Ambidexterity 3

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May 26, 2023

Equivariant spectra

- For a finite groupoid G, put $\mathcal{S}_G = [G, \mathcal{S}]$.
- Any $V\colon G \to \mathcal{L}$ (i.e. $V \in \mathcal{L}_G$) gives $S^V, S^{-V} \in \mathcal{S}_G$.
- Define $\pi_n^G(X) = \varinjlim_{V \in \mathcal{L}_G} [S^V, (S^{-n} \wedge X)(V)]^G$.
- Say $f\colon X \to Y$ is a *genuine equivalence* if $\pi_*^H(f)$ is iso for all subgroupoids $H \leq G$.
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- Two ∞-categories: $\mathcal{S}_G$ has Borel equivalences inverted, $\mathcal{GS}_G$ has genuine equivalences inverted.
- For $X \in \mathcal{S}_G$ and $a, b \in \mathrm{obj}(G)$ we have $G(a,b)_+ \to \mathcal{S}(X_a, X_b)$ and correspondingly $\Sigma^\infty G(a,b)_+ \wedge X_a \to X_b$ and $\Sigma^\infty G(a,b)_+ \to F(X_a, X_b)$.
- The map $p\colon G(a,b) \to 1$ gives a transfer $p^t\colon S^0 \to \Sigma^\infty G(a,b)_+$ in $h\mathcal{S}$.
- Combining these gives a map $S^0 \to F(X_a, X_b)$ or $\nu_{ab}\colon X_a \to X_b$, which is essentially $\sum_{u \in G(a,b)} u_*$.
- These induce $\nu\colon c_!(X) \to c_*(X)$ in $h\mathcal{S}$; iso if G is empty or contractible.
- Now suppose we have $q\colon G \to H$. By applying the above to comma categories, we get $\nu\colon q_!(X) \to q_*(X)$ in $h\mathcal{S}_H$, and this is iso if q is faithful (*Wirthmüller isomorphism*).

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- Say $f\colon X \to Y$ is a *Borel equivalence* if $\pi_*(f_a)\colon \pi_*(X_a) \to \pi_*(Y_a)$ is iso for all $a \in \mathrm{obj}(G)$ (or equivalently, $\pi_*^H(f)$ is iso for all discrete subgroupoids $H \leq G$).
- Two ∞-categories: $\mathcal{S}_G$ has Borel equivalences inverted, $\mathcal{G}\mathcal{S}_G$ has genuine equivalences inverted.
- For $X \in \mathcal{S}_G$ and $a, b \in \mathrm{obj}(G)$ we have $G(a,b)_+ \to \mathcal{S}(X_a, X_b)$ and correspondingly $\Sigma^\infty G(a,b)_+ \wedge X_a \to X_b$ and $\Sigma^\infty G(a,b)_+ \to F(X_a, X_b)$.
- The map $p\colon G(a,b) \to 1$ gives a transfer $p^t\colon S^0 \to \Sigma^\infty G(a,b)_+$ in $h\mathcal{S}$.
- Combining these gives a map $S^0 \to F(X_a, X_b)$ or $\nu_{ab}\colon X_a \to X_b$, which is essentially $\sum_{u \in G(a,b)} u_*$.
- These induce $\nu\colon c_!(X) \to c_*(X)$ in $h\mathcal{S}$; iso if G is empty or contractible.
- Now suppose we have $q\colon G \to H$. By applying the above to comma categories, we get $\nu\colon q_!(X) \to q_*(X)$ in $h\mathcal{S}_H$, and this is iso if q is faithful (*Wirthmüller isomorphism*).

Equivariant spectra

- For a finite groupoid G, put $\mathcal{S}_G = [G, \mathcal{S}]$.
- Any $V\colon G \to \mathcal{L}$ (i.e. $V \in \mathcal{L}_G$) gives $S^V, S^{-V} \in \mathcal{S}_G$.
- Define $\pi_n^G(X) = \varinjlim_{V \in \mathcal{L}_G} [S^V, (S^{-n} \wedge X)(V)]^G$.
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- For a complex inner product space V put $MP'(V) = \{(u, U) \mid u \in U \leq V \oplus V\} \cup \{\infty\}$
- Define an orthogonal ring spectrum MP by $MP(V) = \Omega^{iV} MP'(V \oplus iV)$.
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- Theorem (Quillen): $\pi_1(MP) = 0$ and $\pi_0(MP) = \mathbb{Z}[a_1, a_2, \dots]$ with $a_1 = a_{11},\ a_2 = a_{12},\ a_3 = a_{22} - a_{13},\ a_4 = a_{15},\ a_5 = a_{16} + a_{25} + a_{34}, \dots$
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- For a complex inner product space V put $MP'(V) = \{(u, U) \mid u \in U \leq V \oplus V\} \cup \{\infty\}$
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Morava K and E of BC_m

- Put $U = \mathbb{C}[t] \setminus \{0\}$ so $\mathbb{C}P^\infty = U/\mathbb{C}^\times$
- Define $\phi_m \colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ by $\phi_m([f]) = [f^m]$, so $\phi_m^*(x) = [m]_{F_{MP}}(x) = [m]_{MP}(x) \in MP^0(\mathbb{C}P^\infty)$ (and same for E, K).
- The map $h(s, f)(t) = s + (1 - s)(1 + st)f(t)$ gives a contraction of U.
- Put $C_m = \langle e^{2\pi i/m} \rangle < \mathbb{C}^\times$ and $BC_m = U/C_m$.
- $\mathbb{C}P^\infty$ has a tautological bundle T with $T_{[f]} = \mathbb{C}f$ and $\phi_m^*(T) \simeq T^{\otimes m}$.
- Then $BC_m = E(T^{\otimes m}) \setminus (\text{zero section})$ so $\text{cofibre}(BC_m \to \mathbb{C}P^\infty) = \text{Thom}(T^{\otimes m})$
- Using the Thom isomorphism we get $MP^0(BC_m) = MP^0[\![x]\!]/[m]_{MP}(x)$ and $MP^1(BC_m) = 0$ (and same for E, K).
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- Claim: if G is a finite groupoid and $X\colon G \to \mathcal{T}$ is a finite G-CW complex then $E^*(c_!(X))$ is finitely generated over E^*.
- Can reduce to the case of a group, where $c_!(X) = X_{hG} = (EG \times X)/G$ and $(G/H)_{hG} = EG/H = BH$.
- The CW structure gives $\text{skel}_d(X)/\text{skel}_{d-1}(X) \simeq \Sigma^d \bigvee_i (G/H_i)_+$; so enough to prove $E^*(BH)$ is finitely generated for all isotropy groups H.
- So if all isotropy groups for X are abelian, then $E^*(X_{hG})$ is fg.
- Put $m = |G|$ and $F = \{\text{flags } (W_0 < W_1 < \cdots < W_m = \mathbb{C}[G])\}$. Then $X \times F$ has abelian isotropy so $E^*((X \times F)_{hG})$ is fg.
- Also $(X \times F)_{hG}$ is an iterated projective bundle over X_{hG} so $E^*((X \times F)_{hG})$ is free over $E^*(X_{hG})$ with canonical finite basis; so $E^*(X_{hG})$ is also fg.
- Also: there is an equaliser $E^*(X_{hG}) \to E^*((X \times F)_{hG}) \rightrightarrows E^*((X \times F^2)_{hG})$.
- Similarly $K^i(BG)$ is a finite vector space over $\mathbb{F}_p$ with dual $K_{-i}(BG)$, and $\widetilde{K}^*(BG) = \ker(K^*(BG) \to K^*)$ is a nilpotent ideal.
- (Story for E is more complicated; E_*BG is the wrong object.)
- There are nice calculations for Σ_m, $GL_m(F)$ with $p \nmid |F| < \infty$, groups of small nilpotence class; also the generalised character theory of Hopkins-Kuhn-Ravenel, and a clear picture of the relation with representation theory (*Chern approximation*).

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- Theorem (Greenlees-Sadofsky): for G a finite groupoid, $\nu\colon c_!(c^*(K)) \to c_*(c^*(K))$ is an equivalence.
- Equivalently: if G is a finite group, then $K \wedge BG_+ \to F(BG_+, K)$ is an equivalence, or $K_*(BG) = K^{-*}(BG)$.
- For the proof we work in the genuine equivariant category $\mathcal{GS}_G$.
- Two fixed point functors $\phi^G, \lambda^G\colon \mathcal{GS}_G \to \mathcal{S}$ (geometric, Lewis-May).
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- Recall EG is contractible with free G-action, and $\widetilde{E}G = \operatorname{cof}(EG_+ \to S^0)$; say X is *free* iff $X \in \operatorname{loc}\langle G_+ \rangle$ iff $EG_+ \wedge X \simeq X$ iff $\widetilde{E}G \wedge X = 0$.
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