrightarrow{u\times v}(\mathbb{C}P^\infty)^2\xrightarrow{\mu}\mathbb{C}P^\infty),$$

and we define ab to be the homotopy class of w. One checks that this is well-defined, and gives a group structure with inverses $[u]^{-1}=[\chi\circ u]$. □

Theorem 15.3. *If X is compact, then the natural bijection $\phi\colon[X,\mathbb{C}P^\infty]\to\mathrm{Pic}(X)$ is a group homomorphism, as is the map $\psi\colon\mathrm{Pic}(X)\to H^2(X)$ defined by $\psi(L)=e(L^*)$.*

Remark 15.4. The map ψ is also an isomorphism under mild additional conditions; in particular, this holds when X is a compact manifold. For noncompact X, the theorem remains true after minor adjustment of the definitions. These adjustments make no difference in most cases of interest.

Proof. We have two projections $\pi_0,\pi_1\colon\mathbb{C}P^\infty\times\mathbb{C}P^\infty\to\mathbb{C}P^\infty$. Let L be the tautological line bundle over $\mathbb{C}P^\infty$, and write $L_i=\pi_i^*L$. We first claim that $\mu^*L\simeq L_0\otimes L_1$. Indeed, if $(M_0,M_1)\in\mathbb{C}P^\infty\times\mathbb{C}P^\infty$ then $(\mu^*L)_{(M_0,M_1)}=M_0M_1$ and $(L_0\otimes L_1)_{(M_0,M_1)}=M_0\otimes M_1$. The map $f_0\otimes f_1\mapsto f_0f_1$ is easily seen to give an isomorphism $M_0\otimes M_1\to M_0M_1$, as required. Now suppose we have maps $u_0,u_1\colon X\to\mathbb{C}P^\infty$, and define $w=(u_0\times u_1)\circ\Delta\colon X\to\mathbb{C}P^\infty\times\mathbb{C}P^\infty$ and $u=\mu\circ w\colon X\to\mathbb{C}P^\infty$, so that $[u]=[u_0][u_1]$ in $[X,\mathbb{C}P^\infty]$. Then $\phi([u_0])\phi([u_1])=[(u_0^*L)\otimes(u_1^*L)]=w^*[L_0\otimes L_1]$, whereas $\phi([u_0][u_1])=\phi([u])=[u^*L]=w^*[\mu^*L]$. As $[\mu^*L]=[L_0\otimes L_1]$, we conclude that ϕ is a group homomorphism.

Next, let $x=e(L^*)\in H^2\mathbb{C}P^\infty$ be the usual generator, and write $x_i=\pi_i^*x=e(L_i^*)\in H^2(\mathbb{C}P^\infty\times\mathbb{C}P^\infty)$. We claim that $\mu^*x=x_0+x_1$. Indeed, as $H^0\mathbb{C}P^\infty=\mathbb{Z}$, the Künneth theorem tells us that $H^2(\mathbb{C}P^\infty\times\mathbb{C}P^\infty)=\mathbb{Z}\{x_0,x_1\}$, so $\mu^*(x)=a_0x_0+a_1x_1$ for some uniquely determined integers a_0,a_1. Define $\eta_0,\eta_1\colon\mathbb{C}P^\infty\to\mathbb{C}P^\infty\times\mathbb{C}P^\infty$ by $\eta_0(M)=(M,\mathbb{C})$ and $\eta_1(M)=(\mathbb{C},M)$; it is easy to see that $\eta_i^*x_j=\delta_{ij}x$ and thus that $\eta_i^*\mu^*x=a_ix$. On the other hand, we have $\mu\eta_i=1$ so $\eta_i^*\mu^*x=x$, so $a_0=a_1=1$, so $\mu^*x=x_0+x_1$ as claimed.

Now let u_0, u_1, w and u be as before; we claim that $u^*x=u_0^*x+u_1^*x$. Indeed, $u^*x=w^*\mu^*x=w^*x_0+w^*x_1$, but $w^*x_i=(\pi_iw)^*x=u_i^*x$, which gives the claim. This shows that $\psi\phi$ is a homomorphism, and ϕ is a group isomorphism, so we see that ψ is a homomorphism. □

We now give an application of these ideas. Let V be a Hermitian space of dimension n, and let C be the cyclic subgroup of S^1 generated by $\omega:=\exp(2\pi i/m)$ for some $m>1$. This acts by multiplication on the sphere $S(V)$, and we define $M=S(V)/C$. One checks that if U is an open ball in $S(V)$ of radius less than $|\omega-1|/2$, then the composite $U\to S(V)\to M$ is an open embedding; it follows that M is a compact manifold of dimension $2n-1$. There is an evident quotient map $q\colon M\to S(V)/S^1=PV$, and we write $y=q^*x_V\in H^2M$.

Proposition 15.5. *There is an element $u\in H^{2n-1}M$ such that $H^*M=\mathbb{Z}[y]/(my,y^n)\oplus\mathbb{Z}u$, with $yu=u^2=0$.*

Proof. Let L be the tautological bundle over PV, and write L^m for the m-fold tensor power $L\otimes\ldots\otimes L$. Theorem 15.3 tells us that $e(L^m)=me(L)=-mx_V\in H^2PV$. The inner product on V gives rise to an inner product on L^m, and as usual we define

$$S(L^m)=\{(M,u)\mid M\in PV \text{ and } \|u\|=1\}.$$

Define $\tilde{\phi}\colon S(V)\to S(L^m)$ by $\tilde{\phi}(v)=(\mathbb{C}v,v^{\otimes m})$. If $c\in C$ then $(cv)^{\otimes m}=c^m v^{\otimes m}=v^{\otimes m}$, so $\tilde{\phi}(cv)=\tilde{\phi}(v)$, so we get an induced map $\phi\colon M\to S(L^m)$. One can check that this is a continuous bijection between compact Hausdorff spaces and thus a homeomorphism. This gives us a Gysin sequence

$$H^{k-2}PV\xrightarrow{-mx}H^kPV\xrightarrow{q^*}H^kM\xrightarrow{\delta}H^{k-1}PV\to\ldots,$$

and thus a short exact sequence

$$H^k(PV)/mx\xrightarrow{q^*}H^kM\to\operatorname{ann}(mx,H^{k-1}PV).$$

We know that $H^*(PV)/mx=\mathbb{Z}[x]/(x^n,mx)$ and $\operatorname{ann}(mx,H^*PV)=\mathbb{Z}\{x^{n-1}\}$. This shows that the first term in the short exact sequence is concentrated in even degrees, and the last term is concentrated in odd degrees, so the sequence has a unique splitting. The theorem now follows (the equations $u^2=0$ and $uy=0$ take place in trivial groups). $\square$

16. Cohomology of projective bundles

Let V be a complex vector bundle of dimension n over a space X. Write

$$PV=\{(x,L)\mid\ \in X \text{ and } L\in PV_x\},$$

and let $\pi\colon PV\to X$ be the obvious projection map. Suppose we have a subspace $Y\subseteq X$ and a trivialisation $V|_Y\simeq Y\times W$ say; this gives a bijection $\pi^{-1}Y\to Y\times PW$. We give PV the unique topology such that π is continuous and these bijections are homeomorphisms. The space PV is called the *projective bundle* associated to V. There is a tautological line bundle L over PV with total space

$$EL=\{(x,M,u)\mid x\in X\ ,\ M\in PV_x\ ,\ x\in M\}.$$

We write $x=x_V=e(L^*)\in H^2PV$ for the Euler class of the dual of L. By example 13.7, this is consistent with our previous notation when X is a point and V is just a vector space.

Theorem 16.1. *There is a unique monic polynomial $f_V(t)=\sum_j=0^n c_jt^{n-j}\in H^*(X)[t]$ (with $c_j\in H^{2j}X$ and $c_0=1$) such that*

$$H^*PV=H^*(X)[x]/f_V(x)=H^*(X)\{1,x,\ldots,x^{n-1}\}.$$

Definition 16.2. The element $c_j=c_j(V)\in H^{2j}X$ is called the *j'th Chern class* of V.

Proof. This is very similar to the proof of the Thom isomorphism theorem. Given a subspace $Y\subseteq X$, we put $B^*Y=H^*(\pi^{-1}Y)$, and we allow ourselves to write x^k for $x_V^k|_{\pi^{-1}Y}\in B^{2k}Y$. We also put $A^mY=\bigoplus_{k=0}^{n-1}H^{m-2k}Y$, so that A^*Y is a free module over H^*Y on generators $x_0,\ldots,x_{n-1}$ with $x_k\in A^{2k}Y$. Define $\phi_Y\colon A^*Y\to B^*Y$ to be the map of $H^*(Y)$-modules that sends x_k to x^k. If $V|_Y$ is trivial then $\pi^{-1}Y\simeq X\times PW$ say, and it follows easily from Theorem 10.1 and the Künneth theorem that ϕ_Y is an isomorphism. Next, suppose that Y and Z are open subspaces of X such that ϕ_Y, ϕ_Z and $\phi_{Y\cap Z}$ are isomorphisms. We have a Mayer-Vietoris sequence relating the groups H^*T for $T=Y,Z,Y\cap Z,Y\cup Z$, and by taking the direct sum of n copies of this we get a Mayer-Vietoris sequence relating the groups A^*T. On the other hand, we have open subsets $\pi^{-1}(Y),\pi^{-1}(Z)\subseteq PV$ whose union is $\pi^{-1}(Y\cup Z)$ and whose intersection is $\pi^{-1}(Y\cap Z)$. This gives us a Mayer-Vietoris sequence involving the groups B^*T. We next need to check that the following diagram commutes:

$$\begin{tikzcd}
A^{k-1}Y\times A^{k-1}Z \arrow[r] \arrow[d, "\phi\times\phi"'] & A^{k-1}(Y\cap Z) \arrow[r] \arrow[d, "\phi"'] & A^k(Y\cup Z) \arrow[r] \arrow[d, "\phi"'] & A^kY\times A^kZ \arrow[r] \arrow[d, "\phi\times\phi"] & A^k(Y\cap Z) \arrow[d, "\phi"] \\
B^{k-1}Y\times B^{k-1}Z \arrow[r] & B^{k-1}(Y\cap Z) \arrow[r] & B^k(Y\cup Z) \arrow[r] & B^kY\times B^kZ \arrow[r] & B^k(Y\cap Z)
\end{tikzcd}$$

Because Mayer-Vietoris sequences are natural, we see that everything commutes when restricted to $H^*T\leq A^*T$. Moreover, on the bottom row all maps are maps of modules over H^*PV; as $x\in H^*PV$, it follows that the whole diagram commutes. We now see from the 5-lemma that $\phi_{Y\cup Z}$ is an isomorphism. As in the proof of Theorem 13.4, we conclude that ϕ_X is an isomorphism, so $H^*PV = H^*(X)\{1, x, \ldots, x^{n-1}\}$. This means that there are unique elements $c_j \in H^{2j}X$ such that $-x^n = \sum_{j=1}^n c_j x^{n-j}$. If we put $c_0 = 1$ and $f_V(t) = \sum_{j=0}^n c_j t^{n-j}$ we find that f_V is the unique monic polynomial of degree n such that $f_V(x) = 0$. It is clear that $H^*(X)[t]/f_V(t)$ is freely generated as a module over H^*X by $\{1, t, \ldots, t^{n-1}\}$, and it follows easily that $H^*PV = H^*(X)[x]/f_V(x)$ as rings. □

Proposition 16.3. *If V and W are two complex vector bundles over a space X then $f_{V\oplus W}(t) = f_V(t)f_W(t)$, and thus*

$$c_k(V \oplus W) = \sum_{i+j=k} c_i(V)c_j(W).$$

The proof depends on the following lemma.

Lemma 16.4. *Let $Y = A \cup B$ be a space, and suppose we have elements $a \in H^iY$ and $b \in H^jY$ such that $a|_A = 0$ and $b|_B = 0$. Then $ab = 0$.*

Proof. Recall that we have a long exact sequence

$$\ldots \to H^i(Y, A) \to H^i(Y) \to H^i(A) \to \ldots.$$

This shows that there is an element $a' \in H^i(Y, A)$ that maps to a in H^iY. Similarly, there is an element $b' \in H^j(Y, B)$ that maps to b in H^jY. Next, recall that we have a commutative diagram of product and restriction maps as follows:

$$\begin{tikzcd} H^i(Y,A) \otimes H^j(Y,B) \arrow[r, "\mu"] \arrow[d] & H^{i+j}(Y, A \cup B) \arrow[d] \\ H^iY \otimes H^jY \arrow[r, "\mu"'] & H^{i+j}Y \end{tikzcd}$$

Note that $A \cup B = Y$ so $H^{i+j}(Y, A \cup B) = 0$. By chasing $a' \otimes b'$ round the diagram, we find that $ab = 0$ as claimed. □

Proof of Proposition 16.3. Let $x \in H^2P(V \oplus W)$ be the Euler class of the dual of the tautological bundle. If V and W have dimensions n and m, then $f_{V\oplus W}(t)$ is the unique monic polynomial of degree $n + m$ over H^*X such that $f_{V\oplus W}(x) = 0$. As $f_V(t)f_W(t)$ is a monic polynomial of degree $n + m$, it will thus be enough to show that $f_V(x)f_W(x) = 0$. Note that PV and PW can be thought of as subspaces of $P(V \oplus W)$, with $PV \cap PW = \emptyset$. This means that the spaces $A := P(V \oplus W) \setminus PW$ and $B := P(V \oplus W) \setminus PV$ have $A \cup B = P(V \oplus W)$, so by the lemma it suffices to check that $f_V(x)|_A = 0$ and $f_W(x)|_B = 0$. To do this, we claim that the inclusion $PV \to A$ is a homotopy equivalence. If X is a point, this is corollary 10.3; for general X, we just apply the same constructions in all fibres simultaneously. By definition we have $f_V(x)|_{PV} = 0$, and it follows that $f_V(x)|_A = 0$; by a similar argument we have $f_W(x)|_B = 0$, completing the proof. □

Corollary 16.5. *If we have a short exact sequence $U \rightarrowtail V \twoheadrightarrow W$ of vector bundles over X then $f_V(t) = f_U(t)f_W(t)$.*

Proof. After choosing an inner product on V we get a splitting $V = U \oplus U^\perp$, and the composite $U^\perp \rightarrowtail V \twoheadrightarrow W$ is an isomorphism so $V \simeq U \oplus W$. Proposition 16.3 now tells us that $f_V(t) = f_U(t)f_W(t)$. □

Proposition 16.6. *If L is a complex line bundle over X then $c_1(L) = e(L) = -e(L^*)$ and thus $f_L(t) = t - e(L^*)$.*

Proof. For each $a \in X$, the space PL_a is a single point; it follows that PL can be identified with X, and the tautological bundle over PL can be identified with L itself. This identifies x with $e(L^*)$, so x is a root of the degree one monic polynomial $f(t) = t - e(L^*)$, which proves that $f_L(t) = f(t)$ and that $c_1(L) = -e(L^*)$. We know from Theorem 15.3 that $e(L^*) = -e(L)$. □

Proposition 16.7. *If V is a bundle over X of complex dimension n, then $c_n(V) = e(V) \in H^{2n}X$.*

Proof. The map $(a, v) \mapsto (a, [v, 1])$ gives a homeomorphism $EV \to P(V \oplus \mathbb{C}) \setminus PV$, and one checks that this extends to a homeomorphism $P(V \oplus \mathbb{C})/PV \simeq X^V$. Let q denote the resulting quotient map $P(V \oplus \mathbb{C}) \to X^V$, and let $u \in \widetilde{H}^{2n}X^V$ be the Thom class. We see from Proposition 16.3 that $f_{V\oplus\mathbb{C}}(t) = tf_V(t)$, so $H^*P(V \oplus \mathbb{C}) = H^*(X)[x]/xf_V(x)$ and $H^*PV = H^*(X)[x]/f_V(x)$. This shows that the restriction map $H^*P(V \oplus \mathbb{C}) \to H^*PV$ is surjective, so the long exact sequence of the pair $(P(V \oplus \mathbb{C}), PV)$ tells us that $q^* \colon \widetilde{H}^*X^V \to H^*P(V \oplus \mathbb{C})$ is injective. The image is the ideal generated by $f_V(x)$, which is a free module of rank one over H^*X; all this follows algebraically from our descriptions of $H^*P(V \oplus \mathbb{C})$ and H^*PV. It follows that $q^*(u) = vf_V(x)$ for some $v \in H^0X$. We claim that $v = 1$; it will be enough to show that $v|_{\{a\}} = 1$ for every point $a \in X$, and because all our constructions are natural this is the same as saying that $v = 1$ in the case where X itself is a point (and thus V is just a vector space). In that case one sees from the proof of Lemma 10.4 that q^*u is the element $y_{V\oplus\mathbb{C},n}$ defined there, and we know from Lemmas 10.7 and 10.8 that $y_{V\oplus\mathbb{C},n} = x^n$. On the other hand, when X is a point we know that V is trivial and thus $f_V(t) = t^n$ so $q^*u = x^n = f_V(x)$ as required. We now return to the case of a general space X. Define $j \colon X \to P(V \oplus \mathbb{C})$ by $j(a) = (a, [0 : 1])$, so that $qj \colon X \to X^V$ is the usual inclusion of the zero section. This means that $e(V) = (qj)^*u = j^*q^*u = j^*f_V(x) = f_V(j^*x)$. On the other hand, the restriction of the tautological bundle to jX is trivial, so $j^*x = 0$, and clearly $f_V(0) = c_n$ so we see that $e(V) = c_n$. □

We next discuss an application of Theorem 16.1 to the cohomology of complex hypersurfaces. Let U and V be complex vector spaces of dimensions n and m with $n \leq m$, and let $\beta \colon U \otimes V \to \mathbb{C}$ be a nonzero linear map. We write

$$H(\beta) := \{(L, M) \in PU \times PV \mid \beta(L \otimes M) = 0\}.$$

Spaces of this form are called complex hypersurfaces of degree $(1, 1)$ in $PU \times PV$. One can check that $H(\beta)$ is a manifold if and only if for each $u \in U^\times$ there exists $v \in V^\times$ with $\beta(u, v) \neq 0$, or equivalently the adjoint map $\beta^\# \colon U \to V^*$ is injective. If so, we say that β is nondegenerate and that $H(\beta)$ is a Milnor hypersurface. One can then choose coordinates so that $U = \mathbb{C}^n$ and $V = \mathbb{C}^m$ and $\beta(z, w) = \sum_{i=1}^n z_i w_i$, so in particular $H(\beta)$ is independent of β up to unnatural isomorphism.

We now compute the cohomology of Milnor hypersurfaces. As $H(\beta) \subset PU \times PV$ we have evident projections $PU \xleftarrow{p} H(\beta) \xrightarrow{q} PV$, and we define $y, z \in H^2H(\beta)$ by $y = p^*x_U$ and $z = q^*x_V$.

Proposition 16.8. *If β is nondegenerate then*

$$H^*H(\beta) = \mathbb{Z}[y, z]/\left(y^n, \sum_{i=0}^{n-1} (-y)^i z^{m-1-i}\right) = \mathbb{Z}\{y^i z^j \mid 0 \leq i < n\ ,\ 0 \leq j < m - 1\}.$$

Moreover, we have $z^m = 0$ in this ring.

Proof. As $x_U^n = 0$ and $x_V^m = 0$ we have $y^n = 0$ and $z^m = 0$.

Given $L \in PU$, define $\gamma_L \colon V \to L^*$ by $\gamma_L(v)(u) = \beta(u, v)$. Because β is nondegenerate, we see that γ_L is surjective; we write W_L for its kernel. These spaces fit together to form a bundle W of dimension $m-1$ over PU and it is easy to identify PW with $H(\beta)$, and $x_W \in H^2PW$ with $z \in H^2H(\beta)$. Thus, Theorem 16.1 tells us that

$$\begin{aligned} H^*H(\beta) &= H^*(PU)[z]/f_W(z) \\ &= H^*(PU)\{z^j \mid 0 \leq j < m - 1\} \\ &= \mathbb{Z}\{y^i z^j \mid 0 \leq i < n\ ,\ 0 \leq j < m - 1\}. \end{aligned}$$

To determine $f_W(t)$, observe that we have a short exact sequence $W \to V \xrightarrow{\gamma} L^*$ of bundles over PU, where V refers to the obvious trivial bundle and L to the tautological bundle. Thus, corollary 16.5 tells us that $t^m = f_V(t) = f_W(t)f_{L^*}(t) = f_W(t)(t + y)$. If we put $g(t) = \sum_{i=0}^{n-1}(-y)^i t^{m-1-i}$ then we see by direct calculation that $(t + y)g(t) = t^m$ and it is easy to see that $t + y$ is not a zero divisor in $H^*(PU)[t]$ so $f_W(t) = g(t)$; this proves the proposition. □

17. Monic polynomials

The results of Section 16 indicate that the algebra of monic polynomials is relevant to the understanding of complex vector bundles. In this section, we develop some of this algebra. The main point is that things which work for arbitrary polynomials over a field often work for monic polynomials over any ring.

Let R^* be a graded ring that is commutative in the graded sense. We will consider the graded ring $R[t]$ in which t is given degree 2. Thus, a homogeneous element of $R[t]$ of degree $2m$ has the form $f(t) = \sum_i a_i t^i$ with $|a_i| = 2(m-i)$ and so $a_i = 0$ for $i > m$. The *polynomial degree* of $f(t)$ is the largest k such that $a_k \neq 0$ (or $-\infty$ if $f(t) = 0$). If $f(t)$ has polynomial degree k and $a_k = 1$ we say that $f(t)$ is *monic*. We write $R[t]_{<k}$ for the submodule of polynomials of degree less than k, and $\mathrm{Mon}(R)$ for the set of monic polynomials, and $\mathrm{Mon}_k(R)$ for the set of monic polynomials of polynomial degree k.

Lemma 17.1. *Suppose that $g(t) \in \mathrm{Mon}(R)$ and $f(t) \in R[t]$ with $f(t) \neq 0$. Then $f(t)g(t) \neq 0$ and $\deg(f(t)g(t)) = \deg(f(t)) + \deg(g(t))$.*

Proof. As $g(t)$ is monic we have $g(t) = t^n + \text{ lower terms }$, where $n = \deg(g(t))$. Similarly we have $f(t) = at^m + \text{ lower terms }$ for some $a \neq 0$ in R, where $m = \deg(f(t))$. We then find that $f(t)g(t) = at^{n+m} + \text{ lower terms }$ so $f(t)g(t) \neq 0$ and $\deg(f(t)g(t)) = n + m$ as claimed. □

Proposition 17.2. *If $g(t) \in \mathrm{Mon}_n(R)$ and $f(t) \in R[t]$ then there is a unique way to write $f(t) = q(t)g(t) + r(t)$ with $q(t) \in R[t]$ and $r(t) \in R[t]_{<n}$. In other words, the map $(q(t), r(t)) \mapsto q(t)g(t) + r(t)$ gives an isomorphism $\phi\colon R[t] \times R[t]_{<n} \to R[t]$ of R-modules.*

Proof. The basic point is that one can do long division of polynomials, and one only ever has to divide by the leading coefficient of $g(t)$, which is not a problem if $g(t)$ is monic. However, we will spell out the details.

First suppose that $\phi(q_0, r_0) = \phi(q_1, r_1)$, so $gq_0 + r_0 = gq_1 + r1$, so $r_0 - r_1 = g.(q_1 - q_0)$. Here $r_0 - r_1$ has degree less than n. If $q_1 - q_0$ is nonzero then Lemma 17.1 tells us that the right hand side is nonzero and has degree at least n, which is a contradiction. We must therefore have $q_1 - q_0 = 0$, and thus $r_0 - r_1 = g.(q_1 - q_0) = 0$, so $(q_0, r_0) = (q_1, r_1)$. This means that ϕ is injective.

We next show by induction that $R[t]_{<k} \subseteq \mathrm{image}(\phi)$ for all k. If $f(t) \in R[t]_{<k}$ with $k \leq n$ then $f = \phi(0, f)$ which starts the induction. In general, if $f(t)$ has degree $k \geq n$ we can write $f(t) = at^k + \text{ lower terms }$ for some $a \in R$. We then put $f_1(t) = f(t) - at^{k-n}g(t)$ and note that this has degree strictly less than k. By induction we may therefore assume that $f_1(t) = \phi(q(t), r(t))$ for some q and r, and it follows that $f(t) = \phi(q(t) + at^{k-n}, r(t))$ which gives the induction step. □

Corollary 17.3. *If $g(t) \in \mathrm{Mon}_n(R)$ then the composite*

$$R[t]_{<n} \xrightarrow{inc} R[t] \xrightarrow{proj} R[t]/g(t)$$

is an isomorphism, so $\{1, t, \ldots, t^{n-1}\}$ is a basis for $R[t]/g(t)$ as a module over R.

Proof. Let ψ be the above composite. If $\psi(r_0(t)) = \psi(r_1(t))$ then $r_0(t) - r_1(t)$ has degree less than n and is divisible by $g(t)$ so it must be zero. It follows that ψ is injective. Any element $a \in R[t]/g(t)$ can be written as $a = f(t) + R[t]g(t)$ for some $f(t) \in R[t]$. If we write $f(t) = q(t)g(t) + r(t)$ as before, we find that $r(t) \in R[t]_{<n}$ and $\psi(r(t)) = a$. It follows that ψ is also surjective. □

Corollary 17.4. *Suppose that $f(t) \in R[t]$ and $a \in R$. If $f(a) = 0$ then there is a unique polynomial $q(t) \in R[t]$ with $f(t) = (t - a)q(t)$; otherwise $f(t)$ is not divisible by $t - a$.*

Proof. We apply the proposition with $g(t) = t - a$, giving $f(t) = (t - a)q(t) + r$ for some r. Here r must have degree zero, so it is an element of $R \subset R[t]$. We can now substitute $t = a$ to get $f(a) = r$. If $f(a) = 0$ we conclude that $f(t) = (t - a)q(t)$, and from the uniqueness clause in the proposition we see that there is only one such factorisation. Conversely, if $f(t)$ is divisible by $t - a$ then it is clear that $f(a) = 0$. □

We next discuss a slightly different application of Proposition 17.2. Suppose we have $f(t)$ and $g(t)$ in $R[t]$ as before. For any ideal $I \leq R$, these will then give polynomials in $(R/I)[t]$ which we will continue to call $f(t)$ and $g(t)$. More generally, given a ring homomorphism $\alpha\colon R \to R'$ we can apply α to the coefficients of f and g to get polynomials in $R'[t]$. By a slight abuse of notation we will call these $\alpha(g(t))$ and $\alpha(f(t))$. This construction extends α to give a ring homomorphism $R[t] \to R'[t]$. If $f(t)$ is divisible by $g(t)$ then we can apply α to an equation $f(t) = g(t)q(t)$ to see that $\alpha(f(t))$ is divisible by $\alpha(g(t))$. This argument is not reversible, however: it can happen that $\alpha(f(t))$ is divisible by $\alpha(g(t))$ even though $f(t)$ is not divisible by $g(t)$. We will need a criterion to test whether this is the case.

Proposition 17.5. *Suppose we have $f(t) \in R[t]$ and $g(t) \in \mathrm{Mon}_n(R)$. Write $f(t) = g(t)q(t) + r(t)$ as in Proposition 17.2, and let I be the ideal in R generated by all the coefficients of $r(t)$. Then for any homomorphism $\alpha\colon R \to R'$, the polynomial $\alpha(f(t))$ is divisible by $\alpha(g(t))$ if and only if $\alpha(I) = 0$. In particular, $f(t)$ becomes divisible by $g(t)$ in $(R/J)[t]$ if and only if $I \leq J$.*

Proof. First, it is clear that the homomorphism $\alpha\colon R \to R'$ sends I to zero if and only if the extended homomorphism $\alpha\colon R[t] \to R'[t]$ sends $r(t)$ to zero. If this holds, we can apply α to the equation $f = qg + r$ to see that $\alpha(f) = \alpha(q)\alpha(g)$, so $\alpha(f)$ is divisible by $\alpha(g)$. Conversely, suppose that $\alpha(f) = p\alpha(g)$ for some $p(t) \in R'[t]$. We then have two different divisions $\alpha(f) = \alpha(q)\alpha(g) + r$ and $\alpha(f) = p\alpha(g) + 0$; by the uniqueness clause in Proposition 17.2 we thus have $\alpha(q) = p$ and $\alpha(r) = 0$, so $\alpha(I) = 0$. ☐

Now suppose we have a monic polynomial $f(t) \in \mathrm{Mon}_n(A)$ that may not have any roots; we would like to adjoin some.

Construction 17.6. For $0 \leq k \leq n$ we put $\widetilde{A}_k = A[x_0, \dots, x_{k-1}]$ and

$$g_k(t) = \prod_{i=0}^{k-1} (t - x_k) \in \widetilde{A}_k[t].$$

We then note that there are unique polynomials $f_k(t), r_k(t) \in \widetilde{A}[t]$ such that $\deg(r_k(t)) < k$ and $f(t) = f_k(t)g_k(t) + r_k(t)$. We then let I_k be the ideal in $\widetilde{A}_k$ generated by the coefficients of $r_k(t)$, and put $A_k = \widetilde{A}_k/I_k$. We then have $f(t) = g_k(t)f_k(t)$ in $A_k[t]$. In particular, this means that the elements $x_0, \dots, x_{k-1}$ are roots of $f(t)$ in A_k.

Remark 17.7. It is useful to understand the cases $k = 0$ and $k = n$ in more detail. The first is easy: we just have $A_0 = \widetilde{A}_0 = A$ and $g_0(t) = 1$ and $f_0(t) = f(t)$.

Next, as $f(t)$ and $g_n(t)$ are both monic of degree n we find that $q_n(t) = 1$ and $r_n(t) = f(t) - g_n(t)$. Let c_i denote the coefficient of t^{n-i} in $f(t)$, and note that the corresponding coefficient in $g(t)$ is $(-1)^i\sigma_i$, where σ_i is the i'th elementary symmetric function in the variables $x_0, \dots, x_{n-1}$. It follows that

$$A_n = \frac{A[x_0, \dots, x_{n-1}]}{((-1)^i\sigma_i - c_i \mid 0 \leq i \leq n)}.$$

Here we have constructed A_k all in one go. To analyse its properties, it is convenient to have a slightly different picture that describes A_{k+1} in terms of A_k.

Proposition 17.8. *There is a canonical isomorphism $A_{k+1} = A_k[x_k]/f_k(x_k)$. Thus, A_{k+1} is a free module of rank $n - k$ over A_k, with basis given by the powers $1, x_k, \dots, x_k^{n-k-1}$.*

Proof. Recall that $A_{k+1} = \widetilde{A}_{k+1}/I_{k+1}$. Similarly $A_k = \widetilde{A}_k/I_k$, so

$$A_k[x_k] = \widetilde{A}_k[x_k]/I_k[x_k] = \widetilde{A}_{k+1}/I_k[x_k].$$

It follows that the ring $A'_{k+1} = A_k[x_k]/f_k(x_k)$ can also be described as $\widetilde{A}_{k+1}/I'_{k+1}$, where $I'_{k+1} = I_k[x_k] + (f_k(x_{k+1}))$. It will therefore be enough to show that $I_{k+1} = I'_{k+1}$.

Note that in $A'_{k+1}[t]$ we have $f_k(x_k) = 0$, so $f_k(t) = (t - x_k)h(t)$ for some $h(t)$. We also have $f(t) = g_k(t)f_k(t)$ (because this already holds in $A_k[t]$) and thus $f(t) = g_k(t)(t-x_k)h(t) = g_{k+1}(t)h(t)$. As I_{k+1} is by definition the smallest ideal modulo which $f(t)$ is divisible by $g_{k+1}(t)$, we see that $I_{k+1} \subseteq I'_{k+1}$. Conversely, as $f(t) = g_{k+1}(t)f_{k+1}(t) = g_k(t)(t - x_k)f_{k+1}(t)$ in $A_{k+1}[t]$ we see in particular that $f(t)$ is divisible by $g_k(t)$ in $A_{k+1}[t]$, so we must have $I_k[x_k] \subseteq I_{k+1}$. We can thus regard A_{k+1} as a quotient of $A_k[x_k]$, which gives another factorisation $f(t) = g_k(t)f_k(t)$ in $A_{k+1}[t]$. By subtracting this from our earlier factorisation we get $g_k(t)(f_k(t) - (t - x_k)f_{k+1}(t)) = 0$, but $g_k(t)$ is monic so we must have $f_k(t) - (t - x_k)f_{k+1}(t) = 0$. We can now substitute $t = x_k$ to see that $f_k(x_k) = 0$ in A_{k+1}, so $f_k(x_k) \in I_{k+1}$. It follows that $I'_{k+1} = I_{k+1}$ as claimed, so $A_{k+1} = A_k[x_k]/f_k(x_k)$. The stated basis therefore follows from Corollary 17.3. ☐

Corollary 17.9. *The ring A_k is a free module over A of rank equal to $\prod_{i=0}^{k-1}(n-i)$. The monomials $\prod_{i=0}^{k-1} x_i^{\alpha_i}$ (with $0 \leq \alpha_i < n-i$ for all i) form a basis.*

Proof. This follows by induction on k from the proposition. □

18. Flag manifolds

We now revisit the cohomology of flag manifolds, which was described previously in Example 2.11. Rather than merely proving the result stated there, we will do something more general that turns out to be better suited to inductive proof.

Suppose we start with a Hermitian vector bundle V of dimension n over a space X. We put

$$\mathrm{Flag}_k(V) = \{(x; W_0, \dots, W_k) \mid x \in X,\ 0 = W_0 < \cdots < W_k \leq V_x,\ \dim(W_i) = i\}.$$

If X is a single point then V is just a vector space. If also $k = n$ then $\mathrm{Flag}_k(V)$ is just the space $\mathrm{Flag}(V)$ of complete flags as considered earlier.

To understand the cohomology of $\mathrm{Flag}_k(V)$, we need to consider certain vector bundles. There is an evident projection $\pi\colon \mathrm{Flag}_k(V) \to X$ given by $\pi(x; \underline{W}) = x$; using this we get a bundle π^*V over $\mathrm{Flag}_k(V)$, whose fibre at $(x; \underline{W})$ is V_x. For $0 \leq i < k$, we let L_i denote the line bundle whose fibre at a point $(x; \underline{W})$ is the quotient W_{i+1}/W_i. We also let U_k be the $(n-k)$-dimensional bundle whose fibre at $(x, \underline{W})$ is V_x/W_k. Using the inner product we get a splitting

$$\pi^*V \simeq L_0 \oplus \cdots \oplus L_{k-1} \oplus U_k.$$

Now put $A = H^*(X)$, and let $f(t) = f_V(t) \in A[t]$ be the Chern polynomial of V. Put $x_i = e(L_i^*) \in H^2(\mathrm{Flag}_k(V))$, so $f_{L_i}(t) = t - x_i$. Put $g_k(t) = \prod_{i=0}^{k-1}(t - x_i)$ and $f_k(t) = f_{U_k}(t)$ so $f_k(t)$ and $g_k(t)$ are elements of $H^*(\mathrm{Flag}_k(V))[t]$. The above splitting then gives $f(t) = g_k(t) f_k(t)$. Thus, if we build a ring A_k as in Construction 17.6 we find that there is a unique map $A_k \to H^*(\mathrm{Flag}_k(V))$ that sends each $a \in A = H^*(X)$ to $\pi^*(a) \in H^*(\mathrm{Flag}_k(X))$, and sends $x_i \in A_k$ to $x_i \in H^2(\mathrm{Flag}_k(V))$.

Theorem 18.1. *The above map $A_k \to H^*(\mathrm{Flag}_k(V))$ is an isomorphism.*

Proof. With all the machinery that we have put in place, it is now easy to prove this by induction. The case $k = 0$ is trivial. We saw in Proposition 17.8 that $A_{k+1} = A_k[x_k]/f_k(x_k)$, so it will be enough to show that

$$H^*(\mathrm{Flag}_{k+1}(V)) = H^*(\mathrm{Flag}_k(V))[x_k]/f_k(x_k).$$

Theorem 16.1 tells us that the right hand side here is the cohomology of the projective bundle PU_k, so it will be enough to identify PU_k with $\mathrm{Flag}_{k+1}(V)$. A point of PU_k consists of a point $(x; \underline{W})$ in $\mathrm{Flag}_k(V)$ together with a one-dimensional subspace $M \leq V_x/W_k$. This must have the form $M = W_{k+1}/W_k$ for a unique subspace $W_{k+1} \leq V_x$ of dimension $k+1$ containing W_k, and this gives us a point $(x; W_0, \dots, W_{k+1})$ in $\mathrm{Flag}_{k+1}(V)$. This construction gives the required homeomorphism $PU_k \to \mathrm{Flag}_{k+1}(V)$. □

We can now specialise to the case where X is a point, so $A = \mathbb{Z}$ and V is necessarily constant so $f(t) = t^n$. If we also take $k = n$ and appeal to Remark 17.7 we find that

$$H^*(\mathrm{Flag}(V)) = \mathbb{Z}[x_0, \dots, x_{n-1}]/(\sigma_1, \dots, \sigma_n)$$

just as in Example 2.11.

19. Grassmannians

We now consider grassmannians. As with flag manifolds, it is useful to extend the definition to cover grassmannians of vector bundles rather than just vector spaces. We will therefore assume again that we have a hermitian vector bundle V of dimension n over a base space X, and we put

$$\mathrm{Grass}_k(V) = \{(x, W) \mid x \in X,\ W \leq V_k,\ \dim(W) = k\}.$$

This has a projection $\pi\colon \mathrm{Grass}_k(V) \to X$ given by $\pi(x, W) = x$, and we can use this to define a bundle π^*V over $\mathrm{Grass}_k(V)$, whose fibre at a point (x, W) is V_x. There is also a tautological bundle T, whose fibre at (x, W) is W. This is naturally a subbundle of π^*V, so we can also consider the quotient $(\pi^*V)/T$. We will describe the cohomology of $\mathrm{Grass}_k(V)$ in terms of the Chern classes of these bundles.

For the algebraic counterpart, we again consider a ring A equipped with a polynomial

$$f(t) = \sum_{i=0}^{n} c_i t^{n-i} \in \mathrm{Mon}_n(A),$$

and an integer k with $0 \leq k \leq n$. Put

$$\begin{aligned}
\widetilde{B}_k &= A[a_1, \dots, a_k] \\
\widetilde{B}'_k &= A[a_1, \dots, a_k, b_1, \dots, b_{n-k}] \\
a_0 &= b_0 = 1 \\
g(t) &= \sum_{i=0}^{k} a_i t^{k-i} \in \widetilde{B}_k[t] \subseteq \widetilde{B}'_k[t] \\
h(t) &= \sum_{j=0}^{n-k} b_j t^{k-j} \in \widetilde{B}'_k[t].
\end{aligned}$$

In $\widetilde{B}_k[t]$ we can write $f(t) = g(t)q(t) + r(t)$ with $\deg(r(t)) < k$, and we let J_k be the ideal generated by the coefficients of $r(t)$, and put $B_k = \widetilde{B}_k/J_k$. In $\widetilde{B}'_k$ we let J'_k be the ideal generated by the coefficients of $f(t) - g(t)h(t)$, and we put $B'_k = \widetilde{B}'_k/J_k$.

Lemma 19.1. *There is a natural isomorphism $B'_k \to B_k$.*

Proof. We write $\widetilde{\phi}$ for the evident inclusion $\widetilde{B}_k \to \widetilde{B}'_k$, and $\widetilde{\psi}$ for the ring map $\widetilde{B}'_k \to \widetilde{B}_k$ that is the identity on $\widetilde{B}_k$ and sends b_i to the coefficient of t^{n-k-i} in $q(t)$.

We can apply $\widetilde{\phi}$ to the equation $f(t) = g(t)q(t) + r(t)$ to get an equation in $\widetilde{B}'_k[t]$ and then project modulo J'_k to see that $f(t) = g(t)q(t) + r(t)$ in $B'_k[t]$. On the other hand, we also have $f(t) = g(t)h(t) + 0$ in $B'_k[t]$, by the definition of B'_k. As division by monic polynomials is unique, this means that in $B'_k[t]$ we have $q(t) = h(t)$ and $r(t) = 0$. As J_k is generated by the coefficients of $r(t)$ we see that J_k maps to zero in B'_k, so we have an induced map $\phi\colon B_k = \widetilde{B}_k/J_k \to B'_k$.

In the other direction, we note that $\widetilde{\psi}$ sends $h(t)$ to $q(t)$ and so sends $f(t) - g(t)h(t)$ to $f(t) - g(t)q(t) = r(t)$, which in turn maps to zero in $B_k[t]$. As J'_k is generated by the coefficients of $f(t) - g(t)h(t)$ we see that J'_k maps to 0 in B_k, so there is an induced map $\psi\colon B'_k \to B_k$. It is now easy to check that ϕ and ψ are inverse to each other, as required. $\square$

Theorem 19.2. *If we take $A = H^*(X)$ and $f(t) = f_V(t)$ then there is a natural isomorphism $\theta\colon B_k \to H^*(\mathrm{Grass}_k(V))$, which sends a_i to the i'th Chern class of the tautological bundle T, and b_j to the j'th Chern class of the quotient $(\pi^*W)/T$.*

Proof. To prove this, we note that there is certainly a unique map $\widetilde{\theta}\colon \widetilde{B}'_k \to H^*(\mathrm{Grass}_k(V))$ that agrees with π^* on $A = H^*(X)$ and sends a_i to $c_i(T)$ and b_j to $c_j((\pi^*V)/W)$. This means that $\widetilde{\theta}$ sends $f(t) - g(t)h(t)$ to $f_{\pi^*V}(t) - f_T(t) f_{(\pi^*V)/T}(t)$, which is zero by Proposition 16.3. As J'_k is generated by the coefficients of $f - gh$, we deduce that $\widetilde{\theta}(J'_k) = 0$, so there is an induced map $\theta\colon B'_k \to H^*(\mathrm{Grass}_k(V))$. We must show that this is an isomorphism.

Let A'_k be obtained from B'_k by adjoining a full set of roots for $g(t)$ as in Construction 17.6. This is a free module over B'_k on a basis of size $k!$ including the element 1; it will therefore be enough to show that the induced map $A'_k \to A'_k \otimes_{B'_k} H^*(\mathrm{Grass}_k(V))$ is an isomorphism. Using Theorem 18.1, we can identify $A'_k \otimes_{B'_k} H^*(\mathrm{Grass}_k(V))$ with $H^*(\mathrm{Flag}_k(T))$. A point in $\mathrm{Flag}_k(T)$ consists of a point $(x, W') \in \mathrm{Grass}_k(V)$ together with a complete flag $\underline{W}$ in the space $T_{(x,W')} = W'$. Here W_k is forced (for dimensional reasons) to be the same as W', to it is superfluous to record W' separately. This shows that $\mathrm{Flag}_k(T)$ is the same as $\mathrm{Flag}_k(V)$, so $A'_k \otimes_{B'_k} H^*(\mathrm{Grass}_k(V)) = H^*(\mathrm{Flag}_k(V))$.

On the other hand, to form A'_k we adjoined a monic factor of $f(t)$ of degree k, and then adjoined a full set of roots for that factor. It is equivalent to just adjoint a system of k roots for $f(t)$. Thus, another application of Theorem 18.1 identifies A'_k with $H^*(\mathrm{Flag}_k(V))$. One can check that all these identifications are consistent, so our map $A'_k \to A'_k \otimes_{B'_k} H^*(\mathrm{Grass}_k(V))$ is an isomorphism as required. $\square$

Let $i\colon M \to N$ be a smooth map of compact manifolds, of dimensions m and n respectively. We say that i is an *embedding* if it is injective and for each $a \in M$ the map $i_*\colon \tau_a M \to \tau_{i(a)} N$ is also injective. If so, then the spaces $\nu(i)_a := \tau_{i(a)}N/i_*\tau_a M$ fit together to form a bundle $\nu = \nu(i)$ over M of dimension $n-m$, called the *normal bundle* of i. (After choosing an inner product on $\tau(N)$ we can identify $\nu(i)_a$ with the orthogonal complement $\tau_{i(a)}N \ominus i_*\tau_a M$, which makes the name seem more reasonable.)

We next need some thoughts about the tangent bundle to the total space of a vector bundle. Let V be a smooth vector bundle of dimension d over a smooth manifold M of dimension m, so EV is itself a manifold of dimension $d+m$. We wish to understand the bundle $\tau(EV)$ over EV. Let $\pi\colon EV \to M$ be the projection, so we have bundles π^*V and π^*M over EV.

Proposition 20.1. *There is a canonical short exact sequence of bundles*

$$\pi^*V \to \tau(EV) \to \pi^*\tau(M).$$

Thus, a choice of inner product on $\tau(EV)$ gives a splitting $\tau(EV) = \pi^*(V) \oplus \pi^*\tau(M)$.

Proof. The maps $\pi_*\colon \tau(EV)_a \to \tau(M)_{\pi(a)}$ fit together to give a map $\pi_*\colon \tau(EV) \to \pi^*\tau(M)$. Next, consider a point $a \in EV$, or equivalently a pair (b, v) with $b \in M$ and $v \in V_b$. Suppose we have another vector $w \in \pi^*(V)_a = V_b$. We then have a path $\gamma\colon t \mapsto (b, v + tw)$ in EV with $\gamma(0) = a$, so we can define $\xi(w) = \gamma'(0) \in \tau(EV)_a$. This gives us a map $\xi\colon \pi^*V \to \tau(EV)$. The map $\pi \circ \gamma\colon \mathbb{R} \to M$ is constant so $\pi_* \circ \xi = 0$. All that is left is to prove that the sequence $\pi^*V \xrightarrow{\xi} \tau(EV) \xrightarrow{\pi_*} \pi^*\tau(M)$ is short exact. It will be enough to do this over a small neighbourhood of b for every point $b \in M$. We may choose a small neighbourhood W such that W is diffeomorphic to $\mathbb{R}^m$ and the bundle $V|_W$ is trivial. After replacing M by W we may assume that M is a vector space, and that V is the constant bundle with fibre U for some vector space U. Then $EV = M \times U$ is also a vector space, so the tangent space to EV at a point $a = (b, u)$ can be identified with the vector space $M \times U$. Moreover, we have $\pi^*(V)_a = V_b = U$ and $\pi^*\tau(M)_a = \tau(M)_b = M$ and $\xi(x) = (x, 0)$ and $\pi_*(x, y) = y$. It follows easily that the sequence is exact as claimed. □

Corollary 20.2. *Let $z\colon M \to EV$ be the zero-section (in other words $z(a) = (a, 0)$). Then there is a canonical isomorphism $z^*\tau(EV) = \tau(M) \oplus V$.*

Proof. Note that $\pi z = 1$, so $z^*\pi^*$ is the identity functor. The proposition thus gives a canonical short exact sequence $V \xrightarrow{\xi} z^*\tau(EV) \xrightarrow{\zeta} \tau(M)$, where $\zeta = z^*(\pi_*)$. Moreover, the map z gives rise to a map $z_*\colon \tau(M) \to z^*\tau(EV)$. Using $\pi z = 1$ again, we find that $\zeta z_* = 1\colon \tau(M) \to \tau(M)$. It now follows by general nonsense that the map

$$\begin{pmatrix} z_* \\ \xi \end{pmatrix} : \tau(M) \oplus V \to z^*\tau(EV)$$

is an isomorphism. □

We next need to define tubular neighbourhoods. A tubular neighbourhood of an embedding i as above is an open embedding $j\colon E\nu \to N$ such that $j(a, 0) = i(a)$ and j satisfies a certain differential condition to first order near the zero section, which we now explain. Suppose we have a point $a \in M$ and a vector $v \in \nu_a$. We then have a smooth path $\gamma\colon t \mapsto (a, tv)$ in $E\nu$ and thus a path $j\gamma\colon \mathbb{R} \to N$ with $j\gamma(0) = i(a)$. This gives a vector $w = (j\gamma)'(0) \in \tau_{i(a)}N$, whose image in $\nu_a = \tau_{i(a)}N/i_*\tau_a M$ we denote by $\overline{w}$. The condition is that $\overline{w} = v$ for all v.

If N is an embedded submanifold of Euclidean space, and we use an inner product to identify ν with a subbundle of $i^*\tau(N)$ then the condition just says that $j(a, v) = i(a) + v + O(\|v\|^2)$.

We say that two embeddings $j_0, j_1\colon L \to N$ are *isotopic* if there is a smooth map $k\colon \mathbb{R} \times L \to N$ such that $k(t, a) = j_t(a)$ for $t = 0, 1$ and the maps $a \mapsto k(t, a)$ are embeddings for all t.

Theorem 20.3. *Let $i\colon M \to N$ be a smooth embedding as above. Then there exists a tubular neighbourhood of i, and any two such are isotopic.* □

Definition 20.4. We say that two smooth maps $h\colon L \to N$ and $i\colon M \to N$ are *transverse* if whenever $h(a) = i(b) = c$, the map

$$(h_*, i_*)\colon \tau_a(L) \oplus \tau_b(M) \to \tau_c(N)$$

is surjective. If so, one can check that the pullback $K = L \times_N M = \{(a,b) \in L \times M \mid h(a) = i(b)\}$ is a submanifold of $L \times M$. If we let $L \xleftarrow{p} K \xrightarrow{q} M$ be the projections and write $r = hp = iq$ then there is a natural short exact sequence

$$\tau(K) \xrightarrow{\binom{p_*}{q_*}} p^*\tau(L) \oplus q^*\tau(M) \xrightarrow{(h_* \ -i_*)} r^*\tau(N).$$

We say that a map $h\colon L \to N$ is transverse to a submanifold $N \subseteq M$ if it is transverse to the inclusion map $i\colon N \to M$. Similarly, we say that two submanifolds are transverse if the two inclusion maps are transverse.

Remark 20.5. If L, M and N have dimensions l, m and n with $l + m < n$ then h and i can only be transverse if their images are disjoint, so that $L \times_N M = \emptyset$.

Example 20.6. If V is a vector space and U, W are subspaces then U is transverse to W iff $U + W = V$.

Example 20.7. The curve $t \mapsto (t, t^2 - 1)$ in $\mathbb{R}^2$ is transverse to the x-axis, but the curve $t \mapsto (t, t^2)$ is not. The curve $t \mapsto (t^2, t^3)$ (known as the cuspidal cubic) is not transverse to any curve passing through the origin.

Proposition 20.8. *Let N be a smooth n-manifold, and let V be a smooth vector bundle of dimension d over N. Let s be a section of V such that the corresponding map $s\colon N \to EV$ is transverse to the zero-section $N \subset EV$. Put $M = \{a \in N \mid s(a) = 0\} = s^{-1}N$ (which is a smooth $(m-d)$-manifold), and let $i\colon M \to N$ be the inclusion. Then the normal bundle $\nu = \nu(i)$ is canonically isomorphic to $V|_M$.*

Proof. Let $z\colon N \to EV$ be the zero-section. We then have maps $z_*\colon \tau(N) \to z^*\tau(EV)$ and $s_*\colon \tau(N) \to s^*\tau(EV)$. Note that $s|_M = z|_M$ so $(s^*\tau(EV))|_M = (z^*\tau(EV))|_M$ so we can form the map $s_* - z_*\colon \tau(N)|_M \to (z^*\tau(EV))|_M$. After noting that $\pi s = \pi z = 1\colon N \to N$, we find that $\pi_*(s_* - z_*) = 0$. Proposition 20.1 thus tells us that there is a unique map $\sigma\colon \tau(M)|_M \to V|_M$ such that $s_* - z_* = \xi\sigma$. As $si = zi$ we see that $\sigma i_* = 0$ so we get an induced map $\sigma'\colon \nu = \tau(N)|_M/\tau(M) \to V|_M$. We claim that this is an isomorphism; this can be checked by reducing to the case where N is a vector space and V is trivial, just as in the proof of Proposition 20.1. □

21. The Pontrjagin-Thom construction

Let $i\colon M \to N$ be an embedding of smooth compact manifolds, with normal bundle ν. Let $j\colon E\nu \to N$ be a tubular neighbourhood. Note that M^ν is the one-point compactification $E\nu \cup \{\infty\}$, so we can define a map $q\colon N \to M^\nu$ by sending $j(a)$ to a and everything not in the image of j to ∞. For various reasons it is convenient to consider the space N_+ obtained from N by adding a disjoint basepoint, and extend q by sending the basepoint to ∞. This gives a based map $q\colon N_+ \to M^\nu$, which we call the *Pontrjagin-Thom collapse map.*

Now suppose that M and N are oriented, with orientations $u \in \widetilde{H}^m(M^\tau)$ and $v \in \widetilde{H}^n(N^\tau)$ say. Recall that a choice of inner product gives a splitting $i^*\tau(M) = \tau(N) \oplus \nu$, so we can multiply an orientation of ν by v (as in Remark 13.10) to get an orientation of $i^*\tau(M)$. It is not hard to check that there is a unique orientation w of ν such that $vw = i^*(u)$, and we will always implicitly use this orientation. We then have $\widetilde{H}^*(M^\nu) = H^{*-n+m}(M)w$ and this allows us to define a map $i_!\colon H^k(M) \to H^{k+n-m}(N)$ by $i_!(x) = q^*(xw)$.

Theorem 21.1.
(a) *If i is a diffeomorphism then $i_! = (i^*)^{-1}$.*
(b) *We have $(ji)_! = j_! i_!$ whenever this makes sense.*
(c) *If we have a pullback square as shown below in which i is a smooth embedding and transverse to g and everything is suitably oriented, then we have $g^* i_! = j_! f^*\colon H^*M \to H^*L$.*
(d) *If we use the ring map $i^*\colon H^*N \to H^*M$ to regard H^*M as a module over H^*N, then the map $i_!\colon H^*M \to H^{*+n-m}N$ is a module map.*
(e) *If $i\colon M \to N$ is as in Proposition 20.8, then we have $i_!(1) = e(V) \in H^d(N)$.* □

22. Poincaré duality

Let M be an oriented manifold of dimension n. Then it turns out that there is a natural isomorphism $D\colon H_k(M) \to H^{n-k}(M)$. To define D and study its properties we need two new kinds of products in (co)homology. Let $C_*(X)$ be the singular chain complex of M. Then $C^*(X \times Y)$ is chain homotopy equivalent to $\mathrm{Hom}(C_*(X) \otimes C_*(Y), \mathbb{Z})$. Given an element $a \in C_k(X)$ and a map

$$u\colon (C_*(X) \otimes C_*(Y))_n = \bigoplus_{n=i+j} C_i(X) \otimes C_j(Y) \to \mathbb{Z},$$

we can define a map $a \backslash u\colon C_{n-k}(Y) \to \mathbb{Z}$ by $(a \backslash u)(b) = (-1)^{nk} u(a \otimes b)$. One can checks that this construction is compatible with (co)boundary maps and thus induces a map

$$H_k(X) \otimes H^n(X \times Y) \to H^{n-k}(Y),$$

which we again write as $a \otimes u \mapsto a \setminus u$.

Next, suppose that $a \in C_k(X)$ and $v \in C^m(X)$. We have a diagonal map $\delta\colon C_*(X) \to C_*(X \times X)$, and the right hand side is chain homotopy equivalent to $C_*(X) \otimes C_*(X)$, and we can regard v as a map $C_m(X) \to \mathbb{Z}$. We then have an element $(v \otimes 1)(\delta_*(a)) \in C_{k-m}(X)$, which we denote by va (or sometimes $v \cap a$). One can again check that this construction is compatible with (co)boundary maps and thus induces a map

$$H^m(X) \otimes H_k(X) \to H_{k-m}(X),$$

which we just denote by $v \otimes a \mapsto va$. This makes $H_*(X)$ into a module over $H^*(X)$.

Definition 22.1. Let M be an oriented smooth compact n-dimensional manifold. Let $\delta\colon M \to M^2$ be the diagonal embedding, so ν_δ is isomorphic to the tangent bundle of M and thus is oriented. Define $D\colon H_k(M) \to H^{n-k}(M)$ by $D(a) = a \setminus \delta_!(1)$.

Theorem 22.2. *The map D is an isomorphism of modules over $H^*(M)$.* □

Lemma 22.3. *If $a \in M$ and $i\colon 1 \to M$ is the inclusion of $\{a\}$ and we also let a denote the corresponding generator in H_0M, then $D(a) = i_!(1)$.*

Proof. One sees from the definitions that $D(a) = a \setminus \delta_!(1) = (i \times 1)^* \delta_!(1)$. Now observe that the following square is a transverse pullback, and use the Mackey property.

$$\begin{tikzcd} 1 \arrow[r, "i"] \arrow[d, "i"'] & M \arrow[d, "\delta"] \\ M \arrow[r, "i\times 1"'] & M \times M \end{tikzcd}$$

□

23. The Universal Coefficient Theorem

The natural map $\pi\colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ gives rise to a map $\pi_*\colon \mathrm{Hom}(A, \mathbb{Q}) \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ (for all abelian groups A). We define

$$\mathrm{Ext}(A, \mathbb{Z}) := \mathrm{cok}(\pi_*\colon \mathrm{Hom}(A, \mathbb{Q}) \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})).$$

(One can define $\mathrm{Ext}(A, B)$ for any abelian group B but we shall not do so here.)

Remark 23.1. It is easy to compute $\mathrm{Ext}(A, \mathbb{Z})$ using the following observations.

(1) There is a natural isomorphism $\mathrm{Ext}(A \oplus B, \mathbb{Z}) = \mathrm{Ext}(A, \mathbb{Z}) \oplus \mathrm{Ext}(B, \mathbb{Z})$.
(2) If $F = \mathbb{Z}\{e_\alpha\}$ is a free abelian group then $\mathrm{Hom}(F, B) = \prod_\alpha B$ and it is easy to deduce that $\mathrm{Ext}(F, \mathbb{Z}) = 0$
(3) If T is a torsion group (ie for all $a \in T$ there exists $n > 0$ such that $na = 0$) then $\mathrm{Hom}(T, \mathbb{Q}) = 0$ and so $\mathrm{Ext}(T, \mathbb{Z}) = \mathrm{Hom}(T, \mathbb{Q}/\mathbb{Z})$.
(4) If A is a finitely generated abelian group and tA is the torsion subgroup, one can deduce that $\mathrm{Ext}(A, \mathbb{Z}) = \mathrm{Hom}(tA, \mathbb{Q}/\mathbb{Z})$.
(5) If $A = \mathbb{Z}/n$ and $\alpha\colon A \to \mathbb{Q}/\mathbb{Z}$ is defined by $\alpha(k \bmod n\mathbb{Z}) = k/n \bmod \mathbb{Z}$, then $\mathrm{Ext}(A, \mathbb{Z}) \simeq \mathbb{Z}/n$ generated by α.

Theorem 23.2 (Universal coefficients). *Let X be a space such that H_kX is a finitely generated abelian group for all k. Then there is a natural pairing $H_kX \otimes H^kX \to \mathbb{Z}$, which gives rise to surjective maps $H^kX \to \mathrm{Hom}(H_kX, \mathbb{Z})$ and $H_kX \to \mathrm{Hom}(H^kX, \mathbb{Z})$. These fit into short exact sequences as follows:*

$$\mathrm{Ext}(H_{k-1}X, \mathbb{Z}) \to H^kX \to \mathrm{Hom}(H_kX, \mathbb{Z})$$

$$\mathrm{Ext}(H^{k+1}X, \mathbb{Z}) \to H_kX \to \mathrm{Hom}(H^kX, \mathbb{Z}).$$

24. Morse Theory

Let M be a compact smooth manifold, and let $f\colon M \to \mathbb{R}$ be a smooth function. We say that a point $a \in M$ is a *critical point* of f if $d_a(f) = 0$, and that $t \in \mathbb{R}$ is a *critical value* of f if there exists a critical point a with $f(a) = t$. Let a be a critical point of f, and let u and v be tangent vectors at a. Let $\tilde{u}$ and $\tilde{v}$ be smooth vector fields defined on a neighbourhood of a that agree at a with u and v respectively. One checks that the number $(\tilde{u}(\tilde{v}(f)))(a)$ does not depend on the choice of $\tilde{u}$ and $\tilde{v}$, so we can define $H(f, a)(u, v) = (\tilde{u}(\tilde{v}(f))(a)$. One can also check that $H(f, a)$ is a symmetric bilinear form on $\tau_a(M)$, which we call the *Hessian*. In terms of a system of local coordinates with origin at a, the matrix of $H(f, a)$ is just the matrix of partial derivatives $\partial^2(f)/\partial x_i \partial x_j$ evaluated at 0. We say that a is a *nondegenerate* critical point if the bilinear form $H(f, a)$ is nondegenerate. If so, we define the *index* of a to be the maximum possible dimension of a subspace $U \leq \tau_a(M)$ on which the form $H(f, a)$ is negative definite.

We say that f is a *Morse function* if all its critical points are nondegenerate, and if a, b are distinct critical points then $f(a) \neq f(b)$. In a suitable sense, almost all smooth functions are Morse functions. Because we assume that M is compact, it turns out that a Morse function has only finitely many critical points. It is a remarkable fact that the critical points of a Morse function and their indices carry a great deal of information about the topology of M.

To exploit this information, it is convenient to choose a Riemannian structure on M, in other words a smooth inner product on the tangent bundle, which gives rise to an isomorphism between the tangent bundle and the cotangent bundle. The function f gives rise to a section *df* of the cotangent bundle, which corresponds under our isomorphism to a section ∇f of the tangent bundle, or in other words a vector field on M. If a is a critical point then there is a unique endomorphism H' of $\tau_a(M)$ such that $H(f, a)(u, v) = \langle u, H'v \rangle$. This is easily seen to be self-adjoint, so the eigenvalues are real. As a is assumed to be nondegenerate, none of the eigenvalues are zero. One finds that the index of the critical point is the number of negative eigenvalues, counted with multiplicity.

Lemma 24.1. *Let M be smooth m-manifold, and let $f\colon M \to \mathbb{R}$ be a smooth function, and let a be a nondegenerate critical point of index d. Then one can choose a system of coordinates $x_1, \dots, x_n$ on some neighbourhood U of a such that $x_i(a) = 0$ for all i and $f = -\sum_{i=1}^d x_i^2 + \sum_{i=d+1}^n x_i^2$.*

Theorem 24.2. *Let M be a compact smooth manifold, and let f be a Morse function on M. For $t \in \mathbb{R}$ we put $M_t = \{a \in M \mid f(a) \leq t\}$. If $r < s$ and there are no critical values in the interval $[r, s]$ then the inclusion $M_r \to M_s$ is a homotopy equivalence. If r and s are not critical values and there is a unique critical point $a \in M$ such that $f(a) \in (r, s)$ then we have*

$$H^k(M_s, M_r) = \begin{cases} k = \mathrm{index}(a) & \mathbb{Z} \\ k \neq \mathrm{index}(a) & 0 \end{cases}.$$

The main tool used to prove this theorem is the idea of the flow along a vector field. Let M be a compact smooth manifold. A *(global) flow* on M is a smooth map $\phi\colon \mathbb{R} \times M \to M$ with the property that $\phi(0, a) = a$ and $\phi(s, \phi(t, a)) = \phi(s + t, a)$. Given such a flow, and a point $a \in M$, we get a curve $\gamma(t) = \phi(t, a)$ in M and thus a tangent vector $v_a = \gamma'(0) \in \tau_a(M)$. These vectors fit together to give a smooth vector field v on M. It can be shown that the construction $\phi \mapsto v$ is bijective, so that every vector field has a unique associated flow. (This relies on compactness; on a noncompact manifold, one has instead a partially-defined flow, whose domain is some neighbourhood of $0 \times M$ in $\mathbb{R} \times M$.) The maps and homotopies required to prove the theorem are produced using the flows associated to the vector field ∇f and various modifications of it.

Let V be a complex vector space of dimension m, and let $\phi\colon V \to \mathbb{C}$ be a homogeneous polynomial function of degree d. (This means that there exists a symmetric d-linear map $\psi\colon V^{\otimes d} \to \mathbb{C}$ such that $\phi(v) = \psi(v^{\otimes d})$ for all v, or equivalently that for any basis $\{e_i\}$ of V, the function $f(z_1, \dots, z_m) = \phi(\sum_i z_i e_i)$ is a homogeneous polynomial of degree d in the variables z_i.) Define

$$M(\phi) = \{L \in PV \mid \phi(L) = 0\};$$

this is called a *complex hypersurface of degree* d in PV.

We next give a criterion for when M is a manifold. Recall that the derivative $\phi'\colon V \to V^*$ is characterised by the equation

$$\phi(v + \epsilon w) = \phi(v) + \epsilon\phi'(v)(w) + O(\epsilon^2).$$

This has a special feature in our case: because ϕ is homogeneous of degree d, we have

$$\phi(v + \epsilon v) = (1 + \epsilon)^d \phi(v) = \phi(v) + d\epsilon\phi(v) + O(\epsilon^2),$$

so $\phi'(v)(v) = d\phi(v)$. One can also check that $\phi'(\lambda v) = \lambda^{d-1}\phi'(v)$. It follows that the restriction of the map $\phi'\colon V \to V^*$ to any line $L \leq V$ can be regarded as a linear map $L^{\otimes(d-1)} \to V^*$, whose dual is a linear map $\xi_L\colon V \to L^{\otimes(1-d)}$.

Definition 25.1. We say that ϕ is *regular* if $\phi'(v) \neq 0$ whenever $v \neq 0$. If so, we say that $M(\phi)$ is a *regular hypersurface of degree* d.

Proposition 25.2. *Regular hypersurfaces are submanifolds of* PV*.*

Proof. Let ϕ be a regular polynomial.

Suppose that $u \neq 0$, choose a subspace $W < V$ such that $V = \mathbb{C}u \oplus W$, and define a map $\theta\colon W \to PV$ by $\theta(w) = [u + w]$. It will suffice to check that $\theta^{-1}(M(\phi))$ is a submanifold of W. Define $\chi(w) = \phi(u + w)$, so $\theta^{-1}(M(\phi)) = \{w \mid \chi(w) = 0\}$ and χ' is a non-homogeneous polynomial map from W to W^*. By a well-known argument, it is enough to show that $\chi'(w) \neq 0$ whenever $\chi(w) = 0$. Suppose that $\chi(w) = 0$ and put $v = u + w$ so $\phi(v) = 0$, so $\phi'(v)(v) = d\phi(v) = 0$. It is easy to see that $V = \mathbb{C}v \oplus W$ and $\phi'(v) \neq 0$ by assumption so we must have $\phi'(v)(x) \neq 0$ for some $x \in W$. We also have $\chi'(w)(x) = \phi'(v)(x)$, so $\chi'(w) \neq 0$ as required. □

Remark 25.3. Conversely, if $\dim(V) > 2$ and $M(\phi)$ is a submanifold of PV, one can show that $\phi = \psi^k$ for some regular polynomial ψ and some integer $k > 0$; clearly in this case $M(\phi) = M(\psi)$. If $\dim(V) = 2$ then all hypersurfaces are just finite sets of points and thus are submanifolds.

Proposition 25.4. *All regular hypersurfaces of the same degree in* PV *are diffeomorphic.*

Sketch. Let Φ be the vector space of degee d homogeneous polynomial maps $\phi\colon V \to \mathbb{C}$, and put $R = \{[\phi] \in P\Phi \mid \phi \text{ is regular }\}$ and $S = (P\Phi)\setminus R$. Let N be the complex dimension of Φ, sp $P\Phi$ is a $2(N-1)$-dimensional space. One can check that S is a proper closed subvariety of $P\Phi$ and thus that it has complex dimension at most $N-2$ and real dimension at most $2(N-2)$. Equivalently, S has real codimension at least two. This means that if we have a smooth curve in $P\Phi$, then after wiggling it slightly we can assume that it does not touch S, and one can deduce from this that R is path-connected. Now put

$$E = \{([v], [\phi]) \in PV \times R \mid \phi(v) = 0\} = \coprod_{[\phi] \in R} M(\phi)$$

and let $\pi\colon E \to R$ be the projection. One checks that π is proper (in other words, the preimage of a compact set is compact) and a submersion (in other words, the map $\pi_*\colon \tau(E) \to \pi^*\tau(R)$ is surjective). The *Ehresmann fibration theorem* says that a proper submersion is a fibre bundle, which means in particular that for any $[\phi] \in R$ there is a neighbourhood U of $[\phi]$ such that $M(\psi)$ is diffeomorphic to $M(\phi)$ whenever $[\psi] \in U$. Thus, the diffeomorphism type of $M(\phi)$ is a locally constant function of $[\phi]$, and thus globally constant because R is path-connected. □

We next turn to the cohomology of regular hypersurfaces. Define $\phi\colon \mathbb{C}^{2n+1} \to \mathbb{C}$ by $\phi(z) = \sum_{i=0}^{2n} z_i^d$. One checks easily that this is a regular function, defining a smooth hypersurface $M \subset \mathbb{C}P^{2n}$. (We will not discuss the case of hypersurfaces in $\mathbb{C}P^m$ where m is odd, because they are considerably harder.) Let j be the inclusion map $M \to \mathbb{C}P^{2n}$. We can also define a map $i\colon \mathbb{C}P^{n-1} \to M$ by

$$i[w_0 : \ldots : w_{n-1}] = [w_0 : \xi w_0 : \ldots : w_{n-1} : \xi w_{n-1} : 0],$$

where $\xi = e^{\pi i/d}$. We have the usual generators $x \in H^2\mathbb{C}P^{2n}$ and $x \in H^2\mathbb{C}P^{n-1}$, and the composite $ji\colon \mathbb{C}P^{n-1} \to \mathbb{C}P^{2n}$ comes from a linear injection $\mathbb{C}^n \to \mathbb{C}^{2n+1}$ so $(ji)^*x = x$. It should therefore cause no confusion to write x also for the element $j^*x \in H^2M$. We get one more generator by defining $y = i_!(1) \in H^{2n}M$.

Theorem 25.5. *We have $H^*M = \mathbb{Z}[x,y]/(x^n - dy, y^2)$.*

Proof. The Lefschetz Hyperplane Theorem shows that $H^kM = H^k\mathbb{C}P^{2n}$ and $H_kM = H_k\mathbb{C}P^{2n}$ for $k < 2n = \frac{1}{2}\dim_{\mathbb{R}}(\mathbb{C}P^{2n})$. Thus, for $j < n$ we have $H^{2j}M = \mathbb{Z}\{x^j\}$ and $H^{2j+1}M = 0$. By the Universal Coefficient Theorem, there are unique elements $b_j \in H_{2j}M$ for $j < n$ such that $\langle b_j, x^j\rangle = 1$. It is not hard to check that $xb_j = b_{j-1}$. The Poincaré duality isomorphism $D\colon H_k(M) \to H^{4n-2-k}(M)$ now tells us that $H^{2j}M = \mathbb{Z}\{D(b_{2n-1-j})\}$ for $n \le j < 2n$ and that the remaining odd-dimensional cohomology groups of M also vanish. If we put $z = D(b_{n-1}) \in H^{2n}M$ and recall that $x^kD(b_m) = D(x^kb_m) = D(b_{m-k})$ we find that $\{x^j \mid j < n\} \amalg \{x^jz \mid j < n\}$ is a basis for H^*M over $\mathbb{Z}$. It is clear that $z^2 = 0$ for dimensional reasons. All that is left is to prove that $y = z$ and that $x^n = dy$.

To do this, first note that y spans $H^{2n}M$ so $z = ry$ and $x^n = sy$ for some integers r and s; we need to check that $r = 1$ and $s = d$.

Let k denote the inclusion of a point in $\mathbb{C}P^{n-1}$. Using Lemma 22.3 we see that $k_!(1) = x^{n-1}$ and thus that $x^{n-1}y = x^{n-1}i_!(1) = i_!(x^{n-1}) = i_!k_!(1)$. As ik is the inclusion of a point in M we have $i_!k_!(1) = D(b_0) = D(x^{n-1}b_{n-1}) = x^{n-1}z$. Multiplication by x^{n-1} gives an isomorphism $H^{2n}M \to H^{4n-2}M$, so we conclude that $r = 1$ and $z = y$ as claimed.

Next, we see from part (e) of Theorem 21.1 that $j_!(1) = e(L^{-d}) = dx$, so

$$x^{n-1}j_!(x^n) = j_!(x^{2n-1}) = dx^{2n} \in H^{4n}\mathbb{C}P^{2n}.$$

On the other hand, we have

$$x^{n-1}j_!(y) = x^{n-1}j_!i_!(1) = j_!i_!(x^{n-1}) = j_!i_!k_!(1) = x^{2n}.$$

As $y = sx^n$ we see that $dx^{2n} = sx^{2n}$ so $d = s$ as required. $\square$