Ambidexterity 4

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May 26, 2023

K-local spectra

- Recall that Z is *K-acyclic* if $K_*Z = 0$, and
  X is *K-local* if $[Z, X] = 0$ for all K-acyclic Z (e.g. K is K-local)
- We write $\mathcal{K}$ for the ∞-category of K-local spectra.
- There is a functorial cofibration $CX \to X \to LX$
  where CX is K-acyclic and LX is K-local.
- For objects: $LX = 0$ iff $K_*(X) = 0$. For morphisms: Lf iso iff $K_*(f)$ iso.
- The subcategory $\mathcal{K} \subseteq \mathcal{S}$ is closed under limits. To get colimits in $\mathcal{K}$, construct colimits in $\mathcal{S}$ and apply L.
- In MP_0: put $u_i =$ coeff of x^{p^i} in $[p]_{MP}(x)$ and $I_n = (u_0, \dots, u_{n-1})$.
- For an MP-module M we have $LM = (u_n^{-1}M)^{\wedge}_{I_n}$. Thus E is K-local.
- Deep Theorem: $\mathcal{K} = \text{thick}\langle LM \mid M \in \text{Mod}_{MP}\rangle$
- Deep Theorem: There is a finite spectrum F such that
  $MP_*F = MP_*/(u_0^{i_0}, \dots, u_{n-1}^{i_{n-1}})$ for some $i_0, \dots, i_{n-1}$.
- Put $p_G(X) = \text{cof}(c_!(X) \to c_*(X)) = \text{cof}(L(X_{hG}) \to X^{hG})$ so $p_G(K) = 0$.
- If $M \in \text{Mod}_K$ then $p_G(M) \in \text{Mod}_{p_G(K)}$ so $p_G(M) = 0$.
- If $M \in \text{Mod}_{MP}$ then $L(F \wedge M) \in \text{thick}\langle \text{Mod}_K \rangle$ so $p_G(L(F \wedge M)) = 0$ so $F \wedge p_G(LM) = 0$ so $p_G(LM) = 0$.
- Thus, for $X \in \mathcal{K} = \text{thick}\langle L\,\text{Mod}_{MP}\rangle$ we have $p_G(X) = 0$ i.e. $c_!(X) = c_*(X)$.

Naive groupoid duality

- For a finite groupoid G put $M(G) = \mathbb{Q}\{\pi_0(G)\}$ and $M^*(G) = \mathrm{Hom}(M(G), \mathbb{Q}) = \mathrm{Map}(\pi_0(G), \mathbb{Q})$.
- Define an inner product on $M(G)$ by $([a], [b])_G = |G(a, b)|$ (so $([a], [b]) = 0$ unless $a \simeq b$).
- The induced inner product on $M^*(G)$ is $\langle f, g\rangle_G = \sum_{i=1}^{r} |G(a_i, a_i)|^{-1} f(a_i) g(a_i)$, where $a_1, \dots, a_r$ contains one member of each isomorphism class.
- This is also $\langle f, g\rangle_G = \theta(fg)$, where $\theta(h) = \sum_i |G(a_i, a_i)|^{-1} h(a_i)$.
- Given $q\colon G \to H$ we define $q_!\colon M(G) \to M(H)$ by $q_!([a]) = [q(a)]$, and $q^*\colon M^*(H) \to M^*(G)$ by $q^*(g)(a) = g(q(a))$.
- Define $q^*\colon M(H) \to M(G)$ and $q_!\colon M^*(G) \to M^*(H)$ to be adjoint, so $(q_!(u), v)_H = (u, q^*(v))_G$ and $\langle q_!(f), g\rangle_H = \langle f, q^*(g)\rangle_G$.
- This is compatible with the isomorphisms $M(G) \simeq M^*(G) \simeq \mathrm{Hom}(M(G), \mathbb{Q})$.
- The isomorphism $M(G) \to M^*(G)$ is the isomorphism $\nu\colon c_!(c^*(\mathbb{Q})) \to c_*(c^*(\mathbb{Q}))$ that we considered before.

Classical characters

- For finite G, put $R(G) = \pi_0(\mathcal{V}_G) - \pi_0(\mathcal{V}_G)$ (the *representation ring*).
- A functor $q\colon G \to H$ gives $q_! \colon \mathcal{V}_G \rightleftarrows \mathcal{V}_H \colon q^*$ and then $q_! \colon R(G) \rightleftarrows R(H) \colon q^*$.
- Define $\theta = c_! = c_* \colon R(G) \to R(1) = \mathbb{Z}$ (so for a group we have $\theta([U]) = \dim_{\mathbb{C}}(U^G)$).
- This gives a perfect pairing $\langle u, v\rangle_G = \theta(uv)$ on $R(G)$.
- This has $\langle q_!(u), v\rangle_H = \langle u, q^*(v)\rangle_G$.
- Let L be the subfield of $\mathbb{C}$ generated by all roots of unity.
- If $\alpha \in \mathrm{Aut}(V)$ with $\alpha^m = 1$ then all eigenvalues lie in L so $\mathrm{trace}(\alpha) \in L$.
- Put $\Lambda G = [\mathbb{Z}, G]$ so $\mathrm{obj}(\Lambda G) = \{(a, u) \mid a \in \mathrm{obj}(G),\ u \in G(a, a)\}$ and $(a, u) \simeq (a', u')$ iff there exists $g \in G(a, a')$ with $u' = gug^{-1}$.
- Define $C(G) = L \otimes M^*(\Lambda(G)) = \mathrm{Map}(\pi_0(\Lambda G), L)$, so we have $\theta\colon C(G) \to L$ and $q_! \colon C(G) \rightleftarrows C(H) \colon q^*$ for $q\colon G \to H$.
- For a representation $V\colon G \to \mathcal{V}$ define $\chi(V) \in C(G)$ by $\chi(V)([a, u]) = \mathrm{trace}(u_* \colon V_a \to V_a)$.
- This gives an isomorphism $\chi\colon L \otimes R(G) \to C(G)$, compatible with all structure.

Generalised characters

- Fix a prime p and $n > 0$ and let E be Morava E-theory.
- Then $[p^k]_E(x) = g_k(x)h_k(x)$, where $h_k(x) \in E^0[\![x]\!]^\times$ and $g_k(x) \in E^0[x]$ is a monic polynomial of degree p^{nk} and $E^0(BC_{p^k}) = E^0[x]/g_k(x)$.
- Construct L from $\mathbb{Q} \otimes E^0$ by adjoining a full set of roots of $g_k(x)$ for all k.
- Put $\mathbb{Z}/p^\infty = \varinjlim_k \mathbb{Z}/p^k = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} = \mathbb{Q}/\mathbb{Z}_{(p)} = \mathbb{Q}_p/\mathbb{Z}_p = \bigcup_k \sqrt[p^k]{1} \subset S^1$. (Exercise: $\mathrm{Hom}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty) \simeq \mathbb{Z}_p \simeq \mathrm{Hom}(\mathbb{Z}/p^\infty, S^1)$.)
- Put $\Theta = \{\text{all roots of all } g_k(x)\} \subset L$. This is a group under $+_E$, isomorphic to $(\mathbb{Z}/p^\infty)^n$, analogous to the formal group scheme $\mathbb{G}$.
- Put $\Theta^* = \mathrm{Hom}(\Theta, S^1) \simeq \mathbb{Z}_p^n$, regarded as a groupoid with one object.
- Put $\Lambda G = [\Theta^*, G] = \varinjlim_k [\Theta^*/p^k, G]$, $\;C(G) = L \otimes M^*\Lambda G = \mathrm{Map}(\pi_0 \Lambda G, L)$.
- Recall $E^0(B(\Theta^*/p^k)) = E^0[\![x_1, \dots, x_n]\!]/(g_k(x_1), \dots, g_k(x_n))$; there is a canonical map ϕ_k from this to L.
- Thus any $u\colon \Theta^*/p^k \to G$ gives $\phi_k \circ E^0(Bu)\colon E^0BG \to L$. Assembling these gives $\chi\colon L \otimes_{E^0} E^0(BG) \to C(G)$.
- Theorem (Hopkins, Kuhn, Ravenel): χ is an isomorphism.
- Both sides have inner products, and operators $q_!$ and q^* adjoint to each other, and χ preserves all this.

Proof of the generalised character theorem

- Reduce to the case of a finite group G.
- Generalise: for a finite G-CW complex Z, we have

$$\chi_{G,Z} \colon L \otimes_{E^0} E^*(Z_{hG}) \to L \otimes_{\mathbb{Q}} \left(\prod_{\theta \colon \Theta^* \to G} H^*(Z^{\mathrm{image}(\theta)}; \mathbb{Q}) \right)^G$$

- Prove by calculation that $\theta_{G,Z}$ is iso when $Z = G/A$ with $A \leq G$ abelian. (Here $Z_{hG} = BA$, and $Z^{\mathrm{image}(\theta)}$ is Z (if $\mathrm{image}(\theta) \leq A$) or $\emptyset$ (otherwise).)
- Deduce by Mayer-Vietoris that $\chi_{G,Z}$ is iso if Z has abelian isotropy.
- Let F be the space of complete flags in $\mathbb{C}[G]$, so $Z \times F$ and $Z \times F^2$ have abelian isotropy, and we have an equaliser $E^*(Z_{hG}) \to E^*((Z \times F)_{hG}) \rightrightarrows E^*((Z \times F^2)_{hG})$. Deduce the general case from this.
- Corollary: $\mathbb{Q} \otimes E^0(BG) = u_0^{-1} E^0(BG) \simeq \left(\prod_A \mathbb{Q} \otimes D_A\right)^G$. Here A runs over abelian subgroups $A < G$, and D_A is a certain regular local ring, free of finite rank as an E^0-module.
- Recall $\mathrm{spf}(E^0(BA)) = \mathrm{Hom}(A^*, \mathbb{G})$; morally $\mathrm{spf}(E^0(D_A)) = \mathrm{Inj}(A^*, \mathbb{G})$.
- There is a similar map $u_k^{-1} E^0(BG)/I_k \to \left(\prod_A u_k^{-1} D_{k,A}\right)^G$ for $k > 0$, which is an F-isomorphism (Greenlees-Strickland; see also Stapleton).

The inner product and the trace

- Suppose for simplicity that $E^1(BG) = 0$ and $E^0(BG)$ is free over E^0, say with basis $e_1, \dots, e_r$.
- The pairing on $E^0(BG)$ is $\langle f, g\rangle_G = \theta(fg)$ for some $\theta\colon E^0(BG) \to E^0$.
- We can also define the trace $\tau\colon E^0(BG) \to E^0$, so if $fe_i = \sum_j a_{ij} e_j$ then $\tau(f) = \sum_i a_{ii}$.
- Choose representatives $u_1, \dots, u_m$ of the isomorphism classes in ΛG, with automorphism groups $\Gamma_i = (\Lambda G)(u_i, u_i)$; then on $L \otimes_{E^0} E^0(BG) \simeq C(G)$ we have $\theta(f) = \sum_i |\Gamma_i|^{-1} f(u_i)$ and $\tau(f) = \sum_i f(u_i)$.
- If G is an abelian group then $\Gamma_i \simeq G$ for all i and so $\tau = |G|\theta$ on $C(G)$ or on $E^0(BG)$.

Even more generalised characters

- For a space X, put $\Lambda_0 X = \varinjlim_m [B(\Theta^*/p^m)_+, X]$, so $\Lambda_0 BG = \pi_0(\Lambda G)$ for finite groupoids G.
- Put $C(X) = \mathrm{Map}(\Lambda_0(X), L)$; we still have a ring map $\chi\colon L \otimes_{E^0} E^0(X) \to C(X)$, which is iso for $X = BG$.
- Recall that the Eilenberg-MacLane space $B^d A = K(A, d)$ has $\pi_d(B^d A) = A$ and $\pi_i(B^d A) = 0$ for $i \neq d$ and $[Z, B^d A] = H^d(Z; A)$.
- Note that Θ^*/p^k is similar to $\Theta^* = \mathbb{Z}_p^n$ or $\mathbb{Z}^n$, and $B(\mathbb{Z}^n)$ is the torus $(S^1)^n$, with $H_*(B(\mathbb{Z}^n)) = \lambda^*(\mathbb{Z}^n)$.
- We find that $\Lambda_0(B^d A) = \mathrm{Hom}(\lambda^d \Theta^*, A) \simeq A_{(p)}^{\binom{n}{d}}$ (assuming $|A| < \infty$).
- Claim: if $X \simeq \Omega^2 Z$ for some Z, then $\Lambda_0 X = \mathrm{Hom}(\lambda^*(\Theta^*), \pi_*(X))$.
- Proof uses $\Sigma^2((P \times Q)_+) \simeq \Sigma^2(S^0 \vee P \vee Q \vee (P \wedge Q))$, iterated to split $\Sigma^2 B(\mathbb{Z}^n)_+ = \Sigma^2((S^1)^n_+)$ as a wedge of spheres, together with $[B(\mathbb{Z}^n)_+, X] = [\Sigma^2 B(\mathbb{Z}^n)_+, Z]$.
- Theorem (Lurie): $\chi\colon L \otimes_{E^0} E^0(X) \to C(X)$ is iso if X is a π-finite space/finite ∞-groupoid.

Morava theory of Eilenberg-MacLane spaces

- Recall $\mathrm{spf}(E^0BA) = \mathrm{Hom}(A^*, \mathbb{G}) = \mathrm{Tor}(A, \mathbb{G})$
- Also $\mathrm{Hom}(\lambda^d\Theta^*, A) = \mathrm{Tor}(\Theta, \dots, \Theta, A)^{\Sigma_d}$
- Theorem (essentially Ravenel-Wilson):
  $\mathrm{spf}(E^0(B^dC_{p^k})) = \lambda^d(\ker(p^k.1 \colon \mathbb{G} \to \mathbb{G}))$ ($= 0$ for $d > n$).
- Corollary: for A finite abelian: $\mathrm{spf}(E^0(B^dA)) = \mathrm{Tor}(\mathbb{G}, \dots, \mathbb{G}, A)^{\Sigma_d}$
- Corollary: $\mathrm{spf}(E^0(B^{n+1}\mathbb{Z})) = \mathrm{spf}(E^0(B^n(\mathbb{Z}/p^\infty))) = \mathrm{Tor}(\mathbb{G}, \dots, \mathbb{G})^{\Sigma_n}$, and this is a one-dimensional formal group of height one.
- Equivalently $E^0(B^n(\mathbb{Z}/p^\infty)) = E^0[\![y]\!]$ with $\mathrm{mult}^*(y) = y_0 + y_1 + y_0y_1$.
- Alternatively, $K^0(B^dA)$ is a finite-dimensional Hopf algebra over $\mathbb{F}_p$, and the category of such is equivalent to a category of *Dieudonné modules*.

General ambidexterity

- Theorem (Hopkins-Lurie): any $q\colon X \to Y$ of finite ∞-groupoids/π-finite spaces is ambidextrous, i.e. $q_! \simeq q_*$ as functors $\mathcal{K}(X) \to \mathcal{K}(Y)$.
- There exists m such that all fibres $(q \downarrow b)$ have $\pi_k = 0$ for all $k > m$. Greenlees-Sadofsky gives $m \leq 1$; do $m > 1$ by induction.
- Key case: $c\colon B^m C_p \to 1$ is ambidextrous.
- Assuming this, any q with fibres $B^m C_p$ is ambidextrous. Thus any $B^m A \to 1$ is ambidextrous. Thus any q with fibre $B^m A$ is ambidextrous, such as the Postnikov truncation $X \to X_{<m}$ (if $X = X_{\leq m}$).But $X_{<m} \to 1$ is ambidextrous by induction, so $X \to 1$ is ambidextrous. Thus any q with m-truncated fibres is ambidextrous.
- Further reduction (similar to $m = 1$): enough to show that
  $c_!(c^*(K)) \to c_*(c^*(K))$ is iso for $c\colon B^m C_p \to 1$
  or that $K_*(B^m C_p) \to K^*(B^m C_p)$ is iso
  or that the corresponding pairing on $K^*(B^m C_p)$ is perfect
  or that the corresponding pairing on $E^*(B^m C_p)$ is perfect.
- As with the case $m = 1$, the pairing is given by a map $\theta\colon E^0(B^m C_p) \to E^0$ and we also have a trace map $\tau\colon E^0(B^m C_p) \to E^0$, and these satisfy $\tau = p^k \theta$ for some k.
- One can calculate enough structure of $E^0(B^m C_p)$ to deduce from this that θ gives a perfect pairing.