Ambidexterity 2

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May 26, 2023

First hints of recursion

- Consider $\mathcal{C}$ with all finite (co)limits and functors $c\colon G \to 1$ and $q\colon G \to H$.
- Say G is contractible iff ($G \neq \emptyset$ and $|G(x,y)| = 1$ for all x and y) iff $c\colon G \to 1$ is an equivalence.
- If so, the map $\rho = (c_*(C) \xrightarrow{p_a} C_a \xrightarrow{i_a} c_!(C))$ is independent of $a \in \mathrm{obj}(G)$ and is an isomorphism; so $c_! \simeq c_*$. (This is level -2.)
- All comma categories $(q \downarrow b)$ are contractible iff q is an equivalence; if so, there is a canonical isomorphism $q_! \simeq q_*$.
- If G is empty then there is a unique object in $\mathcal{C}_G$ which is sent by $c_!$ and c_* to the initial and terminal objects of $\mathcal{C}$; so there is a unique morphism $\nu\colon c_! \to c_*$. This is an isomorphism iff $\mathcal{C}$ is pointed. (This is level -1.)
- All comma categories $(q \downarrow b)$ are (empty or contractible) iff q is full and faithful; if so, and $\mathcal{C}$ is pointed, there is a canonical isomorphism $q_! \simeq q_*$.
- Suppose that G is a finite set and $\mathcal{C}$ is pointed. Given $C \in \mathcal{C}_G = \prod_{x \in G} \mathcal{C}$ define $\nu_{xy}\colon C_x \to C_y$ to be 1 if $x = y$ and 0 if $x \neq y$. These give $\nu\colon c_!(C) = \bigoplus_x C_x \to \prod_x C_x = c_*(C)$. For all such maps to be iso, we need $\mathcal{C}$ to be semiadditive. (This is level 0.)
- All comma categories are (equivalent to) finite sets iff q is faithful. If so, and $\mathcal{C}$ is semiadditive, then $q_! \simeq q_*$.

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- This is

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- or, regarding $(f_i)_{i \in I}$ as a morphism $f \colon c^*(C) \to c^*(C')$:

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- Now let G be any finite groupoid, and consider $C \in \mathcal{C}_G$. We want to define $\nu \colon c_!(C) \to c_*(C)$ to give potential ambidexterity at level 1.
- This will use the maps $\nu_{xy} = \sum_{u \in G(x,y)} u_* \colon C_x \to C_y$ for $x, y \in \mathrm{obj}(G)$.
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$$\begin{array}{ccccccc} C & \xrightarrow{\Delta} & \prod_i C & \xleftarrow[\simeq]{\nu} & \bigoplus_i C & & \\ & & {\scriptstyle \prod_i f_i}\downarrow & & \downarrow{\scriptstyle \bigoplus_i f_i} & & \\ & & \prod_i C' & \xleftarrow[\nu]{\simeq} & \bigoplus_i C' & \xrightarrow{\nabla} & C' \end{array}$$

- or, regarding $(f_i)_{i \in I}$ as a morphism $f \colon c^*(C) \to c^*(C')$:

$$\begin{array}{ccccccc} C & \xrightarrow{\eta} & c_*(c^*(C)) & \xleftarrow[\simeq]{\nu} & c_!(c^*(C)) & & \\ & & {\scriptstyle c_*(f)}\downarrow & & \downarrow{\scriptstyle c_!(f)} & & \\ & & c_*(c^*(C')) & \xleftarrow[\nu]{\simeq} & c_!(c^*(C')) & \xrightarrow{\epsilon} & C' \end{array}$$

- Now let G be any finite groupoid, and consider $C \in \mathcal{C}_G$. We want to define $\nu \colon c_!(C) \to c_*(C)$ to give potential ambidexterity at level 1.
- This will use the maps $\nu_{xy} = \sum_{u \in G(x,y)} u_* \colon C_x \to C_y$ for $x, y \in \mathrm{obj}(G)$.
- To form this sum, we are using ambidexterity at level 0.

∞-categories

- ∞-categories are a generalisation of categories/topological categories/model categories.
- We will not give details but only heuristic ideas.
- Given a space X and points $a, b \in X$, put $PX(a,b) = \{u \colon [0,1] \to X \mid u(0) = a,\ u(1) = b\}$.
- Given $v \in PX(b,c)$ and $u \in PX(a,b)$ define $v \circ u \in PX(a,c)$ by $(v \circ u)(t) = u(2t)$ for $0 \leq t \leq 1/2$ and $v(2t-1)$ for $1/2 \leq t \leq 1$.
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  $$\Pi X(a,b) = \pi_0(PX(a,b)) = \{ \text{ pinned homotopy classes of paths from } a \text{ to } b \}$$

  then we get a category: the path groupoid of X.
- A map u from the n-simplex Δ_n to X gives points $a_i = u(e_i)$ for $0 \leq i \leq n$, and paths $u_{ij} \in PX(a_i, a_j)$ for $0 \leq i \leq j \leq n$, plus evidence that u_{ik} is pinned-homotopic to $u_{jk} \circ u_{ij}$.
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(Co)limits in ∞-categories

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Spaces

- Let $\mathcal{T}$ be the category of (compactly generated, weak Hausdorff) spaces.
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Spectra

$\mathcal{L} = \{$ finite dimensional real inner product spaces$\}$; $\mathcal{T}_* = \{$based spaces$\}$;
$S^V = V \cup \{\infty\} \in \mathcal{T}_*$; $S^V \wedge S^W = S^{V \oplus W}$.

- An (orthogonal) spectrum is a continuous functor $X\colon \mathcal{L} \to \mathcal{T}_*$ equipped with maps $S^V \wedge X(W) \to X(V \oplus W)$ subject to various axioms.
- We write $\mathcal{S}$ for the category of spectra, or the corresponding ∞-category.
- For $n \geq 0$ we put $\pi_n(X) = \varinjlim_V [S^n \wedge S^V, X(V)]$ and $\pi_{-n}(X) = \varinjlim_V [S^V, X(\mathbb{R}^n \oplus V)]$.
- A map $f \in \mathcal{S}(X, Y)$ is a weak equivalence iff $\pi_*(f)$ is an isomorphism.
- A based space T gives a spectrum $\Sigma^\infty T$ with $(\Sigma^\infty T)(V) = S^V \wedge T$. We often write T for $\Sigma^\infty T$.
- Let $H(V)$ be the free abelian group generated by V with a suitable topology. This gives an Eilenberg-MacLane spectrum H with $\pi_0(H) = \mathbb{Z}$ and $\pi_n(H) = 0$ for $n \neq 0$.
- Put $MO(V) = \{(u, U) \mid u \in U \leq V \oplus V, \dim(U) = \dim(V)\} \cup \{\infty\}$. This gives a spectrum MO with $\pi_n(MO) = \{$cobordism classes of closed n-manifolds$\}$.

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- Put $(S^{-A})(V) = \{(f, v) \mid f \in \mathcal{L}(A, V),\ v \in V \ominus f(A)\} \cup \{\infty\}$. This gives a spectrum S^{-A} with $\mathcal{S}(S^{-A}, X) = X(A)$ and $S^{-A} \wedge S^{-B} = S^{-(A \oplus B)}$.
- For $X, Y \in \mathcal{S}$ we define $F(X, Y)(V) = \mathcal{S}(S^{-V} \wedge X, Y)$. This gives a spectrum $F(X, Y)$ with $\mathcal{S}(W, F(X, Y)) = \mathcal{S}(W \wedge X, Y)$.
- There is a natural weak equivalence $S^A \wedge S^{-A} \to S^0$.
- We also put $(X \vee Y)(V) = X(V) \vee Y(V)$ and $(X \times Y)(V) = X(V) \times Y(V)$.
- There is a natural inclusion $X \vee Y \to X \times Y$, which is a weak equivalence.

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  - Define $c\colon S^V \wedge Y_+ = (V \times Y) \cup \{\infty\} \to (V \times X) \cup \{\infty\} = S^V \wedge X_+$ by $c(\widetilde{f}(v,x)) = (v,x)$ and $c(v,y) = \infty$ for $(v,y) \notin \text{image}(\widetilde{f})$.
  - We have made this completely natural so it will be compatible with any group actions.
  - Apply $S^{-V} \wedge \Sigma^\infty(-)$ to get $f^t\colon \Sigma^\infty Y_+ \to \Sigma^\infty X_+$.
- For $f: 2 \to 1$, the map $c\colon S^2 \to S^2 \wedge 2_+ = S^2 \vee S^2$ is just the pinch map.

Collapse and transfer

- Consider a map $f\colon X \to Y$ of finite sets, giving $\Sigma^\infty f_+\colon \Sigma^\infty X_+ = \bigoplus_x S^0 \to \bigoplus_y S^0 = \Sigma^\infty Y_+$.
- Because $h\mathcal{S}$ is additive, we can also define $f^t\colon \Sigma^\infty Y_+ \to \Sigma^\infty X_+$, essentially by $f^t(y) = \sum_{f(x)=y} x$.
- For a more precise model:
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