Ambidexterity 2

Neil Strickland

May 26, 2023

First hints of recursion

- Consider $\mathcal{C}$ with all finite (co)limits and functors $c\colon G \to 1$ and $q\colon G \to H$.
- Say G is contractible iff ($G \neq \emptyset$ and $|G(x,y)| = 1$ for all x and y) iff $c\colon G \to 1$ is an equivalence.
- If so, the map $\rho = (c_*(C) \xrightarrow{p_a} C_a \xrightarrow{i_a} c_!(C))$ is independent of $a \in \mathrm{obj}(G)$ and is an isomorphism; so $c_! \simeq c_*$. (This is level -2.)
- All comma categories $(q \downarrow b)$ are contractible iff q is an equivalence; if so, there is a canonical isomorphism $q_! \simeq q_*$.
- If G is empty then there is a unique object in $\mathcal{C}_G$ which is sent by $c_!$ and c_* to the initial and terminal objects of $\mathcal{C}$; so there is a unique morphism $\nu\colon c_! \to c_*$. This is an isomorphism iff $\mathcal{C}$ is pointed. (This is level -1.)
- All comma categories $(q \downarrow b)$ are (empty or contractible) iff q is full and faithful; if so, and $\mathcal{C}$ is pointed, there is a canonical isomorphism $q_! \simeq q_*$.
- Suppose that G is a finite set and $\mathcal{C}$ is pointed. Given $C \in \mathcal{C}_G = \prod_{x \in G} \mathcal{C}$ define $\nu_{xy}\colon C_x \to C_y$ to be 1 if $x = y$ and 0 if $x \neq y$. These give $\nu\colon c_!(C) = \bigoplus_x C_x \to \prod_x C_x = c_*(C)$. For all such maps to be iso, we need $\mathcal{C}$ to be semiadditive. (This is level 0.)
- All comma categories are (equivalent to) finite sets iff q is faithful. If so, and $\mathcal{C}$ is semiadditive, then $q_! \simeq q_*$.

First hints of recursion

- Suppose that $\mathcal{C}$ is semiadditive so we have ambidexterity at level 0.
- Given a finite family of morphisms $f_i \colon C \to C'$ in $\mathcal{C}$, we want to consider $\sum_i f_i \colon C \to C'$.
- This is

$$\begin{array}{ccccc} C & \xrightarrow{\Delta} & \prod_i C & \xleftarrow[\simeq]{\nu} & \bigoplus_i C & & \\ & & {\scriptstyle \prod_i f_i}\downarrow & & \downarrow{\scriptstyle \bigoplus_i f_i} & & \\ & & \prod_i C' & \xleftarrow[\nu]{\simeq} & \bigoplus_i C' & \xrightarrow{\nabla} & C' \end{array}$$

- or, regarding $(f_i)_{i \in I}$ as a morphism $f \colon c^*(C) \to c^*(C')$:

$$\begin{array}{ccccccc} C & \xrightarrow{\eta} & c_*(c^*(C)) & \xleftarrow[\simeq]{\nu} & c_!(c^*(C)) & & \\ & & {\scriptstyle c_*(f)}\downarrow & & \downarrow{\scriptstyle c_!(f)} & & \\ & & c_*(c^*(C')) & \xleftarrow[\nu]{\simeq} & c_!(c^*(C')) & \xrightarrow{\epsilon} & C' \end{array}$$

- Now let G be any finite groupoid, and consider $C \in \mathcal{C}_G$. We want to define $\nu \colon c_!(C) \to c_*(C)$ to give potential ambidexterity at level 1.
- This will use the maps $\nu_{xy} = \sum_{u \in G(x,y)} u_* \colon C_x \to C_y$ for $x, y \in \mathrm{obj}(G)$.
- To form this sum, we are using ambidexterity at level 0.

∞-categories

- ∞-categories are a generalisation of categories/topological categories/model categories.
- We will not give details but only heuristic ideas.
- Given a space X and points $a, b \in X$, put $PX(a,b) = \{u\colon [0,1] \to X \mid u(0) = a,\ u(1) = b\}$.
- Given $v \in PX(b,c)$ and $u \in PX(a,b)$ define $v \circ u \in PX(a,c)$ by $(v \circ u)(t) = u(2t)$ for $0 \le t \le 1/2$ and $v(2t-1)$ for $1/2 \le t \le 1$.
- This is not a category because composition is not associative. If we put

  $$\Pi X(a,b) = \pi_0(PX(a,b)) = \{ \text{ pinned homotopy classes of paths from } a \text{ to } b \ \}$$

  then we get a category: the path groupoid of X.
- A map u from the n-simplex Δ_n to X gives points $a_i = u(e_i)$ for $0 \le i \le n$, and paths $u_{ij} \in PX(a_i, a_j)$ for $0 \le i \le j \le n$, plus evidence that u_{ik} is pinned-homotopic to $u_{jk} \circ u_{ij}$.
- Let $(PX)_n$ be the (discrete) set of all continuous maps $\Delta_n \to X$. These sets, equipped with appropriate structure maps, form an ∞-category. Using this: topological spaces are equivalent to ∞-groupoids.

(Coherent) nerves

- Let $\mathcal{C}$ be an ordinary category.
  - An *n-simplex* of $\mathcal{C}$ is a system of objects c_i for $0 \leq i \leq n$, together with morphisms $u_{ij} \colon c_i \to c_j$ for $i \leq j$, such that $u_{ik} = u_{jk} \circ u_{ij}$ for all $i \leq j \leq k$.
  - We write $(N\mathcal{C})_n$ for the set of all n-simplices of $\mathcal{C}$.
  - These sets, equipped with appropriate structure maps, form an ∞-category. We do not distinguish this from $\mathcal{C}$.
- Let $\mathcal{C}$ be a topological category, so each $\mathcal{C}(a, b)$ is a space.
  - An *n-simplex* of $\mathcal{C}$ is a system of objects c_i for $0 \leq i \leq n$, together with continuous maps $h_{ij} \colon [0, 1]^{j-i-1} \to \mathcal{C}(c_i, c_j)$ for $i < j$, subject to some axioms.
  - From these we extract maps $u_{ij} = h(1, \dots, 1) \in \mathcal{C}(c_i, c_j)$, together with homotopies $u_{ik} \simeq u_{jk} \circ u_{ij}$, and higher homotopies proving the compatibilities of the lower homotopies.
  - We write $(N\mathcal{C})_n$ for the set of all n-simplices of $\mathcal{C}$.
  - These sets, equipped with appropriate structure maps, form an ∞-category. We do not distinguish this from $\mathcal{C}$.
- The above two definitions are compatible: if $\mathcal{C}(a, b)$ is always discrete, then any continuous map $[0, 1]^{j-i-1} \to \mathcal{C}(c_i, c_j)$ is necessarily constant.
- In particular: the category of spaces and the category of orthogonal spectra are topological categories, and so give ∞-categories.
- Another construction in the same spirit gives an ∞-category for any Quillen model category.

More ∞-categories

Let $\mathcal{C}$ be an ∞-category.

- There is a set $\mathcal{C}_0$ of objects, and a set $\mathcal{C}_1$ of morphisms, and a set $\mathcal{C}(a,b)_0 \subseteq \mathcal{C}_1$ of morphisms from a to b, for any pair of objects a and b.
- Given $v \in \mathcal{C}(b,c)_0$ and $u \in \mathcal{C}(a,b)_0$, there need not be a unique composite $v \circ u$. However, there is a family of potential composites, indexed by a subset of $\mathcal{C}_2$.
- In fact, there is a contractible space of potential composites. It is central to the philosophy of ∞-categories that a contractible space of choices is as good as a unique choice.
- The set $\mathcal{C}(a,b)_0$ should be thought of as the set of vertices of a simplicial set or space $\mathcal{C}(a,b)$. More generally, most sets in the theory can be considered as the set of vertices of a naturally defined space.
- There is a homotopy category $h\mathcal{C}$ where $\mathrm{obj}(h\mathcal{C}) = \mathcal{C}_0$ and $\mathrm{mor}(h\mathcal{C})$ is an appropriate quotient of $\mathcal{C}_1$. This construction generalises that homotopy category of a topological category or model category.
- A functor $F\colon \mathcal{C} \to \mathcal{D}$ of ∞-categories consists of compatible maps $F_n\colon \mathcal{C}_n \to \mathcal{D}_n$ for all $n \geq 0$.
- For a space X and a topological category $\mathcal{C}$ considered as ∞-categories, a functor $X \to \mathcal{C}$ is an X-parametrised family of $\mathcal{C}$-objects (or *local system*).

(Co)limits in ∞-categories

- In an ordinary category $\mathcal{C}$, an object t is *terminal* if $|\mathcal{C}(a,t)| = 1$ for all $a \in \mathrm{obj}(\mathcal{C})$.
- In an ∞-category $\mathcal{C}$, an object t is *terminal* if $\mathcal{C}(a,t)$ is contractible for all $a \in \mathcal{C}_0$.
- Dually, an object i is *initial* if $\mathcal{C}(i,b)$ is always contractible.
- In an ordinary category $\mathcal{C}$, given a diagram $u\colon I \to \mathcal{C}$, a *limit* for u is an initial object in the category $\mathrm{Cones}(u)$ of cones over u.
- Now suppose we have ∞-categories I and $\mathcal{C}$, together with a functor $u\colon I \to \mathcal{C}$. One can define an ∞-category $\mathrm{Cones}(u)$ by analogy with the ordinary case, and then define a limit to be a terminal object in $\mathrm{Cones}(u)$.
- We have $h\mathcal{C}(a, \varprojlim_i b_i) = h([I,\mathcal{C}])(c^*(a), b)$ but $h([I,\mathcal{C}]) \to [hI, h\mathcal{C}]$ need not be an equivalence.
- There is a dual treatment for colimits.
- As in ordinary category theory, in many contexts (co)limits exist for formal reasons.
- There is a definition of adjunctions in ∞-category theory. An adjunction $\mathcal{C} \leftrightarrows \mathcal{D}$ induces an adjunction $h\mathcal{C} \leftrightarrows h\mathcal{D}$.

Spaces

- Let $\mathcal{T}$ be the category of (compactly generated, weak Hausdorff) spaces.
- $\mathcal{T}_*$ is the category of spaces $X \in \mathcal{T}$ equipped with a basepoint $0_X \in X$ (sometimes called ∞).
- $X \vee Y = (X \amalg Y)/(0_X \sim 0_Y)$ and $X \wedge Y = (X \times Y)/((x, 0_Y) \sim (0_X, 0_Y) \sim (0_X, y))$.
- $F(X, Y) = \mathcal{T}_*(X, Y)$ with suitable topology (so $F(X, F(Y, Z)) \simeq F(X \wedge Y, Z)$ and $F(X \vee Y, Z) = F(X, Z) \times F(Y, Z)$).
- Sometimes we need to assume that X has the homotopy type of a CW complex and/or distinguish between weak and strong homotopy equivalences; we will not be careful about this.
- We regard $\mathcal{T}$ and $\mathcal{T}_*$ as ∞-categories by the coherent nerve construction.
- Put $\mathcal{L} = \{$ finite dimensional real inner product spaces$\}$
- For $V \in \mathcal{L}$ put $S^V = V \cup \{\infty\} \in \mathcal{T}_*$. These satisfy $S^V \wedge S^W \simeq S^{V \oplus W}$.

Spectra

$\mathcal{L} = \{$ finite dimensional real inner product spaces$\}$; $\mathcal{T}_* = \{$based spaces$\}$;
$S^V = V \cup \{\infty\} \in \mathcal{T}_*$; $S^V \wedge S^W = S^{V \oplus W}$.

- An (orthogonal) spectrum is a continuous functor $X \colon \mathcal{L} \to \mathcal{T}_*$ equipped with maps $S^V \wedge X(W) \to X(V \oplus W)$ subject to various axioms.
- We write $\mathcal{S}$ for the category of spectra, or the corresponding ∞-category.
- For $n \geq 0$ we put $\pi_n(X) = \varinjlim_V [S^n \wedge S^V, X(V)]$ and $\pi_{-n}(X) = \varinjlim_V [S^V, X(\mathbb{R}^n \oplus V)]$.
- A map $f \in \mathcal{S}(X, Y)$ is a weak equivalence iff $\pi_*(f)$ is an isomorphism.
- A based space T gives a spectrum $\Sigma^\infty T$ with $(\Sigma^\infty T)(V) = S^V \wedge T$. We often write T for $\Sigma^\infty T$.
- Let $H(V)$ be the free abelian group generated by V with a suitable topology. This gives an Eilenberg-MacLane spectrum H with $\pi_0(H) = \mathbb{Z}$ and $\pi_n(H) = 0$ for $n \neq 0$.
- Put $MO(V) = \{(u, U) \mid u \in U \leq V \oplus V, \dim(U) = \dim(V)\} \cup \{\infty\}$. This gives a spectrum MO with $\pi_n(MO) = \{$cobordism classes of closed n-manifolds$\}$.

Products of spectra

- A *pairing* from spectra X and Y to Z consists of maps $X(U) \wedge Y(V) \to Z(U \oplus V)$ subject to various axioms.
- There is a spectrum $X \wedge Y$ such that such pairings biject with morphisms $X \wedge Y \to Z$.
- $\Sigma^\infty T \wedge \Sigma^\infty U = \Sigma^\infty (T \wedge U)$
- It is easy to define
  $H(U) \wedge H(V) \to H(U \oplus V)$ and $MO(U) \wedge MO(V) \to MO(U \oplus V)$.
  These give $H \wedge H \to H$ and $MO \wedge MO \to MO$,
  making H and MO into commutative ring spectra.
- Put $(S^{-A})(V) = \{(f, v) \mid f \in \mathcal{L}(A, V),\ v \in V \ominus f(A)\} \cup \{\infty\}$. This gives a spectrum S^{-A} with $\mathcal{S}(S^{-A}, X) = X(A)$ and $S^{-A} \wedge S^{-B} = S^{-(A \oplus B)}$.
- For $X, Y \in \mathcal{S}$ we define $F(X, Y)(V) = \mathcal{S}(S^{-V} \wedge X, Y)$. This gives a spectrum $F(X, Y)$ with $\mathcal{S}(W, F(X, Y)) = \mathcal{S}(W \wedge X, Y)$.
- There is a natural weak equivalence $S^A \wedge S^{-A} \to S^0$.
- We also put $(X \vee Y)(V) = X(V) \vee Y(V)$ and
  $(X \times Y)(V) = X(V) \times Y(V)$.
- There is a natural inclusion $X \vee Y \to X \times Y$, which is a weak equivalence.

Negative spheres and stability

- There are two versions of suspension: $(\Sigma X)(V) = S^1 \wedge X(V)$ and $(\Sigma' X)(V) = X(\mathbb{R} \oplus V)$
- There is a natural map $\Sigma X \to \Sigma' X$, which is a weak equivalence.
- We also put $(\Omega X)(V) = F(S^1, X(V))$ so Ω is right adjoint to Σ giving $\eta\colon X \to \Omega\Sigma X$ and $\epsilon\colon \Sigma\Omega X \to X$.
- By comparing Σ and Σ' we check that η and ϵ are weak equivalences.
- Thus $\Sigma\colon h\mathcal{S} \to h\mathcal{S}$ is an equivalence with inverse Ω.
- The map $\delta\colon S^2 \to S^2 \vee S^2$ gives $\delta^*\colon \Omega^2 Y \times \Omega^2 Y = F(S^2 \vee S^2, Y) \to \Omega^2 Y$, making $\Omega^2 Y$ an abelian group object up to homotopy (naturally).
- Taking $Y = \Sigma^2 X$, we get an abelian group structure on any $X \in h\mathcal{S}$.
- This gives an abelian group structure on $h\mathcal{S}(W, X)$, making $h\mathcal{S}$ an additive category.
- Thus, we have ambidexterity at level 0: given a finite set I and $X \in \mathcal{S}(I)$ we have an equivalence $c_!(X) = \bigvee_i X_i \to \prod_i X_i = c_*(X)$.
- We now write $\bigoplus_i X_i$ for $\bigvee_i X_i$ or $\prod_i X_i$.

Collapse and transfer

- Consider a map $f\colon X \to Y$ of finite sets, giving $\Sigma^\infty f_+ \colon \Sigma^\infty X_+ = \bigoplus_x S^0 \to \bigoplus_y S^0 = \Sigma^\infty Y_+$.
- Because $h\mathcal{S}$ is additive, we can also define $f^t \colon \Sigma^\infty Y_+ \to \Sigma^\infty X_+$, essentially by $f^t(y) = \sum_{f(x)=y} x$.
- For a more precise model:
  - Put $V = \mathbb{R}[X]$, giving $i\colon X \to V \subset S^V$.
  - Put $s(v) = v/\sqrt{2(1 + \|v\|^2)}$, giving a homeomorphism from V to an open ball of radius $1/\sqrt{2}$.
  - Define $\widetilde{f}\colon V \times X \to V \times Y$ by $\widetilde{f}(v, x) = (s(v) + i(x), f(x))$, so $\widetilde{f}$ is an open embedding covering f.
  - Define $c\colon S^V \wedge Y_+ = (V \times Y) \cup \{\infty\} \to (V \times X) \cup \{\infty\} = S^V \wedge X_+$ by $c(\widetilde{f}(v, x)) = (v, x)$ and $c(v, y) = \infty$ for $(v, y) \notin \text{image}(\widetilde{f})$.
  - We have made this completely natural so it will be compatible with any group actions.
  - Apply $S^{-V} \wedge \Sigma^\infty(-)$ to get $f^t \colon \Sigma^\infty Y_+ \to \Sigma^\infty X_+$.
- For $f\colon 2 \to 1$, the map $c\colon S^2 \to S^2 \wedge 2_+ = S^2 \vee S^2$ is just the pinch map.