Ambidexterity 1

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Introduction

- *Ambidexterity* refers to a situation where we have functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, such that G is both left adjoint and right adjoint to F.
- Easy example:
  - Let $q\colon G \to H$ be a functor of finite groupoids.
  - Let $\mathcal{V}_G = [G, \mathcal{V}]$ be the category of complex representations of complex representations of G.
  - This gives $q^*\colon \mathcal{V}_H \to \mathcal{V}_G$, and it is formal that there are left and right adjoints $q_!, q_*\colon \mathcal{V}_G \to \mathcal{V}_H$.
  - We can write down $\nu\colon q_! \to q_*$ and check that it is an isomorphism.
- Now consider the category $\mathcal{A}_G = [G, \mathcal{A}]$ of $\mathbb{Z}$-linear representations. We again have $q_! \vdash q^* \vdash q_*$ and $\nu\colon q_! \to q_*$. To prove that ν is an isomorphism, we divide by some group orders. If q is faithful, these orders are equal to one, so again $q_! \simeq q_*$.
- Alternatively: $q_! \simeq q_*$ on $\mathcal{PA}_G = \{\text{projectives in } \mathcal{A}_G\}$.
- Now consider the ∞-category $\mathcal{S}_G = [G, \mathcal{S}]$ of spectral representations, i.e. functors to the ∞-category of spectra. We can define $\nu\colon q_! \to q_*$ by a transfer construction; cofibre is the *Tate construction*. If q is faithful then Tate $= 0$ so ν is iso (theorem of Wirthmüller).

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- Let K and E be the corresponding Morava K-theory and E-theory, and let $\mathcal{K}$ be the ∞-category of K-local spectra.
- For $q\colon G \to H$ of finite groupoids, we again have $q^*\colon \mathcal{K}_H \to \mathcal{K}_G$ and $q_!, q_*\colon \mathcal{K}_G \to \mathcal{K}_H$.
- We can again use transfers to define $\nu\colon q_! \to q_*$ (with cofibre = Tate).
- In this K-local context Tate $= 0$ so ν is an equivalence (essentially a theorem of Greenlees and Sadofsky).
- In the framework of Lurie: ∞-groupoids are spaces ($X = BG$). Finite ∞-groupoids are π-finite spaces (finite disjoint unions of connected spaces X with $|\prod_k \pi_k(X)| < \infty$).
- $\mathcal{K}(X)$ is the category of functors from the ∞-groupoid X to $\mathcal{K}$, or families of K-local spectra continuously parametrised by points of X.
- For a functor $q\colon X \to Y$ of finite ∞-groupoids (= map of π-finite spaces) we have $q^*\colon \mathcal{K}(Y) \to \mathcal{K}(X)$ and $q_!, q_*\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ with $q_! \vdash q^* \vdash q_*$.
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- Proof: induction on the truncation level m.Greenlees-Sadofsky is $m = 1$. To define ν at level $m + 1$, we need ν to be an isomorphism at level m.

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- Any $V \in \mathcal{V}_G = [G, \mathcal{V}]$ gives a complex vector bundle over BG with total space $c_!(V)$ and thus an element $\phi(V) \in KU^0(BG)$. This gives a ring map from the representation ring $R(G) = \pi_0(\mathcal{V}_G) - \pi_0(\mathcal{V}_G)$ to $KU^0(BG)$.
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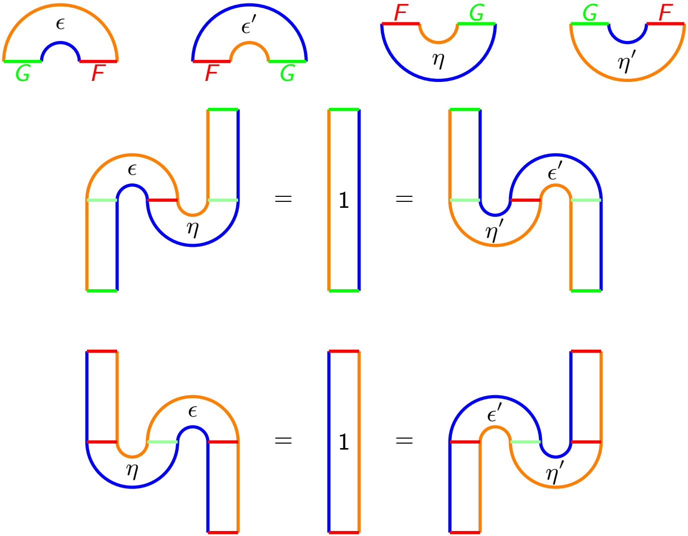
Graphical calculus

$$\mathcal{D} \xleftarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{D}$$



Finite groupoids

- A *groupoid* is a category G in which all morphisms are invertible. In particular:
  - For every object $a \in G$, the set $G(a, a)$ is a group
  - For every morphism $u\colon a \to b$ we have a bijection $G(a, a) \to G(a, b)$ given by $g \mapsto ug$, and a group isomorphism $G(a, a) \to G(b, b)$ given by $g \mapsto ugu^{-1}$.
- A group can be regarded as a groupoid with only one object.
- A groupoid G is *finite* if the set $\pi_0(G)$ of isomorphism classes is finite, and $G(a, b)$ is finite for all a and b.
- If so, we can choose a list $a_1, \ldots, a_r$ containing precisely one object in each isomorphism class; then G is equivalent to the coproduct (in groupoids) of the finite groups $G_i = G(a_i, a_i)$.
- Thus, most questions about finite groupoids can be reduced to questions about finite groups.
- However, the groupoid picture is more natural for many purposes, and leads more easily to the theory for ∞-groupoids.
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Representations of finite groupoids

- Let $\mathcal{A}$ be the category of abelian groups, and let $\mathcal{A}_G$ be the category of functors from G to $\mathcal{A}$.
- Given $A \in \mathcal{A}_G$ write A_x for the value of A at an object x, and $u_* \colon A_x \to A_y$ for the map induced by $u \in G(x, y)$.
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- For more general $q \colon G \to H$ and $b \in \mathrm{obj}(H)$, we have a comma category $(q \downarrow b)$ with objects $\{(x, v) \mid x \in \mathrm{obj}(G),\ v \in H(q(x), b)\}$ and morphisms

$$(q \downarrow b)((x_0, v_0), (x_1, v_1)) = \{g \in G(x_0, x_1) \mid v_1 \circ q(g) = v_0\}.$$

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- One can check that ρ does not depend on the choice of a_k, and that $\rho\nu \colon c_!(A) \to c_!(A)$ is multiplication by d, and that $\nu\rho \colon c_*(A) \to c_*(A)$ is also multiplication by d. Thus ν is iso when d is invertible.
- As d is always invertible in $\mathbb{C}$ we see that ν is iso for $c_!, c_* \colon \mathcal{V}_G \to \mathcal{V}$. When G is a finite group this is the well-known isomorphism $V_G \simeq V^G$.
- By reduction to comma categories: for any $q \colon G \to H$ of finite groupoids we have $q_! \simeq q_* \colon \mathcal{V}_G \to \mathcal{V}_H$.

- For $A \in \mathcal{A}_G$ and $x, y \in \text{obj}(G)$ we define $\nu'_{xy} = \sum_{u \in G(x,y)} u_* \colon A_x \to A_y$.
- For $g \colon x' \to x$ and $h \colon y \to y'$ we find that $\nu'_{x'y'} = h_* \nu_{xy} g_* \colon A_{x'} \to A_{y'}$. Thus, there is a unique $\nu \colon c_!(A) \to c_*(A)$ with $\nu'_{xy} = (A_x \xrightarrow{i_x} c_!(A) \xrightarrow{\nu} c_*(A) \xrightarrow{p_y} A_y)$ for all x, y.

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Representations of finite groupoids

- $d\colon \operatorname{obj}(G) \to \mathbb{N}^+$ by $d(x) = |G(x,x)|$
- $\nu\colon c_!(A) \to c_*(A)$ is an isomorphism in any context where d is invertible.

- G is equivalent to a set iff $d = 1$. If so, then ν is invertible, and we have $c_!(A) \simeq c_*(A) \simeq \bigoplus_{k=1}^{r} A_{a_k}$.
- Exercise: for $q\colon G \to H$, the comma categories $(q \downarrow b)$ are equivalent to sets iff they have $d = 1$ iff $q\colon G(x,y) \to H(q(x), q(y))$ is injective for all $x, y \in \operatorname{obj}(G)$ iff q is a faithful functor.
- Recall that $q_!(A)_b$ and $q_*(A)_b$ are the colimit and limit of the same functor on $(q \downarrow b)$. Thus, if q is faithful then $q_! \simeq q_*$.

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