## Ambidexterity 3

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May 26, 2023

## Equivariant spectra

- For a finite groupoid $G$, put $\mathcal{S}_{G}=[G, \mathcal{S}]$.
- Any $V: G \rightarrow \mathcal{L}$ (i.e. $V \in \mathcal{L}_{G}$ ) gives $S^{V}, S^{-V} \in S_{G}$.
$\Rightarrow$ Define $\pi_{n}^{G}(X)=\lim _{\longrightarrow \in \mathcal{L}_{G}}\left[S^{V},\left(S^{-n} \wedge X\right)(V)\right]^{G}$.
- Say $f: X \rightarrow Y$ is a genuine equivalence if
$\pi_{*}^{H}(f)$ is iso for all subgroupoids $H \leq G$.
$\rightarrow$ Say $f: X \rightarrow Y$ is a Borel equivalence if
$\pi_{*}\left(f_{a}\right): \pi_{*}\left(X_{a}\right) \rightarrow \pi_{*}\left(Y_{a}\right)$ is iso for all $a \in \operatorname{obj}(G)$
(or equivalently, $\pi_{*}^{H}(f)$ is iso for all discrete subgroupoids $H \leq G$ ).
- Two $\infty$-categories: $\mathcal{S}_{G}$ has Borel equivalences inverted, $\mathcal{G} \mathcal{S}_{G}$ has genuine equivalences inverted.
- For $X \in \mathcal{S}_{G}$ and $a, b \in \operatorname{obj}(G)$ we have $G(a, b)+\rightarrow S\left(X_{a}, X_{b}\right)$ and correspondingly $\Sigma^{\infty} G(a, b)_{+} \wedge X_{a} \rightarrow X_{b}$ and $\Sigma^{\infty} G(a, b)_{+} \rightarrow F\left(X_{a}, X_{b}\right)$.
- The map $p: G(a, b) \rightarrow 1$ gives a transfer $p^{t}: S^{0} \rightarrow \Sigma^{\infty} G(a, b)_{+}$in $h \mathcal{S}$.
- Combining these gives a map $S^{0} \rightarrow F\left(X_{a}, X_{b}\right)$ or $\nu_{a b}: X_{a} \rightarrow X_{b}$, which is essentially $\sum_{u \in G(a, b)} u_{*}$.
These induce $\nu: c_{!}(X) \rightarrow c_{*}(X)$ in $h \mathcal{S}$; iso if $G$ is empty or contractible.
- Now suppose we have $q: G \rightarrow H$. By applying the above to comma categories, we get $\nu: q_{!}(X) \rightarrow q_{*}(X)$ in $h S_{H}$, and this is iso if $q$ is faithful (Wirthmüller isomorphism).


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- Any $V: G \rightarrow \mathcal{L}$ (i.e. $V \in \mathcal{L}_{G}$ ) gives $S^{V}, S^{-V} \in \mathcal{S}_{G}$.
- Define $\pi_{n}^{G}(X)=\lim _{\longrightarrow V \in \mathcal{L}_{G}}\left[S^{V},\left(S^{-n} \wedge X\right)(V)\right]^{G}$.
- Say $f: X \rightarrow Y$ is a genuine equivalence if
$\pi_{*}^{H}(f)$ is iso for all subgroupoids $H \leq G$.
- Say $f: X \rightarrow Y$ is a Borel equivalence if
$\pi_{*}\left(f_{a}\right): \pi_{*}\left(X_{a}\right) \rightarrow \pi_{*}\left(Y_{a}\right)$ is iso for all $a \in \operatorname{obj}(G)$
(or equivalently, $\pi_{*}^{H}(f)$ is iso for all discrete subgroupoids $H \leq G$ ).
- Two $\infty$-categories: $\mathcal{S}_{G}$ has Borel equivalences inverted, $\mathcal{G} \mathcal{S}_{G}$ has genuine equivalences inverted.
- For $X \in \mathcal{S}_{G}$ and $a, b \in \operatorname{obj}(G)$ we have $G(a, b)_{+} \rightarrow \mathcal{S}\left(X_{a}, X_{b}\right)$ and correspondingly $\Sigma^{\infty} G(a, b)_{+} \wedge X_{a} \rightarrow X_{b}$ and $\Sigma^{\infty} G(a, b)_{+} \rightarrow F\left(X_{a}, X_{b}\right)$.
- The map $p: G(a, b) \rightarrow 1$ gives a transfer $p^{t}: S^{0} \rightarrow \Sigma^{\infty} G(a, b)_{+}$in $h \mathcal{S}$.
- Combining these gives a map $S^{0} \rightarrow F\left(X_{a}, X_{b}\right)$ or $\nu_{a b}: X_{a} \rightarrow X_{b}$, which is essentially $\sum_{u \in G(a, b)} u_{*}$.
- These induce $\nu: c_{!}(X) \rightarrow c_{*}(X)$ in $h \mathcal{S}$; iso if $G$ is empty or contractible.


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- These induce $\nu: c_{!}(X) \rightarrow c_{*}(X)$ in $h \mathcal{S}$; iso if $G$ is empty or contractible.
- Now suppose we have $q: G \rightarrow H$. By applying the above to comma categories, we get $\nu: q_{!}(X) \rightarrow q_{*}(X)$ in $h \mathcal{S}_{H}$, and this is iso if $q$ is faithful (Wirthmüller isomorphism).


## Periodic complex cobordism

- For a complex inner product space $V$ put $M P^{\prime}(V)=\{(u, U) \mid u \in U \leq V \oplus V\} \cup\{\infty\}$
$\Rightarrow$ Define an orthogonal ring spectrum $M P$ by $M P(V)=\Omega^{i V} M P^{\prime}(V \oplus i V)$.
- Using $\pi_{2+k} M P^{\prime}(\mathbb{C})=\pi_{2+k}\left(S^{0} \vee \mathbb{C} P^{2} \vee S^{4}\right) \rightarrow \pi_{k} M P=M P_{k}=M P^{-k}$ we get $u^{-1} \in M P_{-2}$ and $u \in M P_{2}$ with $u^{-1} u=1$; so $M P_{2 i+j}=M P_{j}$.
$\Rightarrow$ There are natural maps $\Sigma^{V} P(\mathbb{C} \oplus V) \rightarrow M P^{\prime}(V)$ which assemble to give $x: \Sigma^{\infty} \mathbb{C} P^{\infty} \rightarrow M P$ i.e. $x \in M P^{0}\left(\mathbb{C} P^{\infty}\right)$.
$\triangleright$ Using cofibrations $\mathbb{C} P^{n-1} \rightarrow \mathbb{C} P^{n} \rightarrow S^{2 n}$ we get $M P^{*}\left(\mathbb{C} P^{\infty}\right)=M P^{*} \llbracket x \rrbracket$.
- Using Künneth we get $M P^{*}\left(\mathbb{C} P^{\infty} \times \mathbb{C} P^{\infty}\right)=M P^{*}[y, z \rrbracket$.
- We can identify $\mathbb{C} P^{\infty}$ with $(\mathbb{C}[t] \backslash\{0\}) / \mathbb{C}^{\times}$, which is a commutative monoid under multiplication (and a group up to homotopy).
$\Rightarrow$ We have mult* $(x)=F_{M P}(y, z)=\sum_{i, j \geq 0} a_{i j} y^{i} z^{j}$ for some $a_{i j} \in \pi_{0}(M P)$.
- Commutative monoid structure implies $F_{M P}(x, 0)=x$ and $F_{M P}(x, y)=F_{M P}(y, x)$ and $F_{M P}\left(x, F_{M P}(y, z)\right)=F_{M P}\left(F_{M P}(x, y), z\right)$ so $F_{M P}$ is a formal group law.
- Theorem (Quillen): $\pi_{1}(M P)=0$ and $\pi_{0}(M P)=\mathbb{Z}\left[a_{1}, a_{2}, \ldots\right]$ with $a_{1}=a_{11}, a_{2}=a_{12}, a_{3}=a_{22}-a_{13}, a_{4}=a_{15}, a_{5}=a_{16}+a_{25}+a_{34}, \ldots$
- Also: for any formal group law $F$ over any ring $R$, there is a unique $\phi: \pi_{0}(M P) \rightarrow R$ carrying $F_{M P}$ to $F$. So $\pi_{0}(M P)$ is the Lazard ring.


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## Periodic complex cobordism

- For a complex inner product space $V$ put $M P^{\prime}(V)=\{(u, U) \mid u \in U \leq V \oplus V\} \cup\{\infty\}$
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## Morava $K$ and $E$

- Fix a prime $p$ and $n>0$.
- Put $I(x)=\sum_{:>0} x^{p^{n k}} / p^{k} \in \mathbb{Q} \llbracket x \rrbracket, \quad F(x, y)=I^{-1}(I(x)+I(y)) \in F G L(\mathbb{Q})$.
$\Rightarrow$ In fact $F \in \mathrm{FGL}(\mathbb{Z})$ so we can reduce $\bmod p$ to get $F_{K} \in \mathrm{FGL}\left(\mathbb{F}_{p}\right)$.
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- Write $x+_{F} y=F(x, y)$ and $[n]_{F}(x)=x+\cdots+x(n$ terms $)$.
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- Define $E_{0}=\mathbb{Z}_{p} \llbracket u_{1}, \ldots, u_{n-1} \rrbracket$ with $u_{0}=p, u_{n}=1$.
- For $I=\left(i_{1}, \ldots, i_{r}\right)$ in $\{1, \ldots, n\}^{r}$ we put $|I|=r$ and $\|/\|=i_{1}+\cdots+i_{r}$ and $\pi_{t}(l)=\Pi_{s<t} p^{i_{s}}$ and $u_{l}=\prod_{t=1}^{r} u_{i_{t}}^{\pi_{t}(l)}$. Then put $I_{E}(x)=\sum_{1} u_{1} x^{|l|\| \|} / p^{|| |} \in\left(\mathbb{Q} \otimes E_{0}\right) \llbracket x \rrbracket$ and $F_{E}(x, y)=I_{E}^{-1}\left(I_{E}(x)+I_{E}(y)\right)$.
- Using the Functional Equation Lemma: $F_{E} \in \operatorname{FGL}\left(E_{0}\right)$.
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- There is also a ring spectrum $K$ with $K^{0} X=\left(E^{0} X\right) /\left(u_{0}, \ldots, u_{n-1}\right)$ whenever the sequence is regular (and same for $K_{0} X$ ).


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- Put $U=\mathbb{C}[t] \backslash\{0\}$ so $\mathbb{C} P^{\infty}=U / \mathbb{C}^{\times}$
- Define $\phi_{m}: \mathbb{C} P^{\infty} \rightarrow \mathbb{C} P^{\infty}$ by $\phi_{m}([f])=\left[f^{m}\right]$, so $\phi_{m}^{*}(x)=[m]_{F_{M P}}(x)=[m]_{M P}(x) \in M P^{0}\left(\mathbb{C} P^{\infty}\right)$ (and same for $\left.E, K\right)$.
- The map $h(s, f)(t)=s+(1-s)(1+s t) f(t)$ gives a contraction of $U$.

P Put $C_{m}=\left\langle e^{2 \pi i / m}\right\rangle<\mathbb{C}^{\times}$and $B C_{m}=U / C_{m}$.
$\Rightarrow \mathbb{C} P^{\infty}$ has a tautological bundle $T$ with $T_{[f]}=\mathbb{C} f$ and $\phi_{m}^{*}(T) \simeq T^{\otimes m}$.

- Then $B C_{m}=E\left(T^{\otimes m}\right) \backslash$ (zero section) so cofibre $\left(B C_{m} \rightarrow \mathbb{C} P^{\infty}\right)=\operatorname{Thom}\left(T^{\otimes m}\right)$
$\Rightarrow$ Using the Thom isomorphism we get $M P^{0}\left(B C_{m}\right)=M P^{0} \llbracket x \rrbracket /[m]_{M P}(x)$ and $M P^{1}\left(B C_{m}\right)=0($ and same for $E, K)$.
- If $m=p^{k} m_{1}$ with $p \nmid m_{1}$ then $[m]_{K}(x)$ is a unit multiple of $\left[p^{k}\right]_{K}(x)=x^{p^{n k}}$ so $K^{0}\left(B C_{m}\right)=\mathbb{F}_{p}\left\{x^{i} \mid i<p^{n k}\right\}$.
- Similarly $E^{0}\left(B C_{m}\right)=E^{0}\left\{x^{i} \mid i<p^{n k}\right\}$ (free of finite rank over $E^{0}$ ).
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- Define $\phi_{m}: \mathbb{C} P^{\infty} \rightarrow \mathbb{C} P^{\infty}$ by $\phi_{m}([f])=\left[f^{m}\right]$, so $\phi_{m}^{*}(x)=[m]_{F_{M P}}(x)=[m]_{M P}(x) \in M P^{0}\left(\mathbb{C} P^{\infty}\right)$ (and same for $E, K$ ).
- The map $h(s, f)(t)=s+(1-s)(1+s t) f(t)$ gives a contraction of $U$.
- Put $C_{m}=\left\langle e^{2 \pi i / m}\right\rangle<\mathbb{C}^{\times}$and $B C_{m}=U / C_{m}$.
- $\mathbb{C} P^{\infty}$ has a tautological bundle $T$ with $T_{[f]}=\mathbb{C} f$ and $\phi_{m}^{*}(T) \simeq T^{\otimes m}$.
- Then $B C_{m}=E\left(T^{\otimes m}\right) \backslash$ (zero section) so cofibre $\left(B C_{m} \rightarrow \mathbb{C} P^{\infty}\right)=\operatorname{Thom}\left(T^{\otimes m}\right)$
- Using the Thom isomorphism we get $M P^{0}\left(B C_{m}\right)=M P^{0} \llbracket x \rrbracket /[m]_{M P}(x)$ and $M P^{1}\left(B C_{m}\right)=0$ (and same for $E, K$ ).
- If $m=p^{k} m_{1}$ with $p \nmid m_{1}$ then $[m]_{\kappa}(x)$ is a unit multiple of $\left[p^{k}\right]_{\kappa}(x)=x^{p^{n k}}$ so $K^{0}\left(B C_{m}\right)=\mathbb{F}_{p}\left\{x^{i} \mid i<p^{n k}\right\}$.
- Similarly $E^{0}\left(B C_{m}\right)=E^{0}\left\{x^{i} \mid i<p^{n k}\right\}$ (free of finite rank over $E^{0}$ ).
- For $A$ finite abelian: $E^{0}(B A)=E^{0}\left(B C_{m_{1}}\right) \otimes_{E^{0}} \cdots \otimes_{E^{0}} E^{0}\left(B C_{m_{r}}\right)$ is again free of finite rank, and $E^{1} B A=0$.
- Or: $\mathbb{G}:=\operatorname{spf}\left(E^{0}\left(\mathbb{C} P^{\infty}\right)\right)$ is a formal group scheme over $S:=\operatorname{spf}\left(E^{0}\right)$, and $\operatorname{spf}\left(E^{0}(B A)\right)=\operatorname{Hom}\left(A^{*}, \mathbb{G}\right)$, where $A^{*}=\operatorname{Hom}\left(A, S^{1}\right)$.


## Morava $K$ and $E$ of $B G$

- Claim: if $G$ is a finite groupoid and $X: G \rightarrow \mathcal{T}$ is a finite $G-C W$ complex then $E^{*}\left(c_{!}(X)\right)$ is finitely generated over $E^{*}$.
$\Rightarrow$ Can reduce to the case of a group, where $c_{!}(X)=X_{h G}=(E G \times X) / G$ and $(G / H)_{h G}=E G / H=B H$.
- The CW structure gives $\operatorname{skel}_{d}(X) /$ skel $_{d-1}(X) \simeq \Sigma^{d} V_{i}\left(G / H_{i}\right)+$; so enough to prove $E^{*}(B H)$ is finitely generated for all isotropy groups $H$.
- So if all isotropy groups for $X$ are abelian, then $E^{*}\left(X_{h G}\right)$ is fg .
$\triangleright$ Put $m=|G|$ and $F=\left\{\right.$ flags $\left.\left(W_{0}<W_{1}<\cdots<W_{m}=\mathbb{C}[G]\right)\right\}$. Then $X \times F$ has abelian isotropy so $E^{*}\left((X \times F)_{h G}\right)$ is fg .
$\Rightarrow$ Also $(X \times F)_{h G}$ is an iterated projective bundle over $X_{h G}$ so $E^{*}\left((X \times F)_{h G}\right)$ is free over $E^{*}\left(X_{h G}\right)$ with canonical finite basis; so $E^{*}\left(X_{h G}\right)$ is also fg .
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- Similarly $K^{i}(B G)$ is a finite vector space over $\mathbb{F}_{p}$ with dual $K_{-i}(B G)$, and $\widetilde{K}^{*}(B G)=\operatorname{ker}\left(K^{*}(B G) \rightarrow K^{*}\right)$ is a nilpotent ideal.
$\Rightarrow$ (Story for $E$ is more complicated; $E_{*} B G$ is the wrong object.)
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Then $X \times F$ has abelian isotropy so $E^{*}\left((X \times F)_{h G}\right)$ is fg .

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## Tate vanishing

- Theorem (Greenlees-Sadofsky): for $G$ a finite groupoid, $\nu: c_{!}\left(c^{*}(K)\right) \rightarrow c_{*}\left(c^{*}(K)\right)$ is an equivalence.
$\Rightarrow$ Equivalently: if $G$ is a finite group, then $K \wedge B G_{+} \rightarrow F\left(B G_{+}, K\right)$ is an equivalence, or $K_{*}(B G)=K^{-*}(B G)$.
- For the proof we work in the genuine equivariant category $G S_{G}$
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- $\phi^{G} \Sigma^{\infty} X=\Sigma^{\infty} X^{G}$ and $\phi^{G}\left(S^{ \pm V}\right)=S^{ \pm V^{G}}$ and $\phi^{G}(X \wedge Y)=\phi^{G}(X) \wedge \phi^{G}(Y)$.
$\Rightarrow \pi_{*}^{G}(X)=\pi_{*}\left(\lambda^{G}(X)\right)$ and $\mathcal{G} \mathcal{S}\left(c^{*}(X), Y\right)=\mathcal{S}\left(X, \lambda^{G}(Y)\right)$ and if $Y$ is free then $\lambda^{G}(Y) \simeq Y^{h G} \simeq Y_{h G}\left(\right.$ cf $M_{G} \simeq M^{G}$ for free $\mathbb{Z}[G]$-modules $)$
- Recall $E G$ is contractible with free $G$-action, and $\widetilde{E} G=\operatorname{cof}\left(E G_{+} \rightarrow S^{0}\right)$; say $X$ is free iff $X \in \operatorname{loc}\left\langle G_{+}\right\rangle$iff $E G_{+} \wedge X \simeq X$ iff $E G \wedge X=0$.
$\Rightarrow$ As $E G_{+} \wedge K$ is free, we have $\lambda^{G}\left(E G_{+} \wedge K\right)=\left(E G_{+} \wedge K\right)_{h G}=K \wedge B G_{+}$.
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$\Rightarrow E G_{+} \wedge(-)$ converts nonequivariant equivalences to equivariant ones, so $E G_{+} \wedge K \simeq E G_{+} \wedge F\left(E G_{+}, K\right)$. Using this: $\operatorname{cof}(\widetilde{\nu}) \simeq \widetilde{E} G \wedge F\left(E G_{+}, K\right)$.
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For finite $G$, we need $K^{t G}=0$, where $K^{t G}=\widetilde{E}_{G} \wedge F\left(E G_{+}, K\right)=0$.

- For $H<G$ we have $\operatorname{res}_{H}^{G}\left(K^{t G}\right) \simeq K^{t H}$; assume $K^{t H}=0$ by induction.
- We also have $G / H_{+} \wedge K^{t G}=\operatorname{ind}_{H}^{G}\left(\operatorname{res}_{H}^{G}\left(K^{t G}\right)\right)=\operatorname{ind}_{H}^{G}(0)=0$.
- Put $V=\mathbb{C}[G] \ominus \mathbb{C}$ so $V^{G}=0$ so $S(\infty V)^{G}=\emptyset$ so
$S(\infty V)_{+} \in \operatorname{loc}\left\langle G / H_{+} \mid H<G\right\rangle$ so $S(\infty V)_{+} \wedge K^{t G}=0$
- As $S^{\infty V}=\operatorname{cof}\left(S(\infty V)_{+} \rightarrow S^{0}\right)$, we get
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$>\operatorname{Now} \lambda^{G}\left(S^{m V} \wedge F\left(E G_{+}, K\right)\right)=\lambda^{G} F\left(S^{-m V} \wedge E G_{+}, K\right)=F\left(B G_{+}^{-m V}, K\right)$ and $\pi_{*}$ (this) $=K^{*}(B G)$. (Thom class of $\left.-m V\right)$.
- Taking $m \rightarrow \infty$ we get $\pi_{*}^{G}\left(K^{t G}\right)=e(V)^{-1} K^{-*}(B G)$, but $e(V)$ lies in the nilpotent ideal $\widetilde{K}^{0}(B G)$ so inverting $e(V)$ gives 0 .
- As $\pi_{*}^{G}\left(K^{t G}\right)=0$ and $\operatorname{res}_{H}^{G}\left(K^{t G}\right)=0$ for $H<G$ we have $K^{t G}=0$.
- Conclusion: $K^{0}(B G) \simeq K_{0}(B G) \simeq \operatorname{Hom}\left(K^{0}(B G), \mathbb{F}_{p}\right)$, so there is a perfect pairing $K^{0}(B G) \otimes K^{0}(B G) \rightarrow \mathbb{F}_{p}$ (of the form $a \otimes b \mapsto \theta(a b)$ ).
- If $n>1$ then $K^{0}\left(B C_{p^{k}}\right)=\mathbb{F}_{p}[x] / x^{p^{n k}}$ and $\theta(f)=$ coefficient of $x^{p^{n k}-1}$ in $f$.
- For any $G$ the man $\left(K^{0}(B G) \xrightarrow{\delta_{1}} K^{0}(B G) \otimes K^{0}(B G) \xrightarrow{1 \otimes \theta} K^{0}(B G)\right)$
is the identity, and this characterises $\theta$.


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## nilpotent ideal $\widetilde{K}^{0}(B G)$ so inverting $e(V)$ gives 0

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