

Ambidexterity 1

Neil Strickland

May 26, 2023

- ▶ *Ambidexterity* refers to a situation where we have functors $\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$, such that G is both left adjoint and right adjoint to F .
- ▶ Easy example:
 - ▶ Let $q: G \rightarrow H$ be a functor of finite groupoids.
 - ▶ Let $\mathcal{V}_G = [G, \mathcal{V}]$ be the category of complex representations of complex representations of G .
 - ▶ This gives $q^*: \mathcal{V}_H \rightarrow \mathcal{V}_G$, and it is formal that there are left and right adjoints $q_!, q_*: \mathcal{V}_G \rightarrow \mathcal{V}_H$.
 - ▶ We can write down $\nu: q_! \rightarrow q_*$ and check that it is an isomorphism.
- ▶ Now consider the category $\mathcal{A}_G = [G, \mathcal{A}]$ of \mathbb{Z} -linear representations. We again have $q_! \vdash q^* \vdash q_*$ and $\nu: q_! \rightarrow q_*$. To prove that ν is an isomorphism, we divide by some group orders. If q is faithful, these orders are equal to one, so again $q_! \simeq q_*$.
- ▶ Alternatively: $q_! \simeq q_*$ on $\mathcal{PA}_G = \{\text{projectives in } \mathcal{A}_G\}$.
- ▶ Now consider the ∞ -category $\mathcal{S}_G = [G, \mathcal{S}]$ of spectral representations, i.e. functors to the ∞ -category of spectra. We can define $\nu: q_! \rightarrow q_*$ by a transfer construction; cofibre is the *Tate construction*. If q is faithful then $\text{Tate} = 0$ so ν is iso (theorem of Wirthmüller).

- ▶ Now fix a prime p and integer $n > 0$.
- ▶ Let K and E be the corresponding Morava K -theory and E -theory, and let \mathcal{K} be the ∞ -category of K -local spectra.
- ▶ For $q: G \rightarrow H$ of finite groupoids, we again have $q^*: \mathcal{K}_H \rightarrow \mathcal{K}_G$ and $q_!, q_*: \mathcal{K}_G \rightarrow \mathcal{K}_H$.
- ▶ We can again use transfers to define $\nu: q_! \rightarrow q_*$ (with cofibre = Tate).
- ▶ In this K -local context Tate = 0 so ν is an equivalence (essentially a theorem of Greenlees and Sadofsky).
- ▶ In the framework of Lurie: ∞ -groupoids are spaces ($X = BG$).
Finite ∞ -groupoids are π -finite spaces
(finite disjoint unions of connected spaces X with $|\prod_k \pi_k(X)| < \infty$).
- ▶ $\mathcal{K}(X)$ is the category of functors from the ∞ -groupoid X to \mathcal{K} , or families of K -local spectra continuously parametrised by points of X .
- ▶ For a functor $q: X \rightarrow Y$ of finite ∞ -groupoids (= map of π -finite spaces) we have $q^*: \mathcal{K}(Y) \rightarrow \mathcal{K}(X)$ and $q_!, q_*: \mathcal{K}(X) \rightarrow \mathcal{K}(Y)$ with $q_! \vdash q^* \vdash q_*$.
- ▶ Theorem of Hopkins and Lurie (2013): $q_! \simeq q_*$
- ▶ Proof: induction on the truncation level m . Greenlees-Sadofsky is $m = 1$. To define ν at level $m + 1$, we need ν to be an isomorphism at level m .

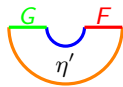
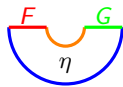
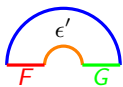
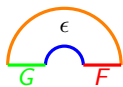
- ▶ There is a category $\mathcal{K}' \supseteq \mathcal{K}$ of *telescopically-local spectra* (also depending on (p, n))
- ▶ Ravenel's Telescope Conjecture (1984) is equivalent to $\mathcal{K} = \mathcal{K}'$.
- ▶ The other conjectures in the same paper were proved by Hopkins, Devinatz and Smith; this is the foundation of chromatic homotopy theory.
- ▶ There is a long history of attempts to disprove the Telescope Conjecture, and recent rumours of progress; but it remains open.
- ▶ The \mathcal{K}' -analogue of the Greenlees-Sadofsky theorem was proved by Kuhn (2003).
- ▶ The \mathcal{K}' -analogue of the Hopkins-Lurie theorem was proved by Carmeli, Schlank and Yanovski (2018).

Ambidexterity and the Atiyah-Segal theorem

- ▶ For a finite groupoid G , we have $c: G \rightarrow 1$ with $c_!, c_*: \mathcal{S}_G \rightarrow \mathcal{S}$ being the homotopy colimit and limit functors.
- ▶ If G is a group, then $c_!(X)$ is the Borel construction $X_{hG} = EG_+ \wedge_G X$ and $c_*(X)$ is the homotopy fixed point spectrum $X^{hG} = F_G(EG_+, X)$.
- ▶ In the K -local case with $E = \text{Morava}$ we have $\pi_*(c_*c^*E) = E^{-*}(BG)$, so the ambidexterity theorem has implications for $E^*(BG)$.
- ▶ Any $V \in \mathcal{V}_G = [G, \mathcal{V}]$ gives a complex vector bundle over BG with total space $c_!(V)$ and thus an element $\phi(V) \in KU^0(BG)$. This gives a ring map from the representation ring $R(G) = \pi_0(\mathcal{V}_G) - \pi_0(\mathcal{V}_G)$ to $KU^0(BG)$.
- ▶ The Atiyah-Segal theorem says that ϕ induces an isomorphism $R(G)_I^\wedge \rightarrow KU^0(BG)$ (where $I = \ker(R(G) \xrightarrow{\dim} \text{Map}(\pi_0(G), \mathbb{Z}))$).
- ▶ If $n = 1$ and G is a finite p -groupoid then $E = KU_p^\wedge$ and $R(G)_p^\wedge = R(G) \otimes \mathbb{Z}_p = E^0(BG)$.
This relates ambidexterity theorems for \mathcal{V}_G to those for \mathcal{K}_G .

Graphical calculus

$$\mathcal{D} \xleftarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{D}$$



- ▶ A *groupoid* is a category G in which all morphisms are invertible. In particular:
 - ▶ For every object $a \in G$, the set $G(a, a)$ is a group
 - ▶ For every morphism $u: a \rightarrow b$ we have a bijection $G(a, a) \rightarrow G(a, b)$ given by $g \mapsto ug$, and a group isomorphism $G(a, a) \rightarrow G(b, b)$ given by $g \mapsto ugu^{-1}$.
- ▶ A group can be regarded as a groupoid with only one object.
- ▶ A groupoid G is *finite* if the set $\pi_0(G)$ of isomorphism classes is finite, and $G(a, b)$ is finite for all a and b .
- ▶ If so, we can choose a list a_1, \dots, a_r containing precisely one object in each isomorphism class; then G is equivalent to the coproduct (in groupoids) of the finite groups $G_i = G(a_i, a_i)$.
- ▶ Thus, most questions about finite groupoids can be reduced to questions about finite groups.
- ▶ However, the groupoid picture is more natural for many purposes, and leads more easily to the theory for ∞ -groupoids.
- ▶ Also: a (finite) set is a (finite) groupoid with only identity morphisms.
- ▶ A groupoid G is equivalent to a set iff $|G(x, y)| \leq 1$ for all x and y , iff $|G(x, x)| = 1$ for all x .

Representations of finite groupoids

- ▶ Let \mathcal{A} be the category of abelian groups, and let \mathcal{A}_G be the category of functors from G to \mathcal{A} .
- ▶ Given $A \in \mathcal{A}_G$ write A_x for the value of A at an object x , and $u_* : A_x \rightarrow A_y$ for the map induced by $u \in G(x, y)$.
- ▶ There is a closed symmetric bimonoidal structure on \mathcal{A}_G , with $(A \oplus B)_x = A_x \oplus B_x$ and $(A \otimes B)_x = A_x \otimes B_x$ and $\underline{\text{Hom}}(A, B)_x = \text{Hom}(A_x, B_x)$ and $\mathbb{1}_x = \mathbb{Z}$.
- ▶ The functor $c: G \rightarrow \mathbf{1}$ gives $c^*: \mathcal{A} \rightarrow \mathcal{A}_G$ with $c^*(A)_x = A$ for all x . This has left adjoint $c_!(A) = \lim_{\rightarrow G} A$ and right adjoint $c_*(A) = \lim_{\leftarrow G} A$.
- ▶ For more general $q: G \rightarrow H$ and $b \in \text{obj}(H)$, we have a comma category $(q \downarrow b)$ with objects $\{(x, v) \mid x \in \text{obj}(G), v \in H(q(x), b)\}$ and morphisms

$$(q \downarrow b)((x_0, v_0), (x_1, v_1)) = \{g \in G(x_0, x_1) \mid v_1 \circ q(g) = v_0\}.$$

- ▶ This is isomorphic to the category of pairs (x, w) with $w \in H(b, q(x))$.
- ▶ For $A \in \mathcal{A}_G$ the construction $(x, v) \mapsto A_x$ gives a functor $(q \downarrow a) \rightarrow \mathcal{A}$ whose colimit and limit are $q_!(A)_b$ and $q_*(A)_b$ respectively (*Kan formula*).
- ▶ Thus, ambidexterity for $q_!$ and q_* relates to (co)limit ambidexterity for the comma categories $(q \downarrow b)$.

Representations of finite groupoids

- ▶ For $A \in \mathcal{A}_G$ and $x, y \in \text{obj}(G)$ we define $\nu'_{xy} = \sum_{u \in G(x,y)} u_* : A_x \rightarrow A_y$.
- ▶ For $g: x' \rightarrow x$ and $h: y \rightarrow y'$ we find that $\nu'_{x'y'} = h_* \nu'_{xy} g_* : A_{x'} \rightarrow A_{y'}$.
Thus, there is a unique $\nu: c_!(A) \rightarrow c_*(A)$ with
 $\nu'_{xy} = (A_x \xrightarrow{i_x} c_!(A) \xrightarrow{\nu} c_*(A) \xrightarrow{p_y} A_y)$ for all x, y .
- ▶ For any $a \in \text{obj}(G)$ we have a projective generator P_a with
 $(P_a)_x = \mathbb{Z}\{G(a, x)\} \simeq \mathbb{Z}\{G(x, a)\}$
so Yoneda gives isomorphisms $\mathcal{A}_G(P_a, A) \simeq A_a \simeq \mathcal{A}_G(A, P_a)$.
Taking A to be constant, we get $c_!(P_a) = c_*(P_a) = \mathbb{Z}$.
One can check that $\nu: c_!(P_a) \rightarrow c_*(P_a)$ is an isomorphism.
- ▶ As ν is iso for projective generators, it is iso for all projectives.

Representations of finite groupoids

- ▶ For $A \in \mathcal{A}_G$ and $x, y \in \text{obj}(G)$ we define $\nu'_{xy} = \sum_{u \in G(x,y)} u_* : A_x \rightarrow A_y$.
 - ▶ For $g: x' \rightarrow x$ and $h: y \rightarrow y'$ we find that $\nu'_{x'y'} = h_* \nu_{xy} g_* : A_{x'} \rightarrow A_{y'}$. Thus, there is a unique $\nu: c_!(A) \rightarrow c_*(A)$ with $\nu'_{xy} = (A_x \xrightarrow{i_x} c_!(A) \xrightarrow{\nu} c_*(A) \xrightarrow{p_y} A_y)$ for all x, y .
-

- ▶ Define $d: \text{obj}(G) \rightarrow \mathbb{N}^+$ by $d(x) = |G(x, x)|$. This satisfies $d(x) = |G(x, y)| = d(y)$ whenever $G(x, y) \neq \emptyset$, so it is constant on isomorphism classes.
- ▶ Now choose a list a_1, \dots, a_r of isomorphism class representatives, and define $\rho = \sum_{k=1}^r (c_*(A) \xrightarrow{p_{a_k}} A_{a_k} \xrightarrow{i_{a_k}} c_!(A))$.
- ▶ One can check that ρ does not depend on the choice of a_k , and that $\rho\nu: c_!(A) \rightarrow c_!(A)$ is multiplication by d , and that $\nu\rho: c_*(A) \rightarrow c_*(A)$ is also multiplication by d . Thus ν is iso when d is invertible.
- ▶ As d is always invertible in \mathbb{C} we see that ν is iso for $c_!, c_*: \mathcal{V}_G \rightarrow \mathcal{V}$. When G is a finite group this is the well-known isomorphism $V_G \simeq V^G$.
- ▶ By reduction to comma categories: for any $q: G \rightarrow H$ of finite groupoids we have $q_! \simeq q_*: \mathcal{V}_G \rightarrow \mathcal{V}_H$.

Representations of finite groupoids

- ▶ $d: \text{obj}(G) \rightarrow \mathbb{N}^+$ by $d(x) = |G(x, x)|$
 - ▶ $\nu: c_!(A) \rightarrow c_*(A)$ is an isomorphism in any context where d is invertible.
-

- ▶ G is equivalent to a set iff $d = 1$. If so, then ν is invertible, and we have $c_!(A) \simeq c_*(A) \simeq \bigoplus_{k=1}^r A_{a_k}$.
- ▶ Exercise: for $q: G \rightarrow H$, the comma categories $(q \downarrow b)$ are equivalent to sets iff they have $d = 1$ iff $q: G(x, y) \rightarrow H(q(x), q(y))$ is injective for all $x, y \in \text{obj}(G)$ iff q is a faithful functor.
- ▶ Recall that $q_!(A)_b$ and $q_*(A)_b$ are the colimit and limit of the same functor on $(q \downarrow b)$. Thus, if q is faithful then $q_! \simeq q_*$.