Ambidexterity 4

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May 26, 2023

K-local spectra

- Recall that Z is *K-acyclic* if $K_*Z = 0$, and X is *K-local* if $[Z, X] = 0$ for all K-acyclic Z (e.g. K is K-local)
- We write $\mathcal{K}$ for the ∞-category of K-local spectra.
- There is a functorial cofibration $CX \to X \to LX$ where CX is K-acyclic and LX is K-local.
- For objects: $LX = 0$ iff $K_*(X) = 0$. For morphisms: Lf iso iff $K_*(f)$ iso.
- The subcategory $\mathcal{K} \subseteq \mathcal{S}$ is closed under limits. To get colimits in $\mathcal{K}$, construct colimits in $\mathcal{S}$ and apply L.
- In MP_0: put $u_i =$ coeff of x^{p^i} in $[p]_{MP}(x)$ and $I_n = (u_0, \dots, u_{n-1})$.
- For an MP-module M we have $LM = (u_n^{-1}M)^{\wedge}_{I_n}$. Thus E is K-local.
- Deep Theorem: $\mathcal{K} = \text{thick}\langle LM \mid M \in \text{Mod}_{MP}\rangle$
- Deep Theorem: There is a finite spectrum F such that $MP_*F = MP_*/(u_0^{i_0}, \dots, u_{n-1}^{i_{n-1}})$ for some $i_0, \dots, i_{n-1}$.
- Put $p_G(X) = \text{cof}(c_!(X) \to c_*(X)) = \text{cof}(L(X_{hG}) \to X^{hG})$ so $p_G(K) = 0$.
- If $M \in \text{Mod}_K$ then $p_G(M) \in \text{Mod}_{p_G(K)}$ so $p_G(M) = 0$.
- If $M \in \text{Mod}_{MP}$ then $L(F \wedge M) \in \text{thick}\langle \text{Mod}_K\rangle$ so $p_G(L(F \wedge M)) = 0$ so $F \wedge p_G(LM) = 0$ so $p_G(LM) = 0$.
- Thus, for $X \in \mathcal{K} = \text{thick}\langle L\,\text{Mod}_{MP}\rangle$ we have $p_G(X) = 0$ i.e. $c_!(X) = c_*(X)$.

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K-local spectra

- Recall that Z is *K-acyclic* if $K_*Z = 0$, and
  X is *K-local* if $[Z, X] = 0$ for all K-acyclic Z (e.g. K is K-local)
- We write $\mathcal{K}$ for the ∞-category of K-local spectra.
- There is a functorial cofibration $CX \to X \to LX$
  where CX is K-acyclic and LX is K-local.
- For objects: $LX = 0$ iff $K_*(X) = 0$. For morphisms: Lf iso iff $K_*(f)$ iso.
- The subcategory $\mathcal{K} \subseteq \mathcal{S}$ is closed under limits. To get colimits in $\mathcal{K}$, construct colimits in $\mathcal{S}$ and apply L.
- In MP_0: put $u_i =$ coeff of x^{p^i} in $[p]_{MP}(x)$ and $I_n = (u_0, \dots, u_{n-1})$.
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- For a finite groupoid G put $M(G) = \mathbb{Q}\{\pi_0(G)\}$ and $M^*(G) = \mathrm{Hom}(M(G), \mathbb{Q}) = \mathrm{Map}(\pi_0(G), \mathbb{Q})$.
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Classical characters

- For finite G, put $R(G) = \pi_0(\mathcal{V}_G) - \pi_0(\mathcal{V}_G)$ (the *representation ring*).
- A functor $q\colon G \to H$ gives $q_!\colon \mathcal{V}_G \rightleftarrows \mathcal{V}_H\colon q^*$ and then $q_!\colon R(G) \rightleftarrows R(H)\colon q^*$.
- Define $\theta = c_! = c_*\colon R(G) \to R(1) = \mathbb{Z}$ (so for a group we have $\theta([U]) = \dim_{\mathbb{C}}(U^G)$).
- This gives a perfect pairing $\langle u, v\rangle_G = \theta(uv)$ on $R(G)$.
- This has $\langle q_!(u), v\rangle_H = \langle u, q^*(v)\rangle_G$.
- Let L be the subfield of $\mathbb{C}$ generated by all roots of unity.
- If $\alpha \in \mathrm{Aut}(V)$ with $\alpha^m = 1$ then all eigenvalues lie in L so $\mathrm{trace}(\alpha) \in L$.
- Put $\Lambda G = [\mathbb{Z}, G]$ so $\mathrm{obj}(\Lambda G) = \{(a, u) \mid a \in \mathrm{obj}(G),\ u \in G(a, a)\}$ and $(a, u) \simeq (a', u')$ iff there exists $g \in G(a, a')$ with $u' = gug^{-1}$.
- Define $C(G) = L \otimes M^*(\Lambda(G)) = \mathrm{Map}(\pi_0(\Lambda G), L)$, so we have $\theta\colon C(G) \to L$ and $q_!\colon C(G) \rightleftarrows C(H)\colon q^*$ for $q\colon G \to H$.
- For a representation $V\colon G \to \mathcal{V}$ define $\chi(V) \in C(G)$ by $\chi(V)([a, u]) = \mathrm{trace}(u_*\colon V_a \to V_a)$.
- This gives an isomorphism $\chi\colon L \otimes R(G) \to C(G)$, compatible with all structure.

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- Then $[p^k]_E(x) = g_k(x)h_k(x)$, where $h_k(x) \in E^0[\![x]\!]^\times$ and $g_k(x) \in E^0[x]$ is a monic polynomial of degree p^{nk} and $E^0(BC_{p^k}) = E^0[x]/g_k(x)$.
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- Thus any $u \colon \Theta^*/p^k \to G$ gives $\phi_k \circ E^0(Bu) \colon E^0BG \to L$. Assembling these gives $\chi \colon L \otimes_{E^0} E^0(BG) \to C(G)$.
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Proof of the generalised character theorem

- Reduce to the case of a finite group G.
- Generalise: for a finite G-CW complex Z, we have

$$\chi_{G,Z} \colon L \otimes_{E^0} E^*(Z_{hG}) \to L \otimes_{\mathbb{Q}} \left(\prod_{\theta \colon \Theta^* \to G} H^*(Z^{\text{image}(\theta)}; \mathbb{Q}) \right)^G$$

- Prove by calculation that $\theta_{G,Z}$ is iso when $Z = G/A$ with $A \leq G$ abelian. (Here $Z_{hG} = BA$, and $Z^{\text{image}(\theta)}$ is Z (if $\text{image}(\theta) \leq A$) or $\emptyset$ (otherwise).)
- Deduce by Mayer-Vietoris that $\chi_{G,Z}$ is iso if Z has abelian isotropy.
- Let F be the space of complete flags in $\mathbb{C}[G]$, so $Z \times F$ and $Z \times F^2$ have abelian isotropy, and we have an equaliser $E^*(Z_{hG}) \to E^*((Z \times F)_{hG}) \rightrightarrows E^*((Z \times F^2)_{hG})$. Deduce the general case from this.
- Corollary: $\mathbb{Q} \otimes E^0(BG) = u_0^{-1} E^0(BG) \simeq \left(\prod_A \mathbb{Q} \otimes D_A \right)^G$. Here A runs over abelian subgroups $A < G$, and D_A is a certain regular local ring, free of finite rank as an E^0-module.
- Recall $\text{spf}(E^0(BA)) = \text{Hom}(A^*, \mathbb{G})$; morally $\text{spf}(E^0(D_A)) = \text{Inj}(A^*, \mathbb{G})$.
- There is a similar map $u_k^{-1} E^0(BG)/I_k \to \left(\prod_A u_k^{-1} D_{k,A} \right)^G$ for $k > 0$, which is an F-isomorphism (Greenlees-Strickland; see also Stapleton).

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- Generalise: for a finite G-CW complex Z, we have

$$\chi_{G,Z} \colon L \otimes_{E^0} E^*(Z_{hG}) \to L \otimes_{\mathbb{Q}} \left(\prod_{\theta \colon \Theta^* \to G} H^*(Z^{\mathrm{image}(\theta)}; \mathbb{Q}) \right)^G$$

- Prove by calculation that $\theta_{G,Z}$ is iso when $Z = G/A$ with $A \leq G$ abelian. (Here $Z_{hG} = BA$, and $Z^{\mathrm{image}(\theta)}$ is Z (if $\mathrm{image}(\theta) \leq A$) or $\emptyset$ (otherwise).)
- Deduce by Mayer-Vietoris that $\chi_{G,Z}$ is iso if Z has abelian isotropy.
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- Suppose for simplicity that $E^1(BG) = 0$ and $E^0(BG)$ is free over E^0, say with basis $e_1, \dots, e_r$.
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- For a space X, put $\Lambda_0 X = \varinjlim_m [B(\Theta^*/p^m)_+, X]$, so $\Lambda_0 BG = \pi_0(\Lambda G)$ for finite groupoids G.
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General ambidexterity

- Theorem (Hopkins-Lurie): any $q\colon X \to Y$ of finite ∞-groupoids/π-finite spaces is ambidextrous, i.e. $q_! \simeq q_*$ as functors $\mathcal{K}(X) \to \mathcal{K}(Y)$.
- There exists m such that all fibres $(q \downarrow b)$ have $\pi_k = 0$ for all $k > m$. Greenlees-Sadofsky gives $m \leq 1$; do $m > 1$ by induction.
- Key case: $c\colon B^m C_p \to 1$ is ambidextrous.
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