# Ambidexterity 3

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May 26, 2023

# Equivariant spectra

- For a finite groupoid $G$, put $\mathcal{S}_G = [G, \mathcal{S}]$.
- Any $V : G \to \mathcal{L}$ (i.e. $V \in \mathcal{L}_G$) gives $S^V, S^{-V} \in \mathcal{S}_G$.
- Define $\pi_n^G(X) = \varinjlim_{V \in \mathcal{L}_G} [S^V, (S^{-n} \wedge X)(V)]^G$.
- Say $f : X \to Y$ is a *genuine equivalence* if $\pi_*^H(f)$ is iso for all subgroupoids $H \leq G$.
- Say $f : X \to Y$ is a *Borel equivalence* if $\pi_*(f_a) : \pi_*(X_a) \to \pi_*(Y_a)$ is iso for all $a \in \mathrm{obj}(G)$ (or equivalently, $\pi_*^H(f)$ is iso for all discrete subgroupoids $H \leq G$).
- Two $\infty$-categories: $\mathcal{S}_G$ has Borel equivalences inverted, $\mathcal{GS}_G$ has genuine equivalences inverted.
- For $X \in \mathcal{S}_G$ and $a, b \in \mathrm{obj}(G)$ we have $G(a,b)_+ \to \mathcal{S}(X_a, X_b)$ and correspondingly $\Sigma^\infty G(a,b)_+ \wedge X_a \to X_b$ and $\Sigma^\infty G(a,b)_+ \to F(X_a, X_b)$.
- The map $p : G(a,b) \to 1$ gives a transfer $p^t : S^0 \to \Sigma^\infty G(a,b)_+$ in $h\mathcal{S}$.
- Combining these gives a map $S^0 \to F(X_a, X_b)$ or $\nu_{ab} : X_a \to X_b$, which is essentially $\sum_{u \in G(a,b)} u_*$.
- These induce $\nu : c_!(X) \to c_*(X)$ in $h\mathcal{S}$; iso if $G$ is empty or contractible.
- Now suppose we have $q : G \to H$. By applying the above to comma categories, we get $\nu : q_!(X) \to q_*(X)$ in $h\mathcal{S}_H$, and this is iso if $q$ is faithful (*Wirthmüller isomorphism*).

# Periodic complex cobordism

- For a complex inner product space $V$ put $MP'(V) = \{(u, U) \mid u \in U \leq V \oplus V\} \cup \{\infty\}$
- Define an orthogonal ring spectrum $MP$ by $MP(V) = \Omega^{iV} MP'(V \oplus iV)$.
- Using $\pi_{2+k} MP'(\mathbb{C}) = \pi_{2+k}(S^0 \vee \mathbb{C}P^2 \vee S^4) \to \pi_k MP = MP_k = MP^{-k}$ we get $u^{-1} \in MP_{-2}$ and $u \in MP_2$ with $u^{-1}u = 1$; so $MP_{2i+j} = MP_j$.
- There are natural maps $\Sigma^V P(\mathbb{C} \oplus V) \to MP'(V)$ which assemble to give $x\colon \Sigma^\infty \mathbb{C}P^\infty \to MP$ i.e. $x \in MP^0(\mathbb{C}P^\infty)$.
- Using cofibrations $\mathbb{C}P^{n-1} \to \mathbb{C}P^n \to S^{2n}$ we get $MP^*(\mathbb{C}P^\infty) = MP^*[\![x]\!]$.
- Using Künneth we get $MP^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = MP^*[\![y, z]\!]$.
- We can identify $\mathbb{C}P^\infty$ with $(\mathbb{C}[t] \setminus \{0\})/\mathbb{C}^\times$, which is a commutative monoid under multiplication (and a group up to homotopy).
- We have $\text{mult}^*(x) = F_{MP}(y, z) = \sum_{i,j \geq 0} a_{ij} y^i z^j$ for some $a_{ij} \in \pi_0(MP)$.
- Commutative monoid structure implies $F_{MP}(x, 0) = x$ and $F_{MP}(x, y) = F_{MP}(y, x)$ and $F_{MP}(x, F_{MP}(y, z)) = F_{MP}(F_{MP}(x, y), z)$ so $F_{MP}$ is a *formal group law*.
- Theorem (Quillen): $\pi_1(MP) = 0$ and $\pi_0(MP) = \mathbb{Z}[a_1, a_2, \dots]$ with $a_1 = a_{11},\ a_2 = a_{12},\ a_3 = a_{22} - a_{13},\ a_4 = a_{15},\ a_5 = a_{16} + a_{25} + a_{34}, \dots$
- Also: for any formal group law $F$ over any ring $R$, there is a unique $\phi\colon \pi_0(MP) \to R$ carrying $F_{MP}$ to $F$. So $\pi_0(MP)$ is the *Lazard ring*.

# Morava $K$ and $E$

- Fix a prime $p$ and $n > 0$.
- Put $l(x) = \sum_{k \geq 0} x^{p^{nk}}/p^k \in \mathbb{Q}[\![x]\!]$, $\quad F(x, y) = l^{-1}(l(x) + l(y)) \in \mathrm{FGL}(\mathbb{Q})$.
- In fact $F \in \mathrm{FGL}(\mathbb{Z})$ so we can reduce mod $p$ to get $F_K \in \mathrm{FGL}(\mathbb{F}_p)$.
- There is a unique $\phi_K \colon MP_0 \to \mathbb{F}_p$ carrying $F_{MP}$ to $F_K$.
- Write $x +_F y = F(x, y)$ and $[n]_F(x) = x + \cdots + x$ ($n$ terms).
- We find that $[p]_K(x) = [p]_{F_K}(x) = x^{p^n}$ i.e. $F_K$ has *height* $n$.
- Define $E_0 = \mathbb{Z}_p[\![u_1, \ldots, u_{n-1}]\!]$ with $u_0 = p$, $u_n = 1$.
- For $I = (i_1, \ldots, i_r)$ in $\{1, \ldots, n\}^r$ we put $|I| = r$ and $\|I\| = i_1 + \cdots + i_r$ and $\pi_t(I) = \prod_{s<t} p^{i_s}$ and $u_I = \prod_{t=1}^{r} u_{i_t}^{\pi_t(I)}$.Then put $l_E(x) = \sum_I u_I x^{p^{\|I\|}}/p^{|I|} \in (\mathbb{Q} \otimes E_0)[\![x]\!]$ and $F_E(x, y) = l_E^{-1}(l_E(x) + l_E(y))$.
- Using the *Functional Equation Lemma:* $F_E \in \mathrm{FGL}(E_0)$.
- This is a *universal deformation* of $F_K$: if $F' \in \mathrm{FGL}(R')$ and $F' = F_K$ mod a nilpotent ideal, then $F'$ comes from $F_E$ (details are complex).
- There is a unique $\phi_E \colon MP_0 \to E_0$ carrying $F_{MP}$ to $F_E$.
- Using Landweber exactness and Brown representability: there is a commutative ring spectrum $E$ with $E_0 X = \pi_0(E \wedge X) = E_0 \otimes_{MP_0} (MP_0 X)$.
- There is also a ring spectrum $K$ with $K^0 X = (E^0 X)/(u_0, \ldots, u_{n-1})$ whenever the sequence is regular (and same for $K_0 X$).

# Morava $K$ and $E$ of $BC_m$

- Put $U = \mathbb{C}[t] \setminus \{0\}$ so $\mathbb{C}P^\infty = U/\mathbb{C}^\times$
- Define $\phi_m \colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ by $\phi_m([f]) = [f^m]$, so $\phi_m^*(x) = [m]_{F_{MP}}(x) = [m]_{MP}(x) \in MP^0(\mathbb{C}P^\infty)$ (and same for $E, K$).
- The map $h(s,f)(t) = s + (1-s)(1+st)f(t)$ gives a contraction of $U$.
- Put $C_m = \langle e^{2\pi i/m} \rangle < \mathbb{C}^\times$ and $BC_m = U/C_m$.
- $\mathbb{C}P^\infty$ has a tautological bundle $T$ with $T_{[f]} = \mathbb{C}f$ and $\phi_m^*(T) \simeq T^{\otimes m}$.
- Then $BC_m = E(T^{\otimes m}) \setminus$ (zero section) so cofibre$(BC_m \to \mathbb{C}P^\infty) = \mathrm{Thom}(T^{\otimes m})$
- Using the Thom isomorphism we get $MP^0(BC_m) = MP^0[\![x]\!]/[m]_{MP}(x)$ and $MP^1(BC_m) = 0$ (and same for $E$, $K$).
- If $m = p^k m_1$ with $p \nmid m_1$ then $[m]_K(x)$ is a unit multiple of $[p^k]_K(x) = x^{p^{nk}}$ so $K^0(BC_m) = \mathbb{F}_p\{x^i \mid i < p^{nk}\}$.
- Similarly $E^0(BC_m) = E^0\{x^i \mid i < p^{nk}\}$ (free of finite rank over $E^0$).
- For $A$ finite abelian: $E^0(BA) = E^0(BC_{m_1}) \otimes_{E^0} \cdots \otimes_{E^0} E^0(BC_{m_r})$ is again free of finite rank, and $E^1 BA = 0$.
- Or: $\mathbb{G} := \mathrm{spf}(E^0(\mathbb{C}P^\infty))$ is a formal group scheme over $S := \mathrm{spf}(E^0)$, and $\mathrm{spf}(E^0(BA)) = \mathrm{Hom}(A^*, \mathbb{G})$, where $A^* = \mathrm{Hom}(A, S^1)$.

# Morava $K$ and $E$ of $BG$

- Claim: if $G$ is a finite groupoid and $X\colon G \to \mathcal{T}$ is a finite $G$-CW complex then $E^*(c_!(X))$ is finitely generated over $E^*$.
- Can reduce to the case of a group, where $c_!(X) = X_{hG} = (EG \times X)/G$ and $(G/H)_{hG} = EG/H = BH$.
- The CW structure gives $\operatorname{skel}_d(X)/\operatorname{skel}_{d-1}(X) \simeq \Sigma^d \bigvee_i (G/H_i)_+$; so enough to prove $E^*(BH)$ is finitely generated for all isotropy groups $H$.
- So if all isotropy groups for $X$ are abelian, then $E^*(X_{hG})$ is fg.
- Put $m = |G|$ and $F = \{\text{flags } (W_0 < W_1 < \cdots < W_m = \mathbb{C}[G])\}$. Then $X \times F$ has abelian isotropy so $E^*((X \times F)_{hG})$ is fg.
- Also $(X \times F)_{hG}$ is an iterated projective bundle over $X_{hG}$ so $E^*((X \times F)_{hG})$ is free over $E^*(X_{hG})$ with canonical finite basis; so $E^*(X_{hG})$ is also fg.
- Also: there is an equaliser $E^*(X_{hG}) \to E^*((X \times F)_{hG}) \rightrightarrows E^*((X \times F^2)_{hG})$.
- Similarly $K^i(BG)$ is a finite vector space over $\mathbb{F}_p$ with dual $K_{-i}(BG)$, and $\widetilde{K}^*(BG) = \ker(K^*(BG) \to K^*)$ is a nilpotent ideal.
- (Story for $E$ is more complicated; $E_*BG$ is the wrong object.)
- There are nice calculations for $\Sigma_m$, $GL_m(F)$ with $p \nmid |F| < \infty$, groups of small nilpotence class; also the generalised character theory of Hopkins-Kuhn-Ravenel, and a clear picture of the relation with representation theory (*Chern approximation*).

## Tate vanishing

- Theorem (Greenlees-Sadofsky): for $G$ a finite groupoid, $\nu\colon c_!(c^*(K)) \to c_*(c^*(K))$ is an equivalence.
- Equivalently: if $G$ is a finite group, then $K \wedge BG_+ \to F(BG_+, K)$ is an equivalence, or $K_*(BG) = K^{-*}(BG)$.
- For the proof we work in the genuine equivariant category $\mathcal{GS}_G$.
- Two fixed point functors $\phi^G, \lambda^G\colon \mathcal{GS}_G \to \mathcal{S}$ (geometric, Lewis-May).
- $\phi^G \Sigma^\infty X = \Sigma^\infty X^G$ and $\phi^G(S^{\pm V}) = S^{\pm V^G}$ and $\phi^G(X \wedge Y) = \phi^G(X) \wedge \phi^G(Y)$.
- $\pi_*^G(X) = \pi_*(\lambda^G(X))$ and $\mathcal{GS}(c^*(X), Y) = \mathcal{S}(X, \lambda^G(Y))$ and if $Y$ is free then $\lambda^G(Y) \simeq Y^{hG} \simeq Y_{hG}$ (cf $M_G \simeq M^G$ for free $\mathbb{Z}[G]$-modules).
- Recall $EG$ is contractible with free $G$-action, and $\widetilde{E}G = \operatorname{cof}(EG_+ \to S^0)$; say $X$ is *free* iff $X \in \operatorname{loc}\langle G_+\rangle$ iff $EG_+ \wedge X \simeq X$ iff $\widetilde{E}G \wedge X = 0$.
- As $EG_+ \wedge K$ is free, we have $\lambda^G(EG_+ \wedge K) = (EG_+ \wedge K)_{hG} = K \wedge BG_+$.
- There is a map $\widetilde{\nu}\colon EG_+ \wedge K \to F(EG_+, K)$ such that $\lambda^G(\widetilde{\nu}) = \nu$.
- $EG_+ \wedge (-)$ converts nonequivariant equivalences to equivariant ones, so $EG_+ \wedge K \simeq EG_+ \wedge F(EG_+, K)$. Using this: $\operatorname{cof}(\widetilde{\nu}) \simeq \widetilde{E}G \wedge F(EG_+, K)$.
- Thus: enough to prove $\widetilde{E}G \wedge F(EG_+, K) = 0$ in $\mathcal{GS}_G$.

# Tate vanishing

For finite $G$, we need $K^{tG} = 0$, where $K^{tG} = \widetilde{E}_G \wedge F(EG_+, K) = 0$.

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- For $H < G$ we have $\mathrm{res}^G_H(K^{tG}) \simeq K^{tH}$; assume $K^{tH} = 0$ by induction.
- We also have $G/H_+ \wedge K^{tG} = \mathrm{ind}^G_H(\mathrm{res}^G_H(K^{tG})) = \mathrm{ind}^G_H(0) = 0$.
- Put $V = \mathbb{C}[G] \ominus \mathbb{C}$ so $V^G = 0$ so $S(\infty V)^G = \emptyset$ so $S(\infty V)_+ \in \mathrm{loc}\langle G/H_+ \mid H < G\rangle$ so $S(\infty V)_+ \wedge K^{tG} = 0$
- As $S^{\infty V} = \mathrm{cof}(S(\infty V)_+ \to S^0)$, we get $K^{tG} \simeq S^{\infty V} \wedge K^{tG} \simeq S^{\infty V} \wedge F(EG_+, K)$.
- Now $\lambda^G(S^{mV} \wedge F(EG_+, K)) = \lambda^G F(S^{-mV} \wedge EG_+, K) = F(BG_+^{-mV}, K)$ and $\pi_*(\text{this}) = K^*(BG).(\text{Thom class of } -mV)$.
- Taking $m \to \infty$ we get $\pi^G_*(K^{tG}) = e(V)^{-1}K^{-*}(BG)$, but $e(V)$ lies in the nilpotent ideal $\widetilde{K}^0(BG)$ so inverting $e(V)$ gives 0.
- As $\pi^G_*(K^{tG}) = 0$ and $\mathrm{res}^G_H(K^{tG}) = 0$ for $H < G$ we have $K^{tG} = 0$.
- Conclusion: $K^0(BG) \simeq K_0(BG) \simeq \mathrm{Hom}(K^0(BG), \mathbb{F}_p)$, so there is a perfect pairing $K^0(BG) \otimes K^0(BG) \to \mathbb{F}_p$ (of the form $a \otimes b \mapsto \theta(ab)$).
- If $n > 1$ then $K^0(BC_{p^k}) = \mathbb{F}_p[x]/x^{p^{nk}}$ and $\theta(f) = $ coefficient of $x^{p^{nk}-1}$ in $f$.
- For any $G$, the map $(K^0(BG) \xrightarrow{\delta_!} K^0(BG) \otimes K^0(BG) \xrightarrow{1\otimes\theta} K^0(BG))$ is the identity, and this characterises $\theta$.